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Reflections in Triangle Geometry

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On the 10th Anniversary of Hyacinthos

Abstract. This paper is a survey of results on reflections in triangle geometry. We work with homogeneous barycentric coordinates with reference to a given triangle ABC and establish various concurrency and perspectivity results related to triangles formed by reflections, in particular the reflection triangle $P^{(a)}P^{(b)}P^{(c)}$ of a point P in the sidelines of ABC, and the triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ of the vertices of ABC in their respective opposite sides. We also consider triads of concurrent circles related to these reflections. In this process, we obtain a number of interesting triangle centers with relatively simple coordinates. While most of these triangle centers have been catalogued in Kimberling's *Encyclopedia of Triangle Centers* [27] (ETC), there are a few interesting new ones. We give additional properties of known triangle centers related to reflections, and in a few cases, exhibit interesting correspondences of cubic curves catalogued in Gibert's *Catalogue of Triangle Cubics* [14] (CTC).

CONTENTS

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Notations. We adopt the usual notations of triangle geometry and work with homogeneous barycentric coordinates with reference to a given triangle ABC with sidelengths a, b, c and angle measures A, B, C. Occasionally, expressions for coordinates are simplified by using Conway's notation:

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \quad S_C = \frac{a^2+b^2-c^2}{2},$$

subject to $S_{AB}+S_{BC}+S_{CA}=S^2$, where S is twice the area of triangle ABC, and S_{BC} stands for S_BS_C etc. The labeling of triangle centers follows ETC [27], except for the most basic and well known ones listed below. References to triangle cubics are made to Gibert's CTC [14].

G	X_2	centroid	O	X_3	circumcenter
H	X_4	orthocenter	N	X_5	nine point center
E_∞	X_{30}	Euler infinity point	E	X_{110}	Euler reflection point
I	X_1	incenter	$G_{\rm e}$	X_7	Gergonne point
$N_{\rm a}$	X_8	Nagel point	$F_{\rm e}$	X_{11}	Feuerbach point
K	X_6	symmedian point	$F_\pm$	X_{13}, X_{14}	Fermat points
$J_\pm$	$X_{15},\ X_{16}$	isodynamic points	W	X_{484}	first Evans perspector

P^*	isogonal conjugate of P
$P^\bullet$	isotomic conjugate of P
P^{-1}	inverse of P in circumcircle
P/Q	cevian quotient
$P_aP_bP_c$	cevian triangle of P
$P^aP^bP^c$	anticevian triangle of P
$P_{[a]}$	pedal of P on BC
$P^{(a)}$	reflection of P in BC
E_t	Point on Euler line dividing OH in the ratio $t:1-t$
$\mathscr{C}(P,Q)$	Bicevian conic through the traces of P and Q on the sidelines

1. The reflection triangle

Let P be a point with the homogeneous barycentric coordinates $(u : v : w)$ in reference to triangle ABC. The reflections of P in the sidelines BC, CA, AB are the points

$$\begin{aligned} P^{(a)} &= (-a^2u : (a^2+b^2-c^2)u + a^2v : (c^2+a^2-b^2)u + a^2w), \\ P^{(b)} &= ((a^2+b^2-c^2)v + b^2u : -b^2v : (b^2+c^2-a^2)v + b^2w), \\ P^{(c)} &= ((c^2+a^2-b^2)w + c^2u : (b^2+c^2-a^2)w + c^2v : -c^2w). \end{aligned}$$

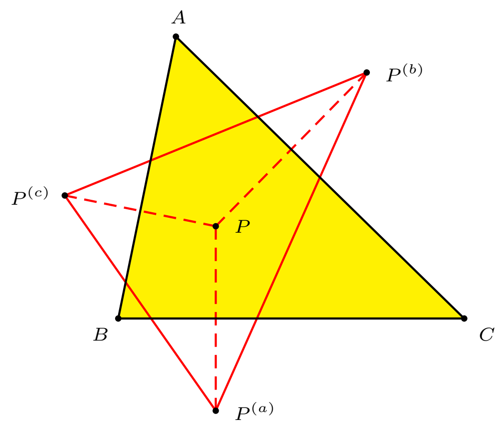


Figure 1. The reflection triangle

We call $P^{(a)}P^{(b)}P^{(c)}$ the reflection triangle of P (see Figure 1). Here are some examples.

(1) The reflection triangle of the circumcenter O is oppositely congruent to ABC at the midpoint of OH, which is the nine-point center N. This is the only reflection triangle congruent to ABC.

(2) The reflection triangle of H is inscribed in the circumcircle of ABC (see Remark (1) following Proposition 2 and Figure 3(b) below).

(3) The reflection triangle of N is homothetic at O to the triangle of reflections (see Propositioin 5 below).

Proposition 1. *The reflection triangle of P is*

(a) *right-angled if and only if P lies on one of the circles with centers* $\begin{array}{c}K^a\\K^b\\K^c\end{array}$ *passing through* $\begin{array}{c}B,\ C\\C,\ A\\A,\ B\end{array}$ *respectively,*

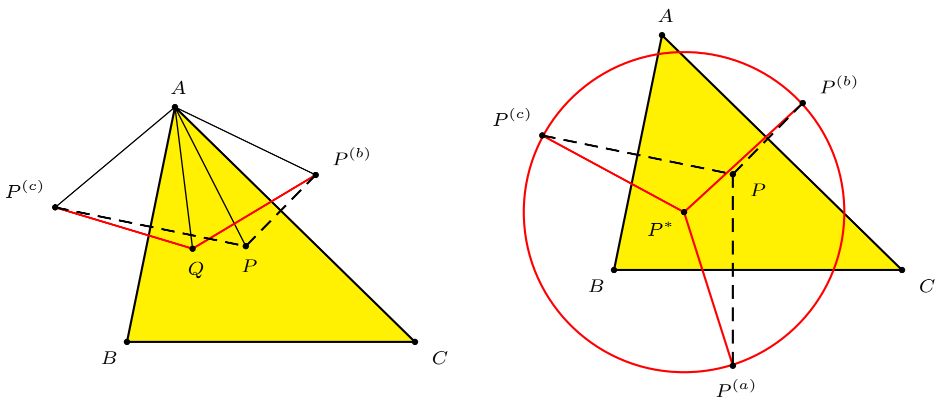
(b) *isosceles if and only if P is on one of the Apollonian circles, each with diameter the feet of the bisectors of an angle on its opposite side,*

(c) *equilateral if and only if P is one of the isodynamic points $J_\pm$,*

(d) *degenerate if and only if P lies on the circumcircle.*

1.1. *Circle of reflections.*

Proposition 2. *The circle $P^{(a)}P^{(b)}P^{(c)}$ has center P^*.*



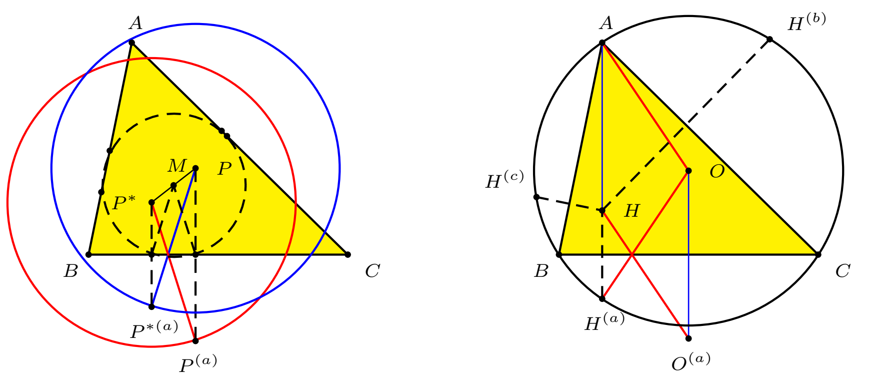
(a) Reflections and isogonal lines (b) circle of reflections

Figure 2. Circle of reflections of P with center P^*

Proof. Let Q be a point on the line isogonal to AP with respect to angle A, *i.e.*, the lines AQ and AP are symmetric with respect to the bisector of angle BAC (see Figure 2(a)). Clearly, the triangles $AQP^{(b)}$ and $AQP^{(c)}$ are congruent, so that Q is equidistant from $P^{(b)}$ and $P^{(c)}$. For the same reason, any point on a line isogonal to BP is equidistant from $P^{(c)}$ and $P^{(a)}$. It follows that the isogonal conjugate P^* is equidistant from the three reflections $P^{(a)}$, $P^{(b)}$, $P^{(c)}$. □

This simple fact has a few interesting consequences.

(1) The circle through the reflections of P and the one through the reflections of P^* are congruent (see Figure 3(a)). In particular, the reflections of the orthocenter H lie on the circumcircle (see Figure 3(b)).



(a) Congruent circles of reflection (b) Reflections of H on circumcircle

Figure 3. Congruence of circles of reflection of P and P^*

(2) The (six) pedals of P and P^* on the sidelines of triangle ABC are concyclic. The center of the common pedal circle is the midpoint of PP^* (see Figure 3(a)). For the isogonal pair O and H, this pedal circle is the nine-point circle.

1.2. *Line of reflections* .

Theorem 3. (a) *The reflections of P in the sidelines are collinear if and only if P lies on the circumcircle. In this case, the line containing the reflections passes through the orthocenter H.*
(b) *The reflections of a line ℓ in the sidelines are concurrent if and only if the line contains the orthocenter H. In this case, the point of concurrency lies on the circumcircle.*

Remarks. (1) Let P be a point on the circumcircle and ℓ a line through the orthocenter H. The reflections of P lies on ℓ if and only if the reflections of ℓ concur at P ([6, 29]). Figure 4 illustrates the case of the Euler line.

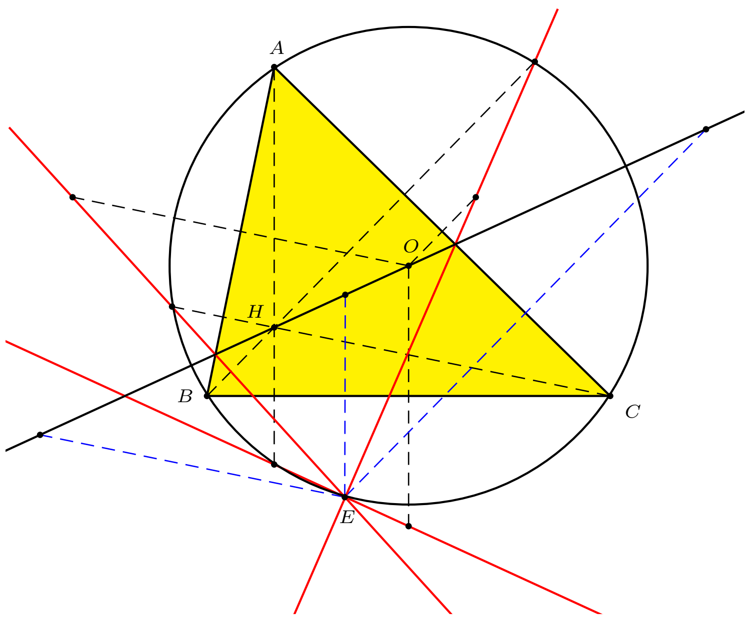


Figure 4. Euler line and Euler reflection point

(2) If $P = \left(\frac{a^2}{v-w} : \frac{b^2}{w-u} : \frac{c^2}{u-v}\right)$ is the isogonal conjugate of the infinite point of a line $ux + vy + wz = 0$, its line of reflections is

$$S_A(v-w)x + S_B(w-u)y + S_C(u-v)z = 0.$$

(3) Let ℓ be the line joining H to $P = (u : v : w)$. The reflections of ℓ in the sidelines of triangle ABC intersect at the point

$$r_0(P) := \left(\frac{a^2}{S_Bv - S_Cw} : \frac{b^2}{S_Cw - S_Au} : \frac{c^2}{S_Au - S_Bv}\right).$$

Clearly, $r_0(P_1) = r_0(P_2)$ if and only if P_1, P_2, H are collinear.

line HP	$r_0(P)$ = intersection of reflections
Euler line	$E = \left(\frac{a^2}{b^2-c^2} : \frac{b^2}{c^2-a^2} : \frac{c^2}{a^2-b^2}\right)$
HI	$X_{109} = \left(\frac{a^2}{(b-c)(b+c-a)} : \frac{b^2}{(c-a)(c+a-b)} : \frac{c^2}{(a-b)(a+b-c)}\right)$
HK	$X_{112} = \left(\frac{a^2}{(b^2-c^2)S_A} : \frac{b^2}{(c^2-a^2)S_B} : \frac{c^2}{(a^2-b^2)S_C}\right)$

Theorem 4 (Blanc [3])**.** *Let ℓ be a line through the circumcenter O of triangle ABC, intersecting the sidelines at X, Y, Z respectively. The circles with diameters AX, BY, CZ are coaxial with two common points and radical axis $\mathscr{L}$ containing the orthocenter H.*

(a) *One of the common points P lies on the nine-point circle, and is the center of the rectangular circum-hyperbola which is the isogonal conjugate of the line ℓ.*

(b) *The second common point Q lies on the circumcircle, and is the reflection of $r_0(P)$ in ℓ.*

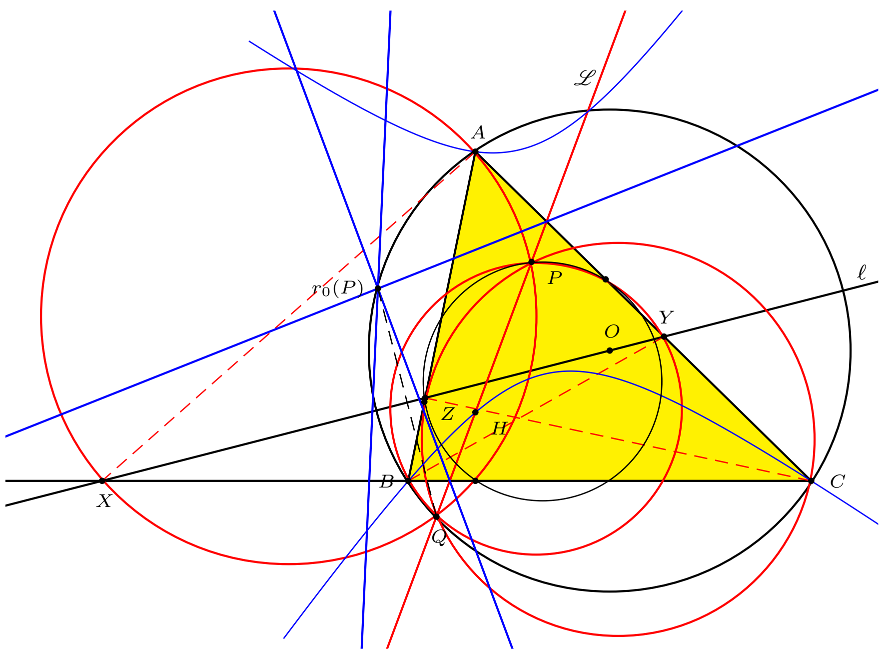


Figure 5. Blanc's theorem

Here are some examples.

Line ℓ	P	Q	$r_0(P)$
Euler line	X_{125}	$X_{476} = \left(\frac{1}{(S_B-S_C)(S^2-3S_{AA})} : \cdots : \cdots\right)$	E
Brocard axis	X_{115}	X_{112}	X_{2715}
OI	X_{11}	$X_{108} = \left(\frac{a}{(b-c)(b+c-a)S_A} : \cdots : \cdots\right)$	X_{2720}

1.3. *The triangle of reflections.* The reflections of the vertices of triangle ABC in their opposite sides are the points

$$\begin{aligned} A^{(a)} &= (-a^2 : a^2+b^2-c^2 : c^2+a^2-b^2), \\ B^{(b)} &= (a^2+b^2-c^2 : -b^2 : b^2+c^2-a^2), \\ C^{(c)} &= (c^2+a^2-b^2 : b^2+c^2-a^2 : -c^2). \end{aligned}$$

We call triangle $A^{(a)}B^{(b)}C^{(c)}$ the triangle of reflections.

Proposition 5. *The triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ is the image of the reflection triangle of N under the homothety $\mathsf{h}(O,2)$.*

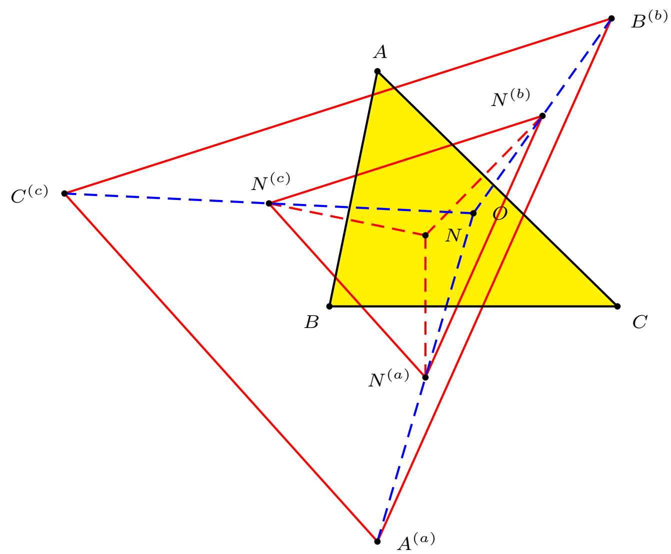


Figure 6. Homothety of triangle of reflections and reflection triangle of N

From this we conclude that
(1) the center of the circle $A^{(a)}B^{(b)}C^{(c)}$ is the point $\mathsf{h}(O,2)(N^*)$, the reflection of O in N^*, which appears as X_{195} in ETC, and
(2) the triangle of reflections is degenerate if and only if the nine-point center N lies on the circumcircle. Here is a simple construction of such a triangle (see Figure 7). Given a point N on a circle $O(R)$, construct

(a) the circle $N\left(\frac{R}{2}\right)$ and choose a point D on this circle, inside the given one (O),

(b) the perpendicular to OD at D to intersect (O) at B and C,

(c) the antipode X of D on the circle (N), and complete the parallelogram $ODXA$ (by translating X by the vector $\mathbf{DO}$).

Then triangle ABC has nine-point center N on its circumcircle. For further results, see [4, p.77], [18] or Proposition 21 below.

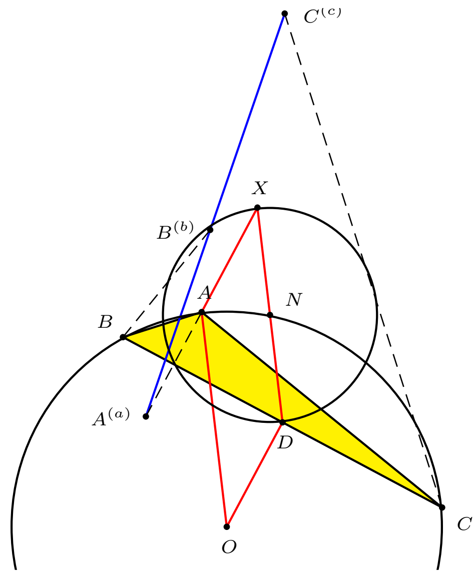


Figure 7. Triangle with degenerate triangle of reflections

2. Perspectivity of reflection triangle

2.1. *Perspectivity with anticevian and orthic triangles.*

Proposition 6 ([10])**.** *The reflection triangle of P is perspective with its anticevian triangle at the cevian quotient $Q = H/P$, which is also the isogonal conjugate of P in the orthic triangle.*

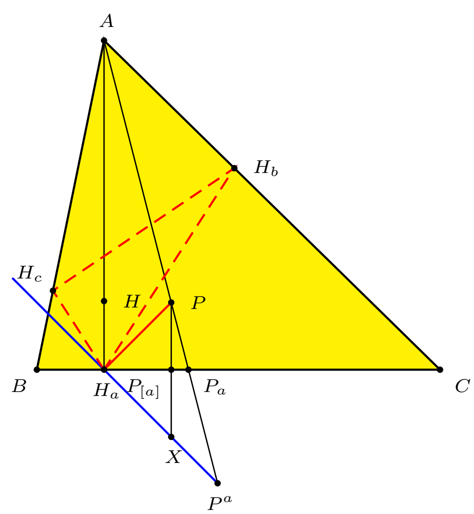


Figure 8. H_aP and $H_aP^{(a)}$ isogonal in orthic triangle

Proof. Let $P_aP_bP_c$ be the cevian triangle of P, and $P^aP^bP^c$ the anticevian triangle. Since P and P^a divide AP_a harmonically, we have $\frac{1}{AP^a}+\frac{1}{AP}=\frac{2}{AP_a}$. If the perpendicular from P to BC intersects the line P^aH_a at X, then

$$\frac{PX}{AH_a}=\frac{PP^a}{AP^a}=\frac{PP_a+P_aP^a}{AP^a}=\frac{PP_a}{AP^a}+\frac{PP_a}{AP}=\frac{2PP_a}{AP_a}=\frac{2PP_{[a]}}{AH_a}.$$

Therefore, $PX=2PP_{[a]}$, and $X=P^{(a)}$. This shows that $P^{(a)}$ lies on the line P^aH_a. Similarly, $P^{(b)}$ and $P^{(c)}$ lie on P^bH_b and P^cH_c respectively. Since the anticevian triangle of P and the orthic triangle are perspective at the cevian quotient H/P, these triangles are perspective with the reflection triangle $P^{(a)}P^{(b)}P^{(c)}$ at the same point.

The fact that $P^{(a)}$ lies on the line H_aP^a means that the lines H_aP^a and H_aP are isogonal lines with respect to the sides H_aH_b and H_aH_c of the orthic triangle; similarly for the pairs H_bP^b, H_bP and H_cP^c, H_cP. It follows that H/P and P are isogonal conjugates in the orthic triangle. □

If $P=(u:v:w)$ in homogeneous barycentric coordinates, then

$$H/P=(u(-S_Au+S_Bv+S_Cw):v(-S_Bv+S_Cw+S_Au):w(-S_Cw+S_Au+S_Bv)).$$

Here are some examples of $(P,\ H/P)$ pairs.

P	I	G	O	H	N	K
H/P	X_{46}	X_{193}	X_{155}	H	X_{52}	X_{25}

2.2. *Perspectivity with the reference triangle.*

Proposition 7. *The reflection triangle of a point P is perspective with ABC if and only if P lies on the Neuberg cubic*

$$\sum_{\text{cyclic}}(S_{AB}+S_{AC}-2S_{BC})u(c^2v^2-b^2w^2)=0. \tag{1}$$

As P traverses the Neuberg cubic, the locus of the perspector Q is the cubic

$$\sum_{\text{cyclic}}\frac{S_Ax}{S^2-3S_{AA}}\left(\frac{y^2}{S^2-3S_{CC}}-\frac{z^2}{S^2-3S_{BB}}\right)=0. \tag{2}$$

The first statement can be found in [30]. The cubic (1) is the famous Neuberg cubic, the isogonal cubic p$K(K,\ E_\infty)$ with pivot the Euler infinity point. It appears as K001 in CTC, where numerous locus properties of the Neuberg cubic can be found; see also [5]. The cubic (2), on the other hand, is the pivotal isocubic p$K(X_{1989},\ X_{265})$, and appears as K060. Given Q on the cubic (2), the point P on the Neuberg cubic can be constructed as the perspector of the cevian and reflection triangles of Q (see Figure 9). Here are some examples of (P, Q) with P on Neuberg cubic and perspector Q of the reflection triangle.

P	O	H	I	W	X_{1157}
Q	N	H	X_{79}	X_{80}	X_{1141}

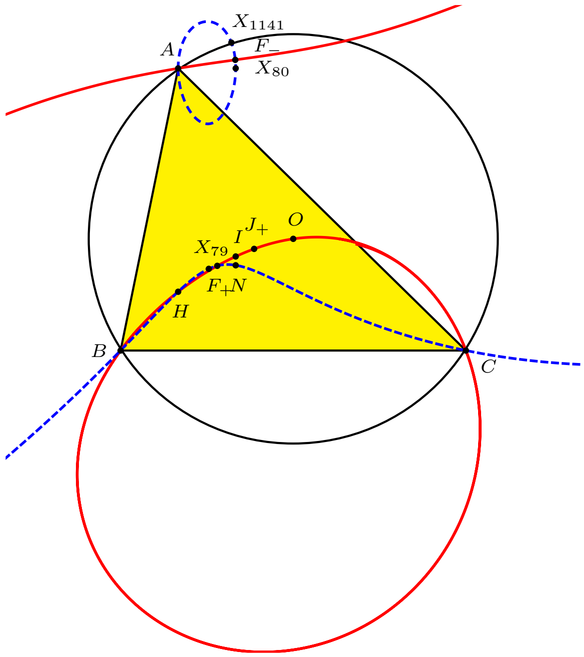


Figure 9. The Neuberg cubic and the cubic K060

Remarks. (1) $X_{79} = \left(\frac{1}{b^2+c^2-a^2+bc} : \frac{1}{c^2+a^2-b^2+ca} : \frac{1}{a^2+b^2-c^2+ab}\right)$ is also the perspector of the triangle formed by the three lines each joining the perpendicular feet of a trace of the incenter on the other two sides (see Figure 10).

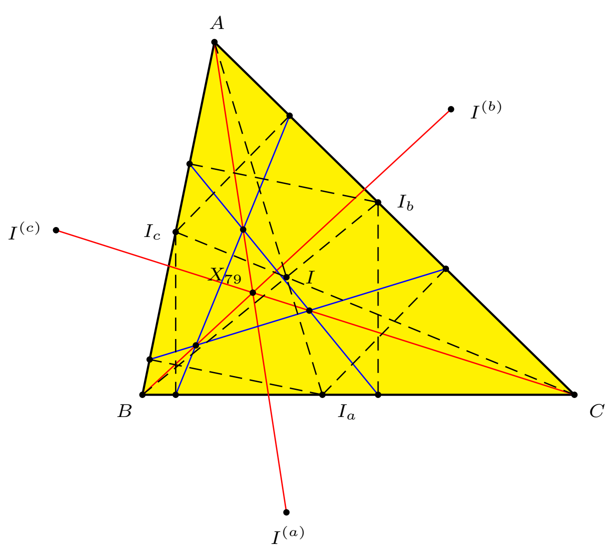


Figure 10. Perspector of reflection triangle of I

(2) For the pair (W, X_{80}),
(i) $W = X_{484} = (a(a^3+a^2(b+c)-a(b^2+bc+c^2)-(b+c)(b-c)^2) : \cdots : \cdots)$ is the first Evans perspector, the perspector of the triangle of reflections $A^{(a)}B^{(b)}C^{(c)}$ and the excentral triangle $I^aI^bI^c$ (see [45]),
(ii) $X_{80} = \left(\frac{1}{b^2+c^2-a^2-bc} : \cdots : \cdots\right)$ is the reflection conjugate of I (see §3 below).
(3) For the pair $(X_{1157},\ X_{1141})$,
(i) $X_{1157} = \left(\frac{a^2(a^6-3a^4(b^2+c^2)+a^2(3b^4-b^2c^2+3c^4)-(b^2-c^2)^2(b^2+c^2))}{a^2(b^2+c^2)-(b^2-c^2)^2} : \cdots : \cdots\right)$ is the inverse of N^* in the circumcircle,
(ii) $X_{1141} = \left(\frac{1}{(S^2+S_{BC})(S^2-3S_{AA})} : \cdots : \cdots\right)$ lies on the circumcircle.

The Neuberg cubic also contains the Fermat points and the isodynamic points. The perspectors of the reflection triangles of
(i) the Fermat points $F_\varepsilon = \left(\frac{1}{\sqrt{3}S_A+\varepsilon S} : \frac{1}{\sqrt{3}S_B+\varepsilon S} : \frac{1}{\sqrt{3}S_C+\varepsilon S}\right)$, $\varepsilon = \pm 1$, are

$$\left(\frac{(S_A+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_A+\varepsilon S)^2} : \frac{(S_B+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_B+\varepsilon S)^2} : \frac{(S_C+\varepsilon\sqrt{3}S)^2}{(\sqrt{3}S_C+\varepsilon S)^2}\right),$$

(ii) the isodynamic points $J_\varepsilon = \left(a^2(\sqrt{3}S_A+\varepsilon S) : b^2(\sqrt{3}S_B+\varepsilon S) : c^2(\sqrt{3}S_C+\varepsilon S)\right)$, $\varepsilon = \pm 1$, are

$$\left(\frac{1}{(S_A+\varepsilon\sqrt{3}S)(\sqrt{3}S_A+\varepsilon S)} : \frac{1}{(S_B+\varepsilon\sqrt{3}S)(\sqrt{3}S_B+\varepsilon S)} : \frac{1}{(S_C+\varepsilon\sqrt{3}S)(\sqrt{3}S_C+\varepsilon S)}\right).$$

The cubic (2) also contains the Fermat points. For these, the corresponding points on the Neuberg cubic are

$$\left(a^2(2(b^2+c^2-a^2)^3-5(b^2+c^2-a^2)b^2c^2-\varepsilon\cdot 2\sqrt{3}b^2c^2S) : \cdots : \cdots\right).$$

2.3. *Perspectivity with cevian triangle and the triangle of reflections.*

Proposition 8. *The reflection triangle of P is perspective with the triangle of reflections if and only if P lies on the cubic* (2)*. The locus of the perspector Q is the Neuberg cubic* (1).

Proof. Note that $A^{(a)}$, $P^{(a)}$ and P_a are collinear, since they are the reflections of A, P and P_a in BC. Similarly, $B^{(b)}$, $P^{(b)}$, P_b are collinear, so are $C^{(c)}$, $P^{(c)}$, P_c. It follows that the reflection triangle of P is perspective with the triangle of reflections if and only if it is perspective with the cevian triangle of P. □

Remark. The correspondence (P, Q) in Proposition 8 is the inverse of the correspondence in Proposition 7 above.

2.4. *Perspectivity of triangle of reflections and anticevian triangles.*

Proposition 9. *The triangle of reflections is perspective to the anticevian triangle of P if and only if P lies on the Napoleon cubic,* i.e., *the isogonal cubic $pK(K, N)$*

$$\sum_{\text{cyclic}} (a^2(b^2+c^2)-(b^2-c^2)^2)u(c^2v^2-b^2w^2)=0. \tag{3}$$

The locus of the perspector Q is the Neuberg cubic (1).

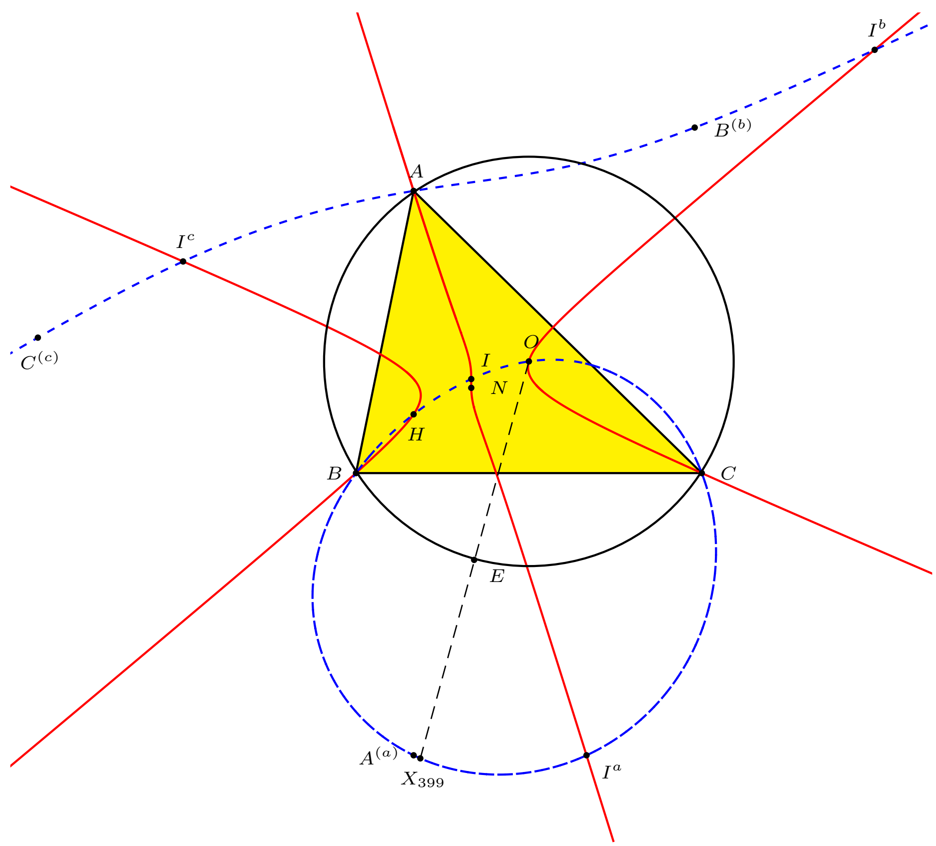


Figure 11. The Napoleon cubic and the Neuberg cubic

P	I	O	N	N^*	X_{195}
Q	W	X_{399}	E_∞	X_{1157}	O

Remarks. (1) For the case of O, the perspector is the Parry reflection point, the triangle center X_{399} which is the reflection of O in the Euler reflection point E. It is also the point of concurrency of reflections in sidelines of lines through vertices parallel to the Euler line (see [34, 35]). In other words, it is the perspector of the triangle of reflections and the cevian triangle of E_∞. The Euler line is the only direction for which these reflections are concurrent.

(2) N^* is the triangle center X_{54} in ETC, called the Kosnita point. It is also the perspector of the centers of the circles OBC, OCA, OAB (see Figure 12).

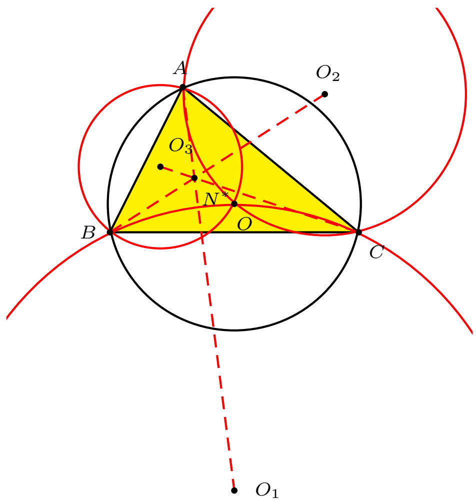


Figure 12. Perspectivity of the centers of the circles OBC, OCA, OAB

3. Reflection conjugates

Proposition 10. *The three circles $P^{(a)}BC$, $P^{(b)}CA$, and $P^{(c)}AB$ have a common point*

$$r_1(P) = \left(\frac{u}{(b^2+c^2-a^2)u(u+v+w)-(a^2vw+b^2wu+c^2uv)} : \cdots : \cdots\right). \tag{4}$$

It is easy to see that $r_1(P) = H$ if and only if P lies on the circumcircle. If $P \neq H$ and P does not lie on the circumcircle, we call $r_1(P)$ the reflection conjugate of P; it is the antipode of P in the rectangular circum-hyperbola $\mathscr{H}(P)$ through P (and the orthocenter H). It also lies on the circle of reflections $P^{(a)}P^{(b)}P^{(c)}$ (see Figure 13).

P	$r_1(P)$	midpoint	hyperbola
I	$X_{80} = \left(\frac{1}{b^2+c^2-a^2-bc} : \cdots : \cdots\right)$	X_{11}	Feuerbach
G	$X_{671} = \left(\frac{1}{b^2+c^2-2a^2} : \cdots : \cdots\right)$	X_{115}	Kiepert
O	$X_{265} = \left(\frac{S_A}{S^2-3S_{AA}} : \cdots : \cdots\right)$	X_{125}	Jerabek
K	$X_{67} = \left(\frac{1}{b^4+c^4-a^4-b^2c^2} : \cdots : \cdots\right)$	X_{125}	Jerabek
X_7	$X_{1156} = \left(\frac{a}{-2a^2+a(b+c)+(b-c)^2} : \cdots : \cdots\right)$	X_{11}	Feuerbach
X_8	$X_{1320} = \left(\frac{a(b+c-a)}{b+c-2a} : \cdots : \cdots\right)$	X_{11}	Feuerbach
X_{13}	X_{14}	X_{115}	Kiepert

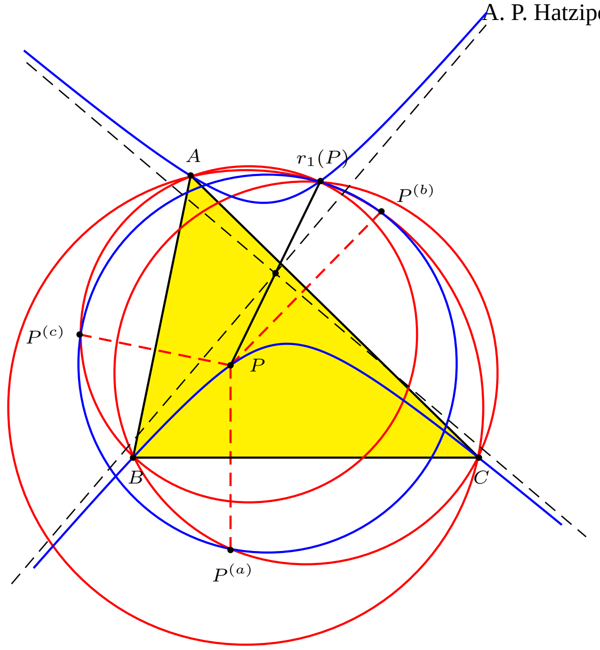


Figure 13. $r_1(P)$ and P are antipodal in $\mathscr{H}(P)$

Remark. $r_1(I) = X_{80}$ is also the perspector of the reflections of the excenters in the respective sidelines (see [42] and Figure 14). In §2.2, we have shown that $r_1(I)$ is the perspector of the reflection triangle of W.

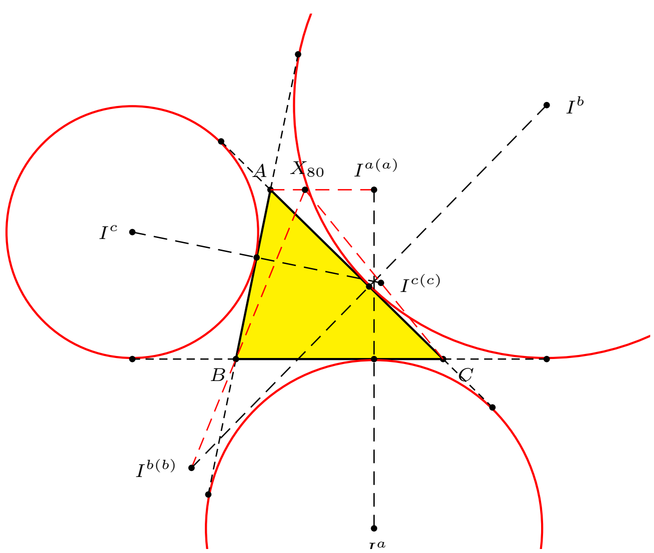


Figure 14. $r_1(I)$ as perspector of reflections of excenters

Proposition 11. *Let $P^{[a]}P^{[b]}P^{[c]}$ be the antipedal triangle of $P = (u : v : w)$. The reflections of the circles $P^{[a]}BC$ in BC, $P^{[b]}CA$ in CA and $P^{[c]}AB$ in AB all contain the reflection conjugate $r_1(P)$.*

Proof. Since B, P, C, and $P^{[a]}$ are concyclic, so are their reflections in the line BC. The circle $P^{[a]}BC$ is identical with the reflection of the circle $P^{(a)}BC$ in BC; similarly for the other two circles. The triad of circles therefore have $r_1(P)$ for a common point. □

Proposition 12. *Let $P_{[a]}P_{[b]}P_{[c]}$ be the pedal triangle of $P = (u : v : w)$. The reflections of the circles $AP_{[b]}P_{[c]}$ in $P_{[b]}P_{[c]}$, $BP_{[c]}P_{[a]}$ in $P_{[c]}P_{[a]}$, and $CP_{[a]}P_{[b]}$ in $P_{[a]}P_{[b]}$ have a common point*

$$\begin{aligned} r_2(P) = \; & (a^2(2a^2b^2c^2u + c^2((a^2+b^2-c^2)^2 - 2a^2b^2)v + b^2((c^2+a^2-b^2)^2 - 2c^2a^2)w) \\ & \cdot (b^2c^2u^2 - c^2(c^2-a^2)uv + b^2(a^2-b^2)uw - a^2(b^2+c^2-a^2)vw) : \cdots : \cdots). \end{aligned}$$

P	$r_2(P)$
G	$(a^2(b^4+c^4-a^4-b^2c^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2+c^2)(b^2-c^2)^2)$ $: \cdots : \cdots)$
I	$(a(b^2+c^2-a^2-bc)(a^3(b+c)-a^2(b^2+c^2)-a(b+c)(b-c)^2+(b^2-c^2)^2)$ $: \cdots : \cdots)$
O	circles coincide with nine-point circle
H	$X_{1986} = \left(\frac{a^2((b^2+c^2-a^2)^2-b^2c^2)(a^4(b^2+c^2)-2a^2(b^4-b^2c^2+c^4)+(b^2+c^2)(b^2-c^2)^2)}{b^2+c^2-a^2} : \cdots : \cdots\right)$
X_{186}	X_{403}

Remarks. (1) For the case of $(H,\ X_{1986})$, see [22].

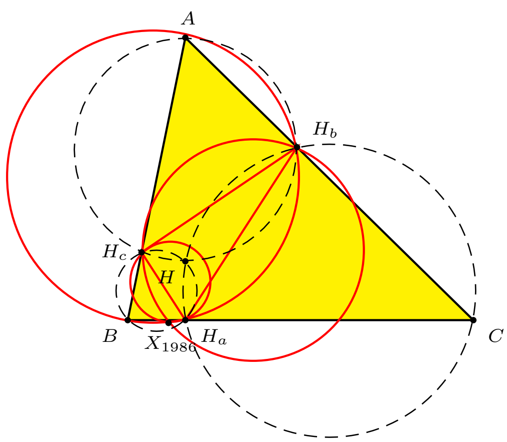


Figure 15. X_{1986} as the common point of reflections of circumcircles of residuals of orthic triangle

(2) For the pair $(X_{186},\ X_{403})$,
(i) X_{186} is the inverse of H in the circumcircle,
(ii) X_{403} is the inverse of H in the nine-point circle (see §4 below).

4. Inversion in the circumcircle

The inverse of P in the circumcircle is the point

$$\begin{aligned} P^{-1} = & \ (a^2(b^2c^2u^2 + b^2(a^2-b^2)wu + c^2(a^2-c^2)uv - a^2(b^2+c^2-a^2)vw) \\ : & \ \ b^2(c^2a^2v^2 + a^2(b^2-a^2)vw - b^2(c^2+a^2-b^2)wu + c^2(b^2-c^2)uv) \\ : & \ \ c^2(a^2b^2w^2 + a^2(c^2-a^2)vw + b^2(c^2-b^2)wu - c^2(a^2+b^2-c^2)uv)). \end{aligned}$$

4.1. *Bailey's theorem.*

Theorem 13 (Bailey [1, Theorem 5])**.** *The isogonal conjugates of P and $r_1(P)$ are inverse in the circumcircle.*

Proof. Let $P = (u : v : w)$, so that $P^* = (a^2vw : b^2wu : c^2uv)$. From the above formula,

$$\begin{aligned} (P^*)^{-1} = & \ \left(a^2vw(a^2vw + (a^2-b^2)uv + (a^2-c^2)wu - (b^2+c^2-a^2)u^2) : \cdots : \cdots\right) \\ = & \ \left(a^2vw(-(b^2+c^2-a^2)u(u+v+w) + a^2vw + b^2wu + c^2uv) : \cdots : \cdots\right). \end{aligned}$$

This clearly is the isogonal conjugate of $r_1(P)$ by a comparison with (4). □

4.2. *The inverses of $A^{(a)}$, $B^{(b)}$, $C^{(c)}$.*

Proposition 14. *The inversive images of $A^{(a)}$, $B^{(b)}$, $C^{(c)}$ in the circumcircle are perspective with ABC at N^*.*

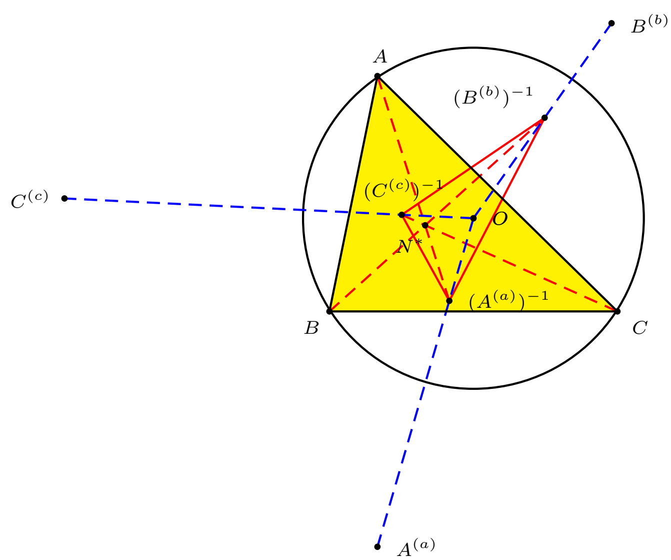


Figure 16. N^* as perspector of inverses of reflections of vertices in opposite sides

Proof. These inversive images are

$$(A^{(a)})^{-1} = (-a^2(S^2 - 3S_{AA}) : b^2(S^2 + S_{AB}) : c^2(S^2 + S_{CA})),$$
$$(B^{(b)})^{-1} = (a^2(S^2 + S_{AB}) : -b^2(S^2 - 3S_{BB}) : c^2(S^2 + S_{BC})),$$
$$(C^{(c)})^{-1} = (a^2(S^2 + S_{CA}) : b^2(S^2 + S_{BC}) : -c^2(S^2 - 3S_{CC})).$$

From these, the triangles ABC and $(A^{(a)})^{-1}(B^{(b)})^{-1}(C^{(c)})^{-1}$ are perspective at

$$N^* = \left(\frac{a^2}{S^2 + S_{BC}} : \frac{b^2}{S^2 + S_{CA}} : \frac{c^2}{S^2 + S_{AB}}\right).$$

□

Corollary 15 (Musselman [32]). *The circles $AOA^{(a)}$, $BOB^{(b)}$, $COC^{(c)}$ are coaxial with common points O and $(N^*)^{-1}$.*

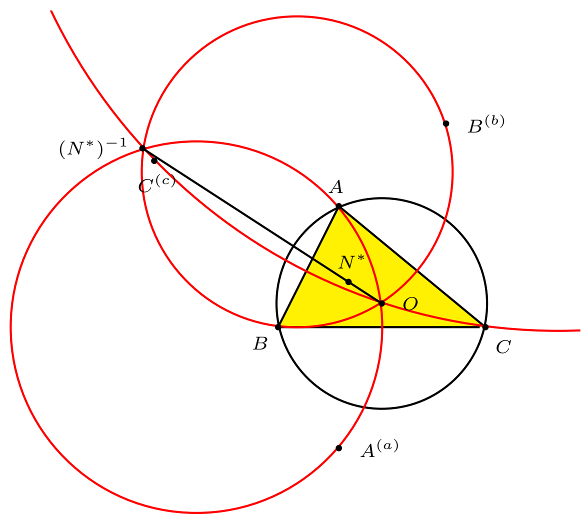


Figure 17. Coaxial circles $APA^{(a)}$, $BOB^{(a)}$, $COC^{(c)}$

Proof. Invert the configuration in Proposition 14 in the circumcircle. □

A generalization of Corollary 15 is the following.

Proposition 16 (van Lamoen [28]). *The circles $APA^{(a)}$, $BPB^{(b)}$ and $CPC^{(c)}$ are coaxial if and only if P lies on the Neuberg cubic.*

Remarks. (1) Another example is the pair (I, W).

(2) If P is a point on the Neuberg cubic, the second common point of the circles $APA^{(a)}$, $BPB^{(b)}$ and $CPC^{(c)}$ is also on the same cubic.

4.3. *Perspectivity of inverses of cevian and anticevian triangles.*

Proposition 17. *The inversive images of P_a, P_b, P_c in the circumcircle form a triangle perspective with ABC if and only if P lies on the circumcircle or the Euler line.*

(a) *If P lies on the circumcircle, the perspector is the isogonal conjugate of the inferior of P. The locus is the isogonal conjugate of the nine-point circle* (see Figure 18).

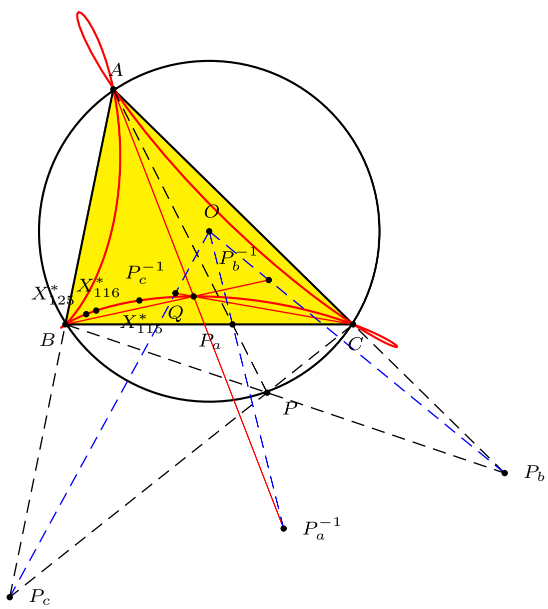


Figure 18. Isogonal conjugate of the nine-point circle

(b) *If P lies on the Euler line, the locus of the perspector is the bicevian conic through the traces of the isogonal conjugates of the Kiepert and Jerabek centers* (see Figure 19).

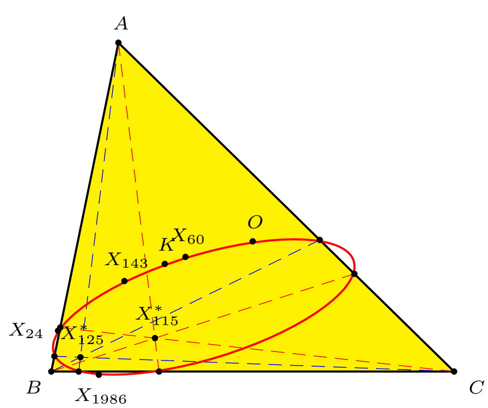


Figure 19. The bicevian conic $\mathscr{C}(X_{115}^*,\ X_{125}^*)$

The conic in Proposition 17(b) has equation

$$\sum_{\text{cyclic}} b^4c^4(b^2-c^2)^4(b^2+c^2-a^2)x^2 - 2a^6b^2c^2(c^2-a^2)^2(a^2-b^2)^2yz = 0.$$

P	O	G	H	N	X_{21}	H^{-1}
Q	O	K	X_{24}	X_{143}	X_{60}	X_{1986}

Remarks. (1) X_{21} is the Schiffler point, the intersection of the Euler lines of IBC, ICA, IAB (see [21]). Here is another property of X_{21} relating to reflections discovered by L. Emelyanov [11]. Let X be the reflection of the touch point of A-excircle in the line joining the other two touch points; similarly define Y and Z. The triangles ABC and XYZ are perspective at the Schiffler point (see Figure 20).

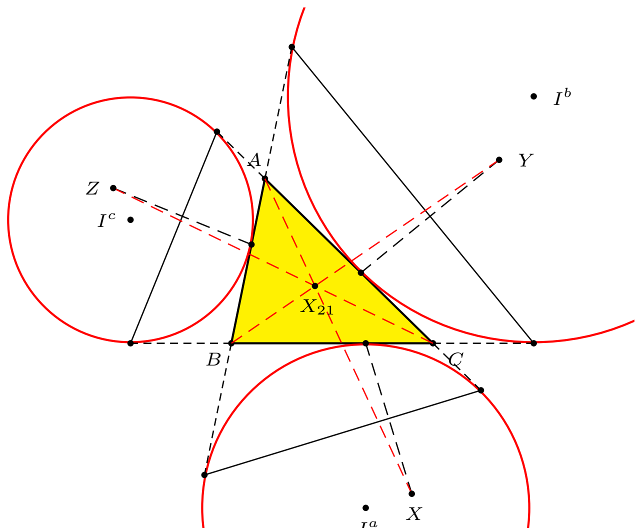


Figure 20. Schiffler point and reflections

(2) $X_{24} = \left(\frac{a^2(S_{AA}-S^2)}{S_A} : \frac{b^2(S_{BB}-S^2)}{S_B} : \frac{c^2(S_{CC}-S^2)}{S_C}\right)$ is the perspector of the orthic-of-orthic triangle (see [26]).

(3) X_{143} is the nine-point center of the orthic triangle.

(4) $X_{60} = \left(\frac{a^2(b+c-a)}{(b+c)^2} : \frac{b^2(c+a-b)}{(c+a)^2} : \frac{c^2(a+b-c)}{(a+b)^2}\right)$ is the isogonal conjugate of the outer Feuerbach point X_{12}.

Proposition 18. *The inversive images of P^a, P^b, P^c in the circumcircle form a triangle perspective with ABC if and only if P lies on*
(1) *the isogonal conjugate of the circle $S_Ax^2+S_By^2+S_Cz^2=0$, or*
(2) *the conic*

$$b^2c^2(b^2-c^2)x^2 + c^2a^2(c^2-a^2)y^2 + a^2b^2(a^2-b^2)z^2 = 0.$$

Remarks. (1) The circle $S_Ax^2 + S_By^2 + S_Cz^2 = 0$ is real only when ABC contains an obtuse angle. In this case, it is the circle with center H orthogonal to the circumcircle.

(2) The conic in (2) is real only when ABC is acute. It has center E and is homothetic to the Jerabek hyperbola, with ratio $\sqrt{\frac{1}{2\cos A\cos B\cos C}}$.

5. Dual triads of concurrent circles

Proposition 19. *Let* $\begin{smallmatrix} X, & Y, & Z \\ X', & Y' & Z' \end{smallmatrix}$ *be two triads of points. The triad of circles* $XY'Z'$*,* $YZ'X'$ *and* $ZX'Y'$ *have a common point if and only if the triad of circles* $X'YZ$*,* $Y'ZX$ *and* $Z'XY$ *have a common point.*

Proof. Let Q be a common point of the triad of circles $XY'Z'$, $YZ'X'$, $ZX'Y'$. Inversion with respect to a circle, center Q transforms the six points X, Y, Z, X', Y', Z' into x, y, z, x', y', z' respectively. Note that $xy'z'$, $yz'x'$ and $zx'y'$ are lines bounding a triangle $x'y'z'$. By Miquel's theorem, the circles $x'yz$, $y'zx$ and $z'xy$ have a common point q'. Their inverses $X'YZ$, $Y'ZX$ and $Z'XY$ have the inverse Q' of q' as a common point. □

Proposition 20 (Musselman [31])**.** *The circles* $AP^{(b)}P^{(c)}$*,* $BP^{(c)}P^{(a)}$*,* $CP^{(a)}P^{(b)}$ *intersect at the point* $r_0(P)$ *on the circumcircle.*

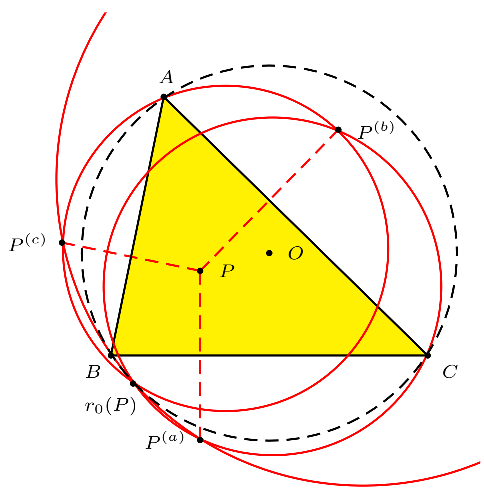


Figure 21. The circles $AP^{(b)}P^{(c)}$, $BP^{(c)}P^{(a)}$, $CP^{(a)}P^{(b)}$ intersect on the circumcircle

5.1. *Circles containing* $A^{(a)}$, $B^{(b)}$, $C^{(c)}$.

5.1.1. *The triad of circles* $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$, $A^{(a)}B^{(b)}C$. Since the circles $A^{(a)}BC$, $AB^{(b)}C$ and $ABC^{(c)}$ all contain the orthocenter H, it follows that that the circles $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$ and $A^{(a)}B^{(b)}C$ also have a common point. This is the point $X_{1157} = (N^*)^{-1}$ (see [41, 18]). The radical axes of the circumcircle with each of these circles bound the anticevian triangle of N^* (see Figure 22).

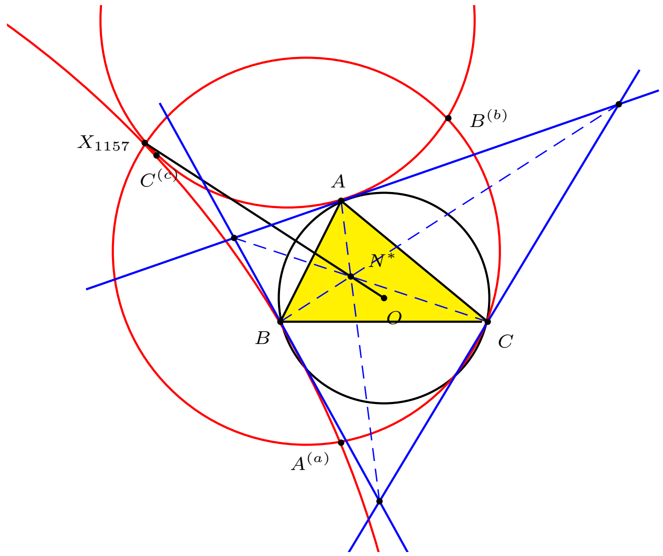


Figure 22. Concurrency of circles $AB^{(b)}C^{(c)}$, $A^{(a)}BC^{(c)}$, $A^{(a)}B^{(b)}C$

5.1.2. *The tangential triangle* . The circles $K^aB^{(b)}C^{(c)}$, $A^{(a)}K^bC^{(c)}$, $A^{(a)}B^{(b)}K^c$ have X_{399} the Parry reflection point as a common point. On the other hand, the circles $A^{(a)}K^bK^c$, $B^{(b)}K^cK^a$, $C^{(c)}K^aK^b$ are concurrent. (see [35]).

5.1.3. *The excentral triangle* . The circles $A^{(a)}I^bI^c$, $I^aB^{(b)}I^c$, $I^aI^bC^{(c)}$ also have the Parry reflection point X_{399} as a common point (see Figure 23).

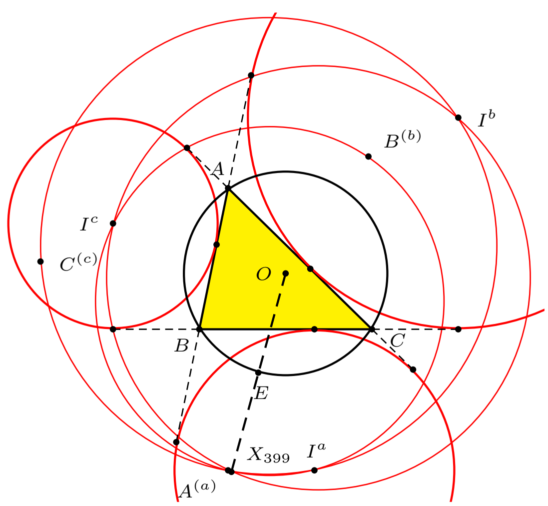


Figure 23. The Parry reflection point X_{399}

The Parry reflection point X_{399}, according to Evans [12], is also the common point of the circles $II^aA^{(a)}$, $II^bB^{(b)}$ and $II^cC^{(c)}$.

By Proposition 19, the circles $I^aB^{(b)}C^{(c)}$, $A^{(a)}I^bC^{(c)}$ and $A^{(a)}B^{(b)}I^c$ have a common point as well. Their centers are perspective with ABC at the point

$$(a(a^2(a+b+c) - a(b^2 - bc + c^2) - (b+c)(b-c)^2) : \cdots : \cdots)$$

on the OI line.

5.1.4. *Equilateral triangles on the sides* . For $\varepsilon = \pm 1$, let A_ε, B_ε, C_ε be the apices of the equilateral triangles erected on the sides BC, CA, AB of triangle ABC respectively, on opposite or the same sides of the vertices according as $\varepsilon = 1$ or -1. Now, for $\varepsilon = \pm 1$, the circles $A^{(a)}B_\varepsilon C_\varepsilon$, $B^{(b)}C_\varepsilon A_\varepsilon$, $C^{(c)}A_\varepsilon B_\varepsilon$ are concurrent at the superior of the Fermat point $F_{-\varepsilon}$ (see [36]).

5.1.5. *Degenerate triangle of reflections* .

Proposition 21 ([18, Theorem 4])**.** *Suppose the nine-point center N of triangle ABC lies on the circumcircle.*
(1) *The reflection triangle $A^{(a)}B^{(b)}C^{(c)}$ degenerates into a line $\mathcal{L}$.*
(2) *If X, Y, Z are the centers of the circles BOC, COA, AOB, the lines AX, BY, CZ are all perpendicular to $\mathcal{L}$.*
(3) *The circles $AOA^{(a)}$, $BOB^{(b)}$, $COC^{(c)}$ are mutually tangent at O. The line joining their centers is the parallel to $\mathcal{L}$ through O.*
(4) *The circles $AB^{(b)}C^{(c)}$, $BC^{(c)}A^{(a)}$, $CA^{(a)}B^{(b)}$ pass through O.*

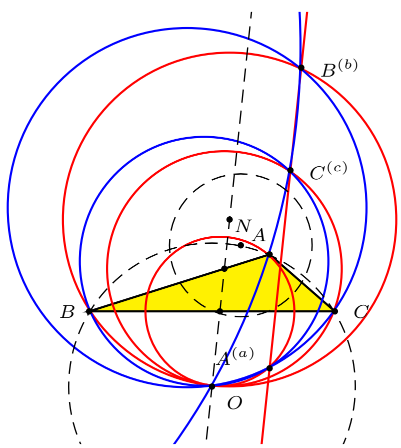


Figure 24. Triangle with degenerate triangle of reflections

5.2. *Reflections in a point.*

Proposition 22. *Given $P = (u : v : w)$, let X, Y, Z be the reflections of A, B, C in P.*
(a) *The circles AYZ, BZX, CXY have a common point a point*

$$r_3(P) = \left(\frac{1}{c^2v(w+u-v) - b^2w(u+v-w)} : \cdots : \cdots \right)$$

which is also the fourth intersection of the circumcircle and the circumconic with center P (see Figure 25).

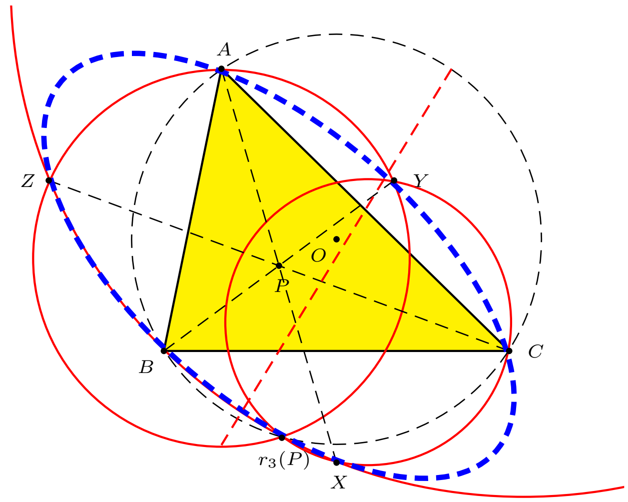


Figure 25. Circles AYZ, BZX, CXY through $r_3(P)$ on circumcircle and circumconic with center P

(b) *The circles XBC, YCA and ZAB intersect have a common point*

$$r_4(P) = \left(\frac{v+w-u}{2a^2vw-(v+w-u)(bw+cv)} : \cdots : \cdots\right)$$

which is the antipode of $r_3(P)$ on the circumconic with center P (see Figure 26). *It is also the reflection conjugate of the superior of P.*

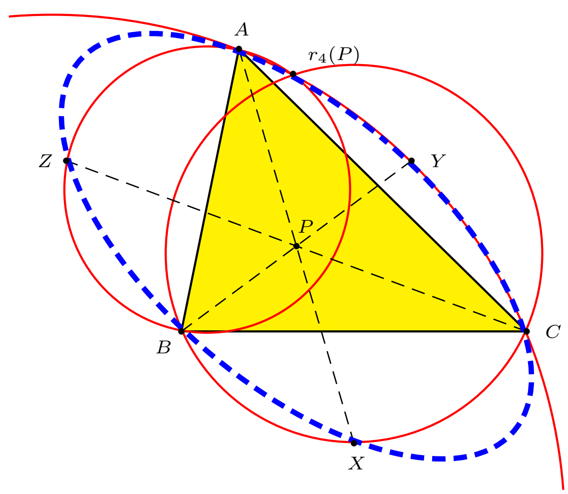


Figure 26. Circles XBC, YCA, ZAB through $r_4(P)$ circumconic with center P

(c) *For a given Q on the circumcircle, the locus of P for which $r_3(P) = Q$ is the bicevian conic $\mathscr{C}(G, Q)$.*

Here are some examples of $r_3(P)$ and $r_4(P)$.

P	G	I	N	K	X_9	X_{10}	X_{2482}	X_{214}	X_{1145}
$r_3(P)$	X_{99}	X_{100}	E	E	X_{100}	X_{100}	X_{99}	X_{100}	X_{100}
$r_4(P)$	$r_1(G)$	X_{1320}	$r_1(O)$	X_{895}	X_{1156}	X_{80}	G	I	N_a

6. Reflections and Miquel circles

6.1. *The reflection of I in O.* If X, Y, Z are the points of tangency of the excircles with the respective sides, the Miquel point of the circles AYZ, BZX, CXY is the reflection of I in O, which is X_{40} in ETC. It is also the circumcenter of the excentral triangle.

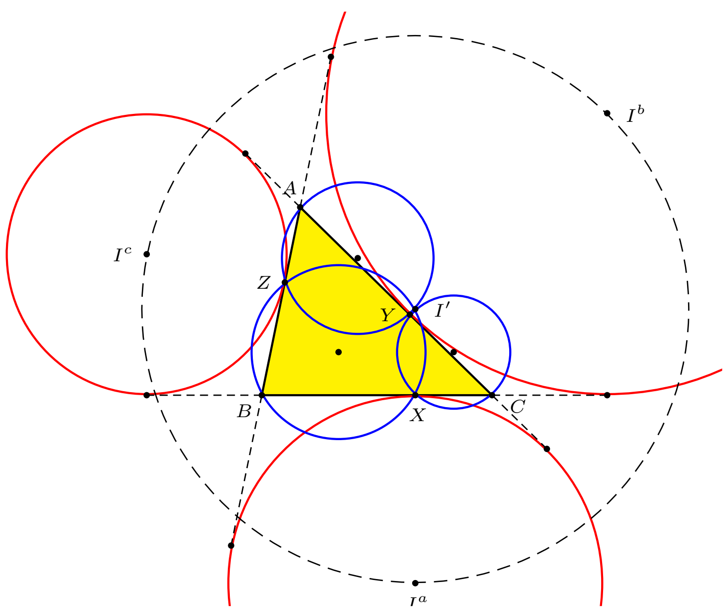


Figure 27. Reflection of I in O as a Miquel point

6.2. *Miquel circles.* For a real number t, we consider the triad of points

$$X_t = (0 : 1-t : t), \quad Y_t = (t : 0 : 1-t), \quad Z_t = (1-t : t : 0)$$

on the sides of the reference triangle. The circles AY_tZ_t, BZ_tX_t and CX_tY_t intersect at the Miquel point

$$\begin{aligned} M_t = & \big(a^2(b^2t^2 + c^2(1-t)^2 - a^2t(1-t)) \\ & : b^2(c^2t^2 + a^2(1-t)^2 - b^2t(1-t)) \\ & : c^2(a^2t^2 + b^2(1-t)^2 - c^2t(1-t))\big). \end{aligned}$$

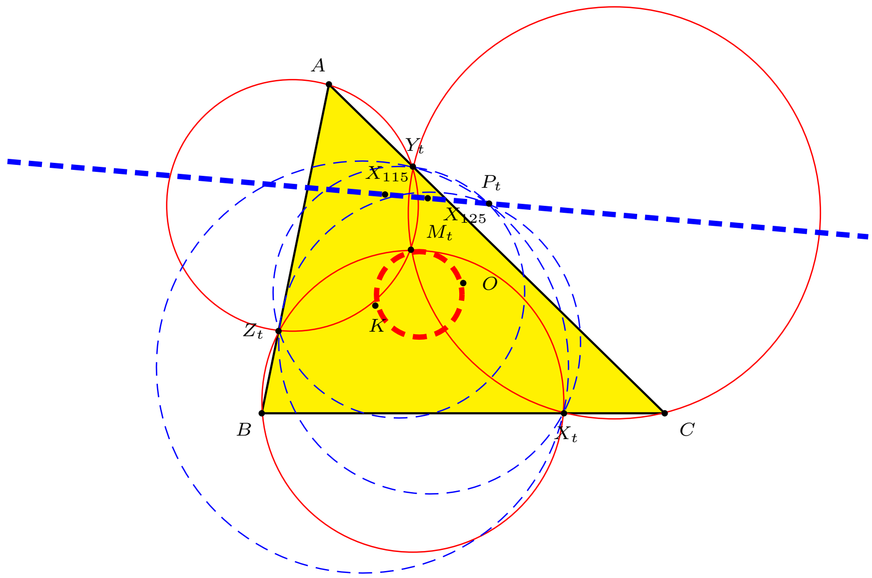


Figure 28. Miquel circles and their reflections

The locus of M_t is the Brocard circle with diameter OK, as is evident from the data in the table below; see Figure 28 and [37, 17].

t	M_t	P_t
0	$\Omega = \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}$	$\frac{1}{c^2-a^2} : \frac{1}{a^2-b^2} : \frac{1}{b^2-c^2}$
$\frac{1}{2}$	O	$X_{115} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$
1	$\Omega' = \frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}$	$\frac{1}{a^2-b^2} : \frac{1}{b^2-c^2} : \frac{1}{c^2-a^2}$
∞	K	$((b^2-c^2)(b^2+c^2-2a^2) : \cdots : \cdots)$
$\frac{a^2b^2-c^4}{(b^2-c^2)(a^2+b^2+c^2)}$	$B_1 = a^2 : c^2 : b^2$	$-(b^4-c^4) : b^2(c^2-a^2) : c^2(a^2-b^2)$
$\frac{c^2a^2-b^4}{(c^2-a^2)(a^2+b^2+c^2)}$	$B_2 = c^2 : b^2 : a^2$	$a^2(b^2-c^2) : -(c^4-a^4) : c^2(a^2-b^2)$
$\frac{a^2b^2-c^4}{(a^2-b^2)(a^2+b^2+c^2)}$	$B_3 = b^2 : a^2 : c^2$	$a^2(b^2-c^2) : b^2(c^2-a^2) : -(a^4-b^4)$

6.3. *Reflections of Miquel circles.* Let A_t, B_t, C_t be the reflections of A in Y_tZ_t, B in Z_tX_t, C in X_tY_t. The circles $A_tY_tZ_t$, $B_tZ_tX_t$ and $C_tX_tY_t$ also have a common point

$$\begin{aligned} P_t = \; & ((b^2-c^2)((c^2-a^2)t + (a^2-b^2)(1-t)) \\ & : (c^2-a^2)((a^2-b^2)t + (b^2-c^2)(1-t)) \\ & : (a^2-b^2)((b^2-c^2)t + (c^2-a^2)(1-t))). \end{aligned}$$

For $t = \frac{1}{2}$, all three reflections coincide with the nine-point circle. However, P_t approaches the Kiepert center $X_{115} = ((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$ as $t \to \frac{1}{2}$. The locus of P_t is the line

$$\frac{x}{b^2-c^2} + \frac{y}{c^2-a^2} + \frac{z}{a^2-b^2} = 0,$$

which clearly contains both the Kiepert center X_{115} and the Jerabek center X_{125} (see Figure 28). This line is the radical axis of the nine-point circle and the pedal circle of G. These two centers are the common points of the two circles (see Figure 29).

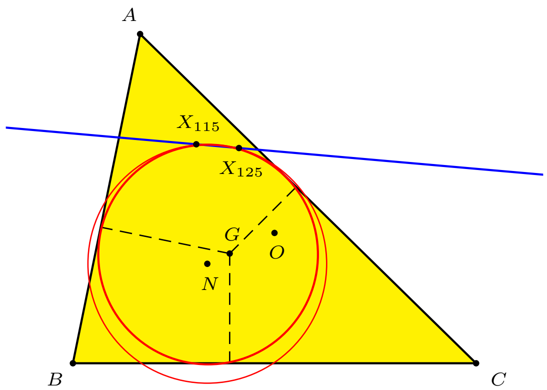


Figure 29. X_{115} and X_{125} as the intersections of nine-point circle and pedal circle of G

6.4. *Reflections of circles of anticevian residuals.* Consider points X^t, Y^t, Z^t such that A, B, C divide Y^tZ^t, Z^tX^t, X^tY^t respectively in the ratio $1-t:t$. Figure 30 shows the construction of these points from X_t, Y_t, Z_t and the midpoints of the sides. Explicitly,

$$\begin{aligned} X^t &= (-t(1-t):(1-t)^2:t^2), \\ Y^t &= (t^2:-t(1-t):(1-t)^2), \\ Z^t &= ((1-t)^2:t^2:-t(1-t)). \end{aligned}$$

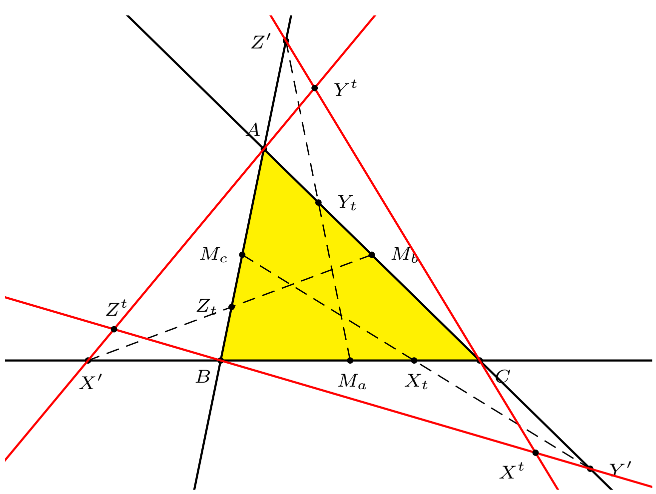


Figure 30. Construction of $X^tY^tZ^t$ from $X_tY_tZ_t$

The circles X^tBC, Y^tCA, Z^tAB intersect at the isogonal conjugate of M_t. The locus of the intersection is therefore the isogonal conjugate of the Brocard circle. On the other hand, the reflections of the circles X_tBC, Y_tCA, Z_tAB intersect at the point

$$\left(\frac{1}{(b^2+c^2-2a^2)t+(a^2-b^2)}:\frac{1}{(c^2+a^2-b^2)t+(b^2-c^2)}:\frac{1}{(a^2+b^2-c^2)t+(c^2-a^2)}\right),$$

which traverses the Steiner circum-ellipse.

7. Reflections of a point in various triangles

7.1. *Reflections in the medial triangle.* If $P = (u : v : w)$, the reflections in the sides of the medial triangle are

$$\begin{aligned} X' =&\ ((S_B+S_C)(v+w) : S_Bv - S_C(w-u) : S_Cw + S_B(u-v)),\\ Y' =&\ (S_Au + S_C(v-w) : (S_C+S_A)(w+u) : S_Cw - S_A(u-v)),\\ Z' =&\ (S_Au - S_B(v-w) : S_Bv + S_A(w-u) : (S_A+S_B)(u+v)). \end{aligned}$$

Proposition 23. *The reflection triangle of P in the medial triangle is perspective with ABC if and only if P lies on the Euler line or the nine-point circle of ABC.*

(a) *If P lies on the Euler line, the perspector traverses the Jerabek hyperbola.*

(b) *If P lies on the nine-point circle, the perspector is the infinite point which is the isogonal conjugate of the superior of P.*

Remarks. (1) If $P = E_t$, then the perspector $Q = E^*_{t'}$, where

$$t' = \frac{a^2b^2c^2(1-t)}{a^2b^2c^2(1-t) - 4S_{ABC}(1-2t)}.$$

P	G	O	H	N	X_{25}	X_{403}	X_{427}	X_{429}	X_{442}	E_∞
Q	$H^\bullet$	X_{68}	H	O	X_{66}	X_{74}	K	X_{65}	X_{72}	X_{265}

(2) For $P = G$, these reflections are the points

$$X' = (2a^2 : S_B : S_C),\quad Y' = (S_A : 2b^2 : S_C),\quad Z' = (S_A : S_B : 2c^2).$$

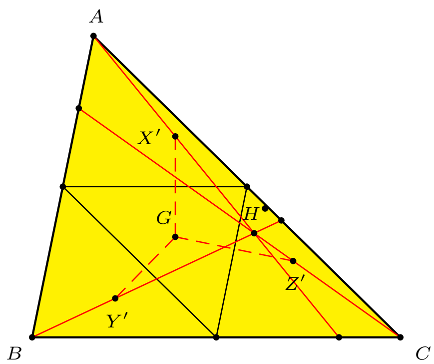
They are trisection points of the corresponding $H^\bullet-$cevian (see Figure 31(a)). The perspector of $X'Y'Z'$ is $X_{69} = H^\bullet$.

(3) If $P = N$, the circumcenter of the medial triangle, the circle through its reflections in the sides of the medial triangle is congruent to the nine-point circle and has center at the orthocenter of the medial triangle, which is the circumcenter O of triangle ABC. These reflections are therefore the midpoints of the circumradii OA, OB, OC (see Figure 31(b)).

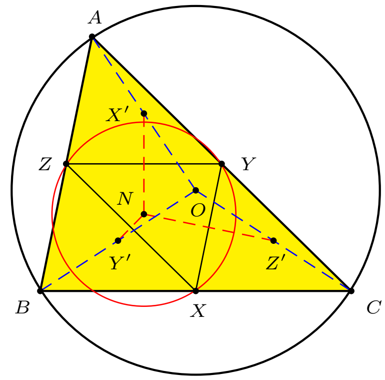
(4) $X_{25} = \left(\frac{a^2}{b^2+c^2-a^2} : \frac{b^2}{c^2+a^2-b^2} : \frac{c^2}{a^2+b^2-c^2}\right)$ is the homothetic center of the tangential and orthic triangles. It is also the perspector of the tangential triangle and the reflection triangle of K. In fact,

$$A'X' : X'H_a = a^2 : S_A,\quad B'Y' : Y'H_b = b^2 : S_B,\quad C'Z' : Z'H_c = c^2 : S_C.$$

(5) $X_{427} = \left(\frac{b^2+c^2}{b^2+c^2-a^2} : \frac{c^2+a^2}{c^2+a^2-b^2} : \frac{a^2+b^2}{a^2+b^2-c^2}\right)$ is the inverse of X_{25} in the orthocentroidal circle. It is also the homothetic center of the orthic triangle and the



(a) Reflections of G



(b) Reflections of N

Figure 31. Reflections in the medial triangle

triangle bounded by the tangents to the nine-point circle at the midpoints of the sidelines (see [7]).

(6) If P is on the nine-point circle, it is the inferior of a point P' on the circumcircle. In this case, the perspector Q is the infinite point which is the isogonal conjugate of P'. In particular, for the Jerabek center $J = X_{125}$ (which is the inferior of the Euler reflection point $E = X_{110}$), the reflections are the pedals of the vertices on the Euler line. The perspector is the infinite point of the perpendicular to the Euler line (see Figure 32).

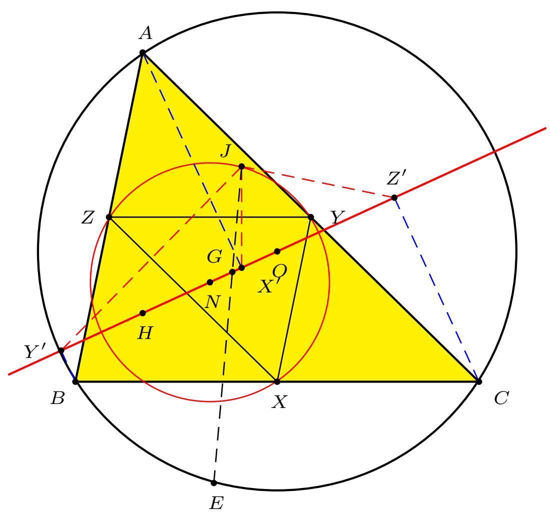


Figure 32. Reflections of Jerabek center in medial triangle

Proposition 24. *The reflections of AP, BP, CP in the respective sidelines of the medial triangle are concurrrent* (i.e., triangle $X'Y'Z'$ is perspective with the orthic triangle) *if and only if P lies on the Jerabek hyperbola of ABC. As P traverses the Jerabek hyperbola, the locus of the perspector is the Euler line* (see Figure 33).

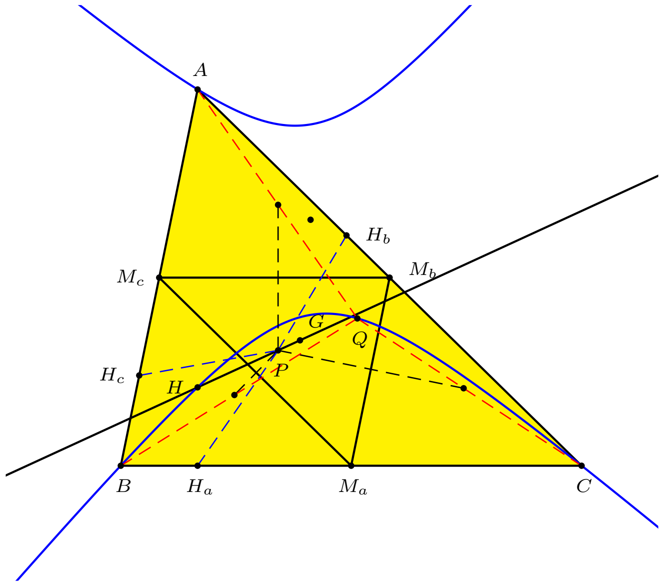


Figure 33. Reflections in medial triangle

Remark. The correspondence is the inverse of the correspondence in Proposition 23(a).

7.2. *Reflections in the orthic triangle.*

Proposition 25. *The reflection triangle of P in the orthic triangle $H_aH_bH_c$ is perspective with ABC if and only if P lies on the cubic*

$$\sum_{\text{cyclic}} \frac{u}{b^2+c^2-a^2}\left(f(c,a,b)v^2 - f(b,c,a)w^2\right) = 0. \tag{5}$$

where

$$f(a,b,c) = a^4(b^2+c^2) - 2a^2(b^4-b^2c^2+c^4) + (b^2+c^2)(b^2-c^2)^2.$$

The locus of the perspector Q is the cubic

$$\sum_{\text{cyclic}} \frac{a^2(S^2-3S_{AA})x}{b^2+c^2-a^2}\left(c^4(S^2-S_{CC})y^2 - b^4(S^2-S_{BB})z^2\right) = 0. \tag{6}$$

Remarks. (1) The cubic (5) is the isocubic p$K(X_{3003},\ H)$, labeled K339 in TCT.
(2) The cubic (6) is the isocubic p$K(X_{186},\ X_{571})$ (see Figure 34).
(3) Here are some correspondences of

P	H	O	X_{1986}
Q	X_{24}	O	X_{186}

The reflection triangle of H in the orthic triangle is homothetic to ABC at $X_{24} = \left(\frac{a^2(S_{AA}-S^2)}{S_A} : \frac{b^2(S_{BB}-S^2)}{S_B} : \frac{c^2(S_{CC}-S^2)}{S_C}\right)$.

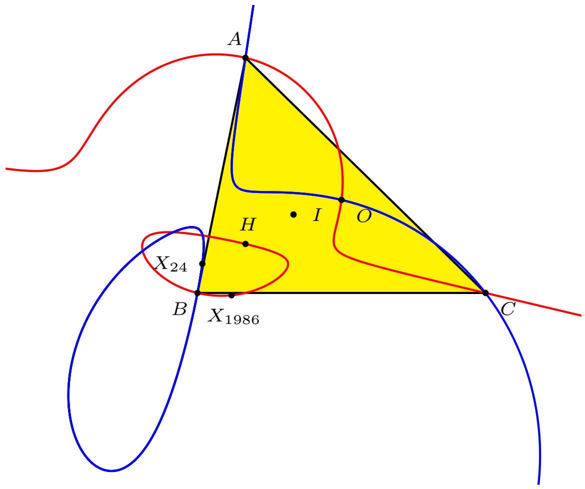


Figure 34. The cubics K339 and p$K(X_{186},\ X_{571})$

7.3. *Reflections in the pedal triangle.*

Proposition 26. *The reflection triangle of P in its pedal triangle are perspective with*
(a) ABC *if and only if P lies on the orthocubic cubic*

$$\sum_{\text{cyclic}} S_{BC}x(c^2y^2 - b^2z^2) = 0, \tag{7}$$

(b) *the pedal triangle if and only if P lies on the Neuberg cubic* (1).

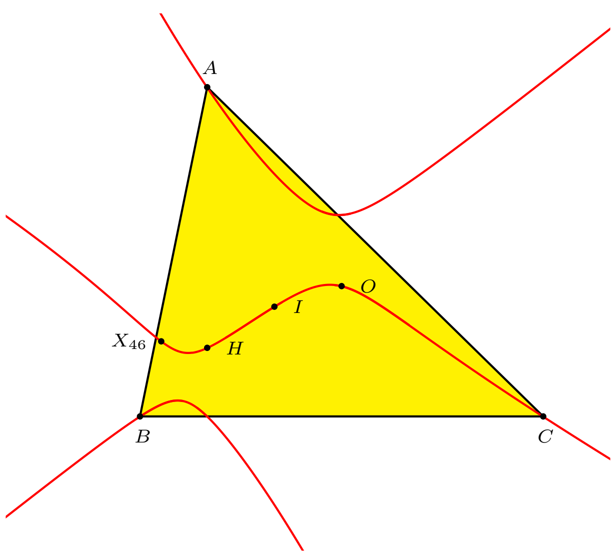


Figure 35. The orthocubic cubic

Remarks. (1) The orthocubic defined by (7) is the curve K006 in CTC.

(2) Both cubics contain the points I, O, H. Here are the corresponding perspectors.

P	I	O	H
perspector with ABC	I	X_{68}	X_{24}
perspector with pedal triangle	I	O	

The missing entry is the perspector of the orthic triangle and the reflection triangle of H in the orthic triangle; it is the triangle center

$$(a^2S_{BC}(3S^2 - S_{AA})(a^2b^2c^2 + 2S_A(S^2 + S_{BC})) : \cdots : \cdots).$$

7.4. *Reflections in the reflection triangle.*

Proposition 27. *The reflections of P in the sidelines of its reflection triangle are perspective with*
(a) ABC *if and only if P lies on the Napoleon cubic* (3).
(b) *the reflection triangle if and only if P lies on the Neuberg cubic* (1).

Remark. Both cubics contain the points I, O, H. Here are the corresponding perspectors.

P	I	O	H
perspector with ABC	I	X_{265}	X_{186}
perspector with reflection triangle	I	O	

The missing entry is the perspector of $H^{(a)}H^{(b)}H^{(c)}$ and the reflection triangle of H in $H^{(a)}H^{(b)}H^{(c)}$; it is the triangle center

$$(a^2S_{BC}(a^2b^2c^2(3S^2 - S_{AA}) + 8S_A(S^2 + S_{BC})(S^2 - S_{AA})) : \cdots : \cdots).$$

8. Reflections in lines

8.1. *Reflections in a line.*

Proposition 28. *Let ℓ be a line through the circumcenter O, and $A'B'C'$ be the reflection of ABC in ℓ. $A'B'C'$ is orthologic to ABC at the fourth intersection of the circumcircle and the rectangular circum-hyperbola which is the isogonal conjugate of ℓ* (see Figure 36).

Remarks. (1) By symmetry, if $A'B'C'$ is orthologic to ABC at Q, then ABC is orthologic to $A'B'C'$ at the reflection of Q in the line ℓ.

Line ℓ	Q	Q'
Euler line	$X_{74} = \left(\frac{a^2}{S^2-3S_{BC}} : \ : \ \right)$	X_{477}
Brocard axis	$X_{98} = \left(\frac{1}{S_{BC}-S_{AA}} : \cdots : \cdots\right)$	X_{2698}
OI	$X_{104} = \left(\frac{a}{a^2(b+c)-2abc-(b+c)(b-c)^2} : \cdots : \cdots\right)$	X_{953}

(2) The orthology is valid if ℓ is replaced by an arbitrary line.

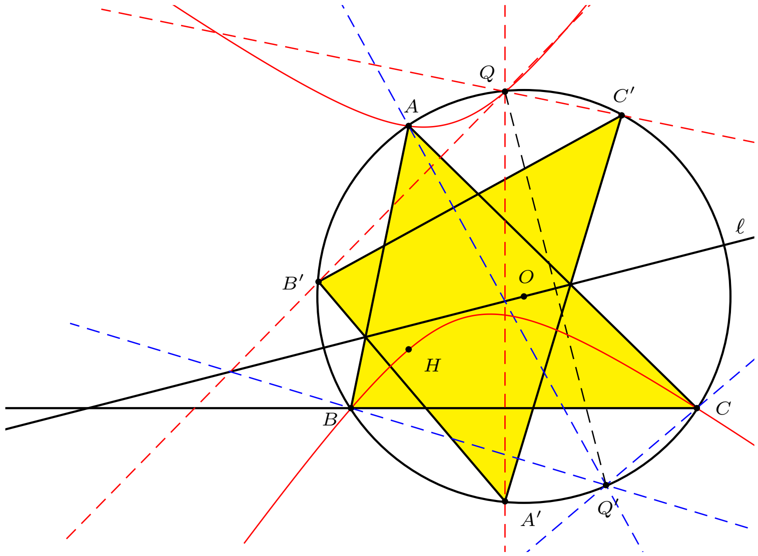


Figure 36. Orthology of triangles symmetric in ℓ

Proposition 29. *Let ℓ be a line through a given point P, and A', B', C' the reflections of A, B, C in ℓ. The lines $A'P$, $B'P$, $C'P$ intersect the sidelines BC, CA, AB respectively at X, Y, Z. The points X, Y, Z are collinear, and the line $\mathscr{L}$ containing them envelopes the inscribed conic with P as a focus* (see Figure 37).

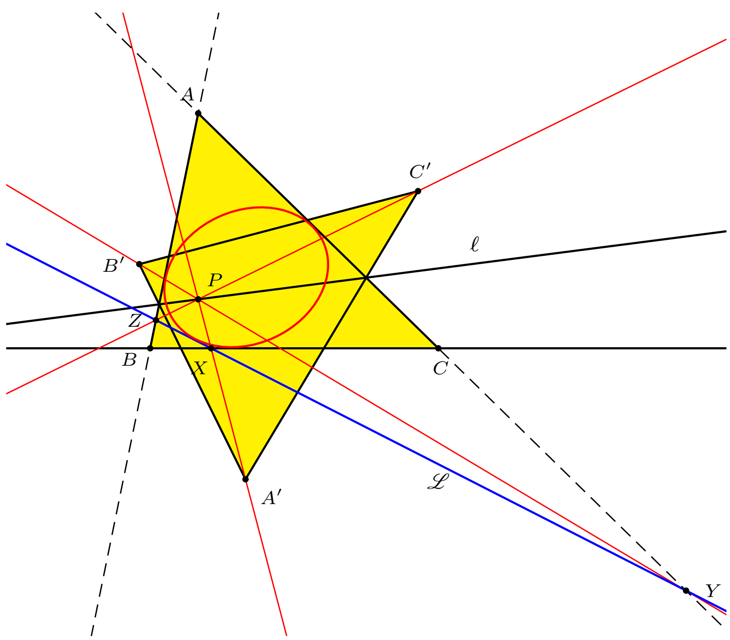


Figure 37. Line $\mathscr{L}$ induced by reflections in ℓ

Proof. Let ℓ be the line joining $P = (u : v : w)$ and $Q = (x : y : z)$. The line $\mathscr{L}$ containing X, Y, Z is

$$\sum_{\text{cyclic}} \frac{u\mathbb{X}}{(b^2u^2 + 2S_Cuv + a^2v^2)(uz - wx)^2 - (a^2w^2 + 2S_Bwu + c^2u^2)(vx - uy)^2} = 0,$$

equivalently with line coordinates

$$\left(\frac{u}{(b^2u^2 + 2S_Cuv + a^2v^2)(uz - wx)^2 - (a^2w^2 + 2S_Bwu + c^2u^2)(vx - uy)^2} : \cdots : \cdots\right).$$

Now, the inscribed conic $\mathscr{C}$ with a focus at $P = (u : v : w)$ has center the midpoint between P and P^* and perspector

$$\left(\frac{1}{u(c^2v^2 + 2S_Avw + b^2w^2)} : \frac{1}{v(a^2w^2 + 2S_Bwu + c^2u^2)} : \frac{1}{w(b^2u^2 + 2S_Cuv + a^2w^2)}\right).$$

Its dual conic is the circumconic

$$\sum_{\text{cyclic}} \frac{u(c^2v^2 + 2S_Avw + b^2w^2)}{\mathbb{X}} = 0,$$

which, as is easily verified, contains the line $\mathscr{L}$ (see [38, §10.6.4]). This means that $\mathscr{L}$ is tangent to the inscribed conic $\mathscr{C}$. □

Remarks. (1) For the collinearity of X, Y, Z, see [23].

(2) The line $\mathscr{L}$ touches the inscribed conic $\mathscr{C}$ at the point

$$\left(\frac{1}{u(c^2v^2 + 2S_Avw + b^2w^2)}\left(\frac{(uz - wx)^2}{a^2w^2 + 2S_Bwu + c^2u^2} - \frac{(vx - uy)^2}{b^2u^2 + 2S_Cuv + a^2v^2}\right)^2 : \cdots : \cdots\right).$$

(i) If $P = I$, then the line $\mathscr{L}$ is tangent to the incircle. For example, if ℓ is the OI-line, then $\mathscr{L}$ touches the incircle at

$$X_{3025} = (a^2(b - c)^2(b + c - a)(a^2 - b^2 + bc - c^2) : \cdots : \cdots).$$

(ii) If P is a point on the circumcircle, then the conic $\mathscr{C}$ is an inscribed parabola, with focus P and directrix the line of reflections of P (see §1.2). If we take ℓ to be the diameter OP, then the line $\mathscr{L}$ touches the parabola at the point

$$(a^4(b^2 - c^2)(S^2 - 3S_{AA})^2 : \cdots : \cdots).$$

(3) Let ℓ be the Euler line. The two lines $\mathscr{L}$ corresponding to O and H intersect at

$$X_{3258} = ((b^2 - c^2)^2(S^2 - 3S_{BC})(S^2 - 3S_{AA}) : \cdots : \cdots)$$

on the nine-point circle, the inferior of X_{476}, the reflection of E in the Euler line (see [15]). More generally, for isogonal conjugate points P and P^* on the Macay cubic K003, *i.e.*, p$K(K, O)$, the two corresponding lines $\mathscr{L}$ with respect to the line PP^* intersect at a point on the common pedal circle of P and P^*. For other results, see [24, 16].

8.2. *Reflections of lines in cevian triangle.*

Proposition 30 ([9])**.** *The reflection triangle of $P = (u : v : w)$ in the cevian triangle of P is perspective with ABC at*

$$r_5(P) = \left(u\left(-\frac{a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right). \qquad (8)$$

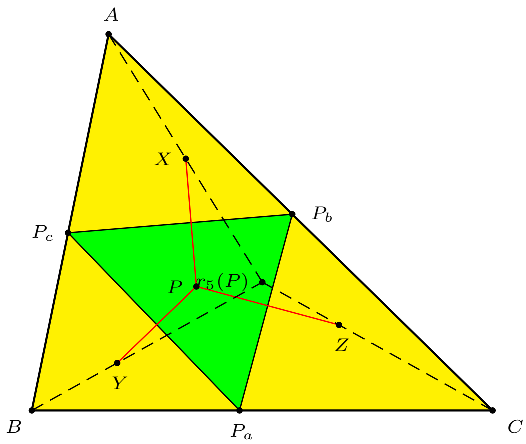


Figure 38. Reflections in sides of cevian triangle

Proof. Relative to the triangle $P_aP_bP_c$, the coordinates of P are $(v + w : w + u : u + v)$. Similarly, those of A, B, C are

$$(-(v+w) : w+u : u+v), \quad (v+w : -(w+u) : u+v), \quad (v+w : w+u : -(u+v)).$$

Triangle ABC is the anticevian triangle of P relative to $P_aP_bP_c$. The perspectivity of ABC and the reflection triangle of P in $P_aP_bP_c$ follows from Proposition 6.

The reflection of P in the line P_bP_c is the point

$$X = \left(u\left(\frac{3a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} - \frac{b^2+c^2-a^2}{vw} + \frac{2(c^2+a^2-b^2)}{wu} + \frac{2(a^2+b^2-c^2)}{uv}\right)\right.$$
$$\left. : v\left(\frac{a^2}{u^2} - \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{c^2+a^2-b^2}{wu}\right) : \ w\left(\frac{a^2}{u^2} + \frac{b^2}{v^2} - \frac{c^2}{w^2} + \frac{a^2+b^2-c^2}{uv}\right)\right).$$

Similarly, the coordinates of the reflections Y of P in P_cP_a, and Z of P in P_aP_b can be written down. From these, it is clear that the lines AX, BY, CZ intersect at the point with coordinates given in (8). □

The triangle XYZ is clearly orthologic with the cevian triangle $P_aP_bP_c$, since the perpendiculars from X to P_bP_c, Y to P_cP_a, and Z to P_aP_b intersect at P. It follows that the perpendiculars from P_a to YZ, P_b to ZX, and P_c to XY are also concurrent. The point of concurrency is

$$r_6(P) = \left(u\left(\frac{a^2}{u^2} + \frac{b^2}{v^2} + \frac{c^2}{w^2} + \frac{b^2+c^2-a^2}{vw}\right) : \cdots : \cdots\right).$$

In fact, P_a, P_b, P_c lie respectively on the perpendicular bisectors of YZ, ZX, XY. The point $r_6(P)$ is the center of the circle XYZ (see Figure 39). As such, it is the isogonal conjugate of P in its own cevian triangle.

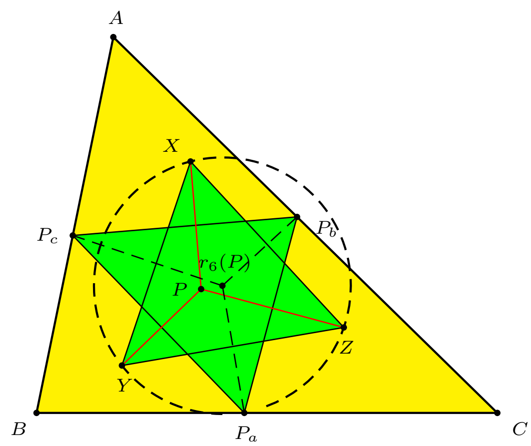


Figure 39. Circumcircle of reflections in cevian triangle

P	I	G	H	G_e	X_{99}	X_{100}	E
$r_5(P)$	X_{35}	$H^\bullet$	X_{24}	X_{57}	X_{115}^*	F_e^*	X_{125}^*
$r_6(P)$		X_{141}	H	X_{354}		X_{1618}	

Remarks. (1) In ETC, $r_5(P)$ is called the Orion transform of P.

(2) $X_{35} = (a^2(b^2+c^2-a^2+bc) : b^2(c^2+a^2-b^2+ca) : c^2(a^2+b^2-c^2+ab))$ divides OI in the ratio $R : 2r$. On the other hand,

$$r_6(I) = (a^2(b^2+c^2-a^2+3bc) : b^2(c^2+a^2-b^2+3ca) : c^2(a^2+b^2-c^2+3ab))$$

divides OI in the ratio $3R : 2r$ (see also Remark (3) following Proposition 31 below).

8.3. *Reflections of sidelines of cevian triangles.* Let P be a point with cevian triangle $P_aP_bP_c$. It is clear that the lines BC, P_bP_c, and their reflections in one another concur at a point on the trilinear polar of P (see Figure 40).

This is the same for line CA, P_cP_a and their reflections in one another; similarly for AB and P_aP_b. Therefore, the following four triangles are line-perspective at the trilinear polars of P:

(i) ABC,

(ii) the cevian triangle of P,

(iii) the triangle bounded by the reflections of P_bP_c in BC, P_cP_a in CA, P_aP_b in AB,

(iv) the triangle bounded by the reflections of BC in P_bP_c, CA in P_cP_a, AB in P_aP_b.

It follows that these triangles are also vertex-perspective (see [25, Theorems 374, 375]. Clearly if P is the centroid G, these triangles are all homothetic at G.

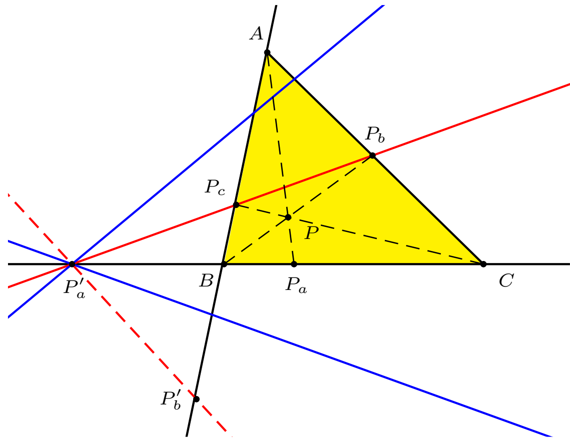


Figure 40. Reflections of sidelines of cevian triangle

Proposition 31. *Let $P_aP_bP_c$ be the cevian triangle of $P = (u : v : w)$.*
(a) *The reflections of P_bP_c in BC, P_cP_a in CA, and P_aP_b in AB bound a triangle perspective with ABC at*

$$r_7(P) = \left(\frac{a^2}{u((c^2+a^2-b^2)v + (a^2+b^2-c^2)w)} : \cdots : \cdots\right).$$

(b) *The reflections of BC in P_bP_c, CA in P_cP_a, and AB in P_aP_b bound a triangle perspective with ABC at*

$$r_8(P) = \left(\frac{a^2vw + u(S_Bv + S_Cw)}{-3a^2v^2w^2 + b^2w^2u^2 + c^2u^2v^2 - 2uvw(S_Au + S_Bv + S_Cw)} : \cdots : \cdots\right).$$

Here are some examples.

P	I	O	H	K	X_{19}	E	X_{393}
$r_7(P)$	X_{21}	X_{1105}	O	$N^\bullet$	X_{1444}	X_{925}	$H^\bullet$

Remarks. (1) The pair $(X_{19},\ X_{1444})$.
(i) $X_{19} = \left(\frac{a}{S_A} : \frac{b}{S_B} : \frac{c}{S_C}\right)$ is the Clawson point. It is the perspector of the triangle bounded by the common chords of the circumcircle with the excircles.
(ii) $X_{1444} = \left(\frac{aS_A}{b+c} : \frac{bS_B}{c+a} : \frac{cS_C}{a+b}\right)$ is the intersection of X_3X_{69} and X_7X_{21}.

(2) $X_{393} = \left(\frac{1}{S_{AA}} : \frac{1}{S_{BB}} : \frac{1}{S_{CC}}\right)$ is the barycentric square of the orthocenter. Let $H_aH_bH_c$ be the orthic triangle, and A_b, A_c the pedals of H_a on CA and AB respectively, and $A' = BA_c \cap CA_b$. Similarly define B' and C'. The lines AA', BB', CC' intersect at X_{393} (see [40]).

(3) The coordinates of $r_8(P)$ are too complicated to list here. For $P = I$, the incenter, note that
(i) $r_8(I) = X_{942} = (a(a^2(b+c) + 2abc - (b+c)(b-c)^2) : \cdots : \cdots)$, and
(ii) the reflections of BC in P_aP_b, CA in P_cP_a, and AB in P_aP_b form a triangle perspective with $P_aP_bP_c$ at $r_6(I)$ which divides OI in the ratio $3R : 2r$.

8.4. *Reflections of H in cevian lines.*

Proposition 32 (Musselman [33]). *Given a point P, let X, Y, Z be the reflections of the orthocenter H in the lines AP, BP, CP respectively. The circles APX, BPY, CPZ have a second common point*

$$r_9(P) = \left(\frac{1}{-2S^2vw + S_A(a^2vw + b^2wu + c^2uv)} : \cdots : \cdots \right).$$

Remark. $r_9(P)$ is also the second intersection of the rectangular circum-hyperbola $\mathscr{H}(P)$ (through H and P) with the circumcircle (see Figure 41).

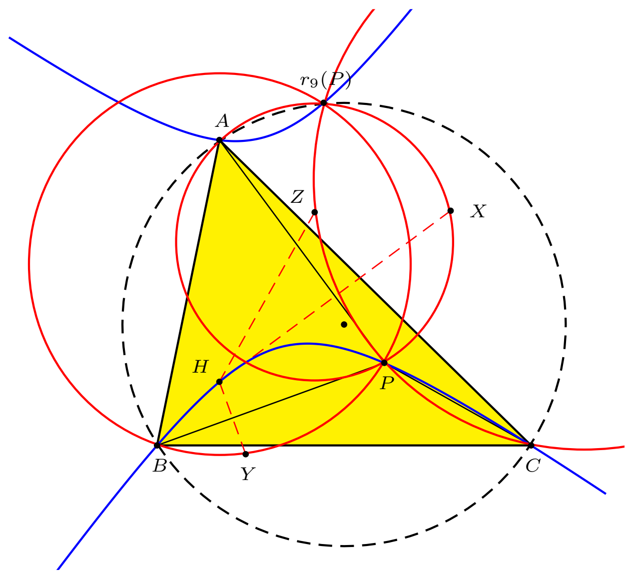


Figure 41. Triad of circles through reflections of H in three cevian lines

8.5. *Reflections in perpendicular bisectors.*

Proposition 33 ([8]). *Given a point P with reflections X, Y, Z in the perpendicular bisectors of BC, CA, AB respectively, the triangle XYZ is perspective with ABC if and only if P lies on the circumcircle or the Euler line.*

(a) *If P is on the circumcircle, the lines AX, BY, CZ are parallel. The perspector is the isogonal conjugate of P* (see Figure 42).

(b) *If $P = E_t$ on the Euler line, then the perspector is $E^*_{t'}$ on the Jerabek hyperbola, where*

$$t' = \frac{a^2b^2c^2(1+t)}{a^2b^2c^2(1+t) - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)t}$$

(see Figure 43).

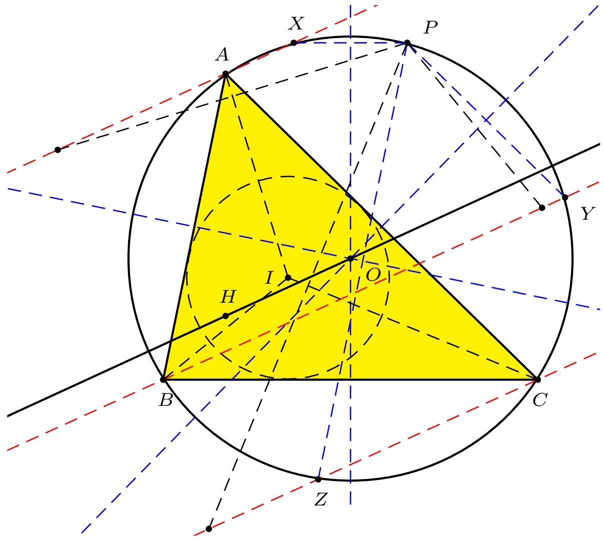


Figure 42. Reflections of P on circumcircle in perpendicular bisectors

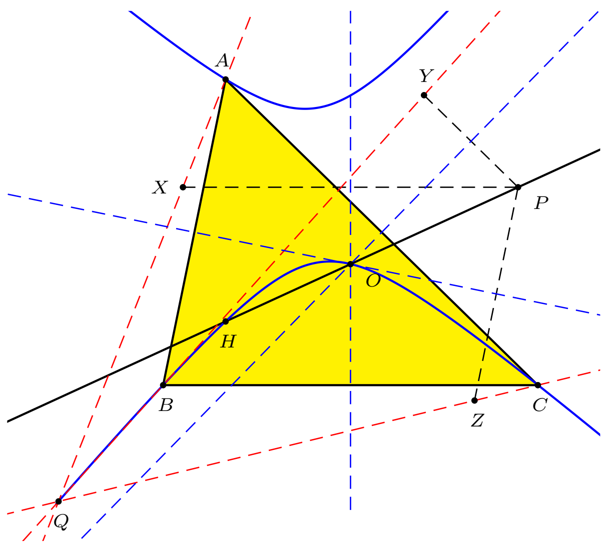


Figure 43. Reflections of P on Euler line in perpendicular bisectors

8.6. *Reflections in altitudes.* Let X, Y, Z be the reflections of P in the altitudes of triangle ABC. The lines AX, BY, CZ are concurrent (at a point Q) if and only if P lies on the reflection conjugate of the Euler line. The perspector lies on the same cubic curve (see Figure 44). This induces a conjugation on the cubic.

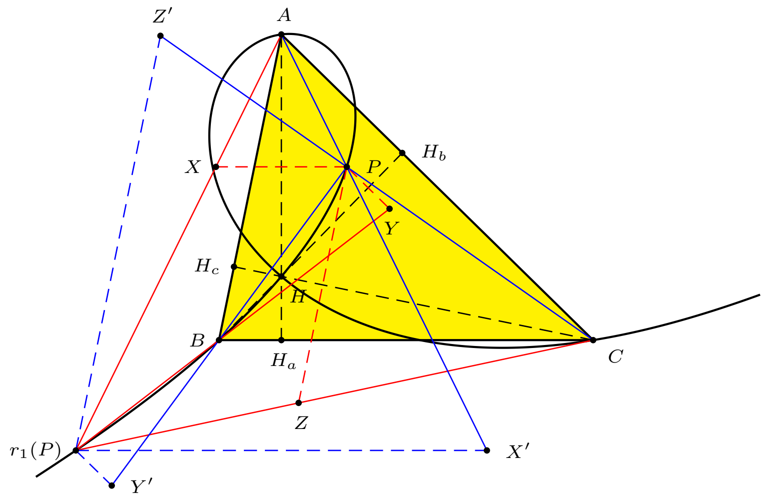


Figure 44. Reflections in altitudes and the reflection conjugate of the Euler line

Proposition 34. *The reflections of $r_1(E_t)$ in the altitudes are perspective with ABC at $r_1(E_{t'})$ if and only if*

$$tt' = \frac{a^2b^2c^2}{a^2b^2c^2 - (b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

9. Reflections of lines in the cevian triangle of incenter

Let $I_aI_bI_c$ be the cevian triangle of I.

Proposition 35 ([20, 44])**.** *The reflections of I_bI_c in AI_a, I_cI_a in BI_b, and I_aI_b in CI_c bound a triangle perspective with ABC at*

$$X_{81} = \left(\frac{a}{b+c} : \frac{b}{c+a} : \frac{c}{a+b}\right)$$

(see Figure 45).

Proof. The equations of these reflection lines are

$$\begin{aligned} -bcx + c(c+a-b)y + b(a+b-c)z &= 0, \\ c(b+c-a)x - cay + a(a+b-c)z &= 0, \\ b(b+c-a)x + a(c+a-b)y - abz &= 0. \end{aligned}$$

The last two lines intersect at the point

$$(-a(b^2+c^2-a^2-bc) : b(a+b)(b+c-a) : c(c+a)(b+c-a)).$$

With the other two points, this form a triangle perspective with ABC at X_{81} with coordinates indicated above. □

Remark. X_{81} is also the homothetic center of ABC and the triangle bounded by the three lines each joining the perpendicular feet of a trace of an angle bisector on the other two angle bisectors ([39]).

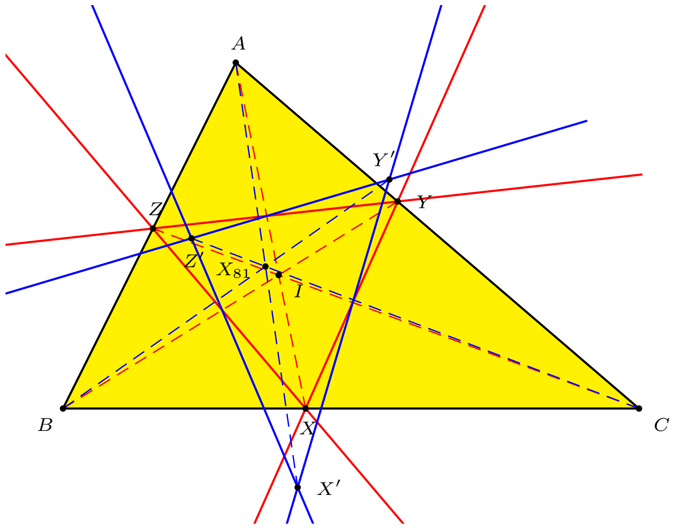


Figure 45. Reflections in the cevian triangle of incenter

Proposition 36. *The reflections of BC in AI_a, CA in BI_b, and AB in CI_c bound a triangle perspective with $I_aI_bI_c$ at*

$$X_{55} = (a^2(b+c-a) : b^2(c+a-b) : c^2(a+b-c)).$$

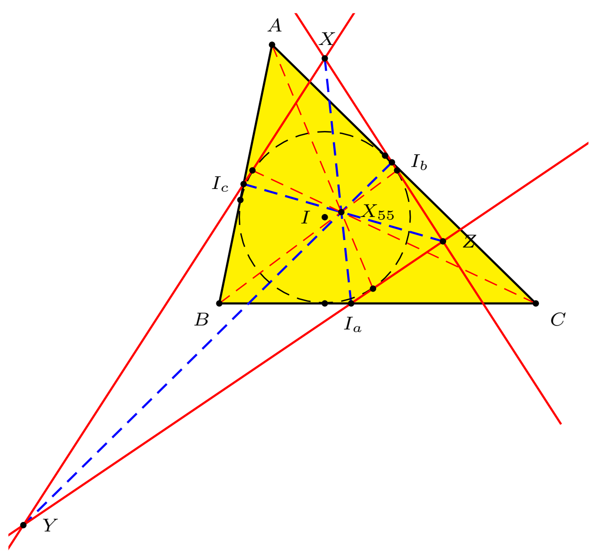


Figure 46. Reflections in angle bisectors

Proposition 37 ([43])**.** *The reflections of AI_a in I_bI_c, BI_b in I_cI_a, and CI_c in I_aI_b are concurrent at a point with coordinates*

$$\begin{aligned}&(a(a^6+a^5(b+c)-4a^4bc-a^3(b+c)(2b^2+bc+2c^2)\\&\quad -a^2(3b^4-b^2c^2+3c^4)+a(b+c)(b-c)^2(b^2+3bc+c^2)+2(b-c)^2(b+c)^4)\\&:\cdots:\cdots)\end{aligned}$$

(see Figure 47).

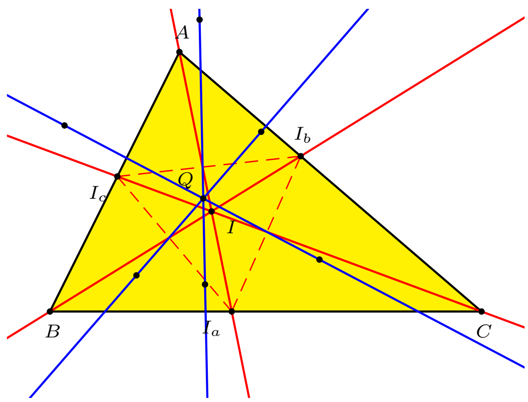


Figure 47. Reflections of angle bisectors in the sidelines of cevian triangle of incenter

10. Reflections in a triangle of feet of angle bisectors

Let P be a given point. Consider the bisectors of angles BPC, CPA, APB, intersecting the sides BC, CA, AB at D_a, D_b, D_c respectively (see Figure 48).

Proposition 38. *The reflections of the lines AP in D_bD_c, BP in D_cD_a, and CP in D_aD_b are concurrent.*

Proof. Denote by x, y, z the distances of P from A, B, C respectively. The point D_a divides BC in the ratio $y:z$ and has homogeneous barycentric coordinates $(0:z:y)$. Similarly, $D_b=(z:0:x)$ and $D_c=(y:x:0)$. These can be regarded as the traces of the isotomic conjugate of the point $(x:y:z)$. Therefore, we consider a more general situation. Given points $P=(u:v:w)$ and $Q=(x:y:z)$, let $D_aD_bD_c$ be the cevian triangle of $Q^\bullet$, the isotomic conjugate of Q. Under what condition are the reflections of the cevians AP, BP, CP in the lines D_bD_c, D_cD_a, D_aD_b concurrent?

The line D_bD_c being $-x\mathbb{X}+y\mathbb{Y}+z\mathbb{Z}=0$, the equation of the reflection of the cevian AP in D_bD_c is

$$\begin{aligned}&(-x((c^2+a^2-b^2)x-(b^2+c^2-a^2)y+2c^2z)v+x((a^2+b^2-c^2)x+2b^2y-(b^2+c^2-a^2)z)w)\mathbb{X}\\&+(y((c^2+a^2-b^2)x-(b^2+c^2-a^2)y+2c^2z)v+(a^2x^2-b^2y^2+c^2z^2+(c^2+a^2-b^2)zx)w)\mathbb{Y}\\&-((a^2x^2+b^2y^2-c^2z^2+(a^2+b^2-c^2)xy)v+z((a^2+b^2-c^2)x+2b^2y-(b^2+c^2-a^2)z)w)\mathbb{Z}\\&=0.\end{aligned}$$

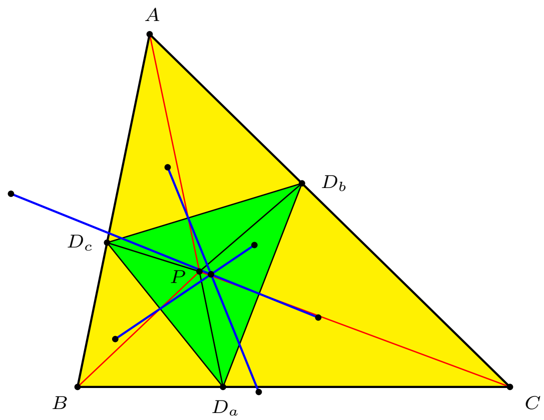


Figure 48. Reflections in a triangle of feet of angle bisectors

By permutating cyclically u, v, w; x, y, z; $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, we obtain the equations of the reflections of BP in D_cD_a and CP in D_aD_b. The condition for the concurrency of the three lines is $F = 0$, where F is a cubic form in u, v, w with coefficients which are sextic forms in x, y, z given in the table below.

term	coefficient
vw^2	$a^2zx(-a^2x^2 + b^2y^2 + c^2z^2 + (b^2 + c^2 - a^2)yz)\cdot$ $(a^2x^2 - 3b^2y^2 + c^2z^2 + (b^2 + c^2 - a^2)yz + (c^2 + a^2 - b^2)zx - (a^2 + b^2 - c^2)xy)$
v^2w	$-a^2xy(-a^2x^2 + b^2y^2 + c^2z^2 + (b^2 + c^2 - a^2)yz)\cdot$ $(a^2x^2 + b^2y^2 - 3c^2z^2 + (b^2 + c^2 - a^2)yz - (c^2 + a^2 - b^2)zx + (a^2 + b^2 - c^2)xy)$
wu^2	$b^2xy(a^2x^2 - b^2y^2 + c^2z^2 + (c^2 + a^2 - b^2)zx)\cdot$ $(a^2x^2 + b^2y^2 - 3c^2z^2 - (b^2 + c^2 - a^2)yz + (c^2 + a^2 - b^2)zx + (a^2 + b^2 - c^2)xy)$
w^2u	$-b^2yz(a^2x^2 - b^2y^2 + c^2z^2 + (c^2 + a^2 - b^2)zx)\cdot$ $(-3a^2x^2 + b^2y^2 + c^2z^2 - (b^2 + c^2 - a^2)yz + (c^2 + a^2 - b^2)zx - (a^2 + b^2 - c^2)xy)$
uv^2	$c^2yz(a^2x^2 + b^2y^2 - c^2z^2 + (a^2 + b^2 - c^2)xy)\cdot$ $(-3a^2x^2 + b^2y^2 + c^2z^2 + (b^2 + c^2 - a^2)yz - (c^2 + a^2 - b^2)zx + (a^2 + b^2 - c^2)xy)$
u^2v	$-c^2zx(a^2x^2 + b^2y^2 - c^2z^2 + (a^2 + b^2 - c^2)xy)\cdot$ $(a^2x^2 - 3b^2y^2 + c^2z^2 - (b^2 + c^2 - a^2)yz + (c^2 + a^2 - b^2)zx + (a^2 + b^2 - c^2)xy)$
uvw	$\sum_{\text{cyclic}} a^4x^5((a^2 + b^2 - c^2)y - (c^2 + a^2 - b^2)z)$ $+\sum_{\text{cyclic}} a^2x^4((a^2 + b^2 - c^2)^2y^2 - (c^2 + a^2 - b^2)^2z^2)$ $+\sum_{\text{cyclic}} a^2x^3yz(((c^2 - a^2)^2 + 3b^2(c^2 + a^2) - 4b^4)y$ $-((a^2 - b^2)^2 + 3c^2(a^2 + b^2) - 4c^4)z)$

By substituting
x^2 by $c^2v^2 + (b^2 + c^2 - a^2)vw + b^2w^2$,
y^2 by $a^2w^2 + (c^2 + a^2 - b^2)wu + c^2u^2$, and
z^2 by $b^2u^2 + (a^2 + b^2 - c^2)uv + a^2v^2$,
which are proportional to the squares of the distances AP, BP, CP respectively, with the help of a computer algebra system, we verify that $F = 0$. Therefore we conclude that the reflections of AP, BP, CP in the sidelines of $D_aD_bD_c$ do concur. □

In the proof of Proposition 38, if we take $Q = G$, the centroid, this yields Proposition 24. On the other hand, if $Q = X_8$, the Nagel point, we have the following result.

Proposition 39. *The locus of P for which the reflections of the cevians AP, BP, CP in the respective sidelines of the intouch triangle is the union of the circumcircle and the line OI:*

$$\sum_{\text{cyclic}} bc(b-c)(b+c-a)\mathbb{X} = 0.$$

(a) *If P is on the circumcircle, the cevians are parallel, with infinite point the isogonal conjugate of P* (see Figure 49).

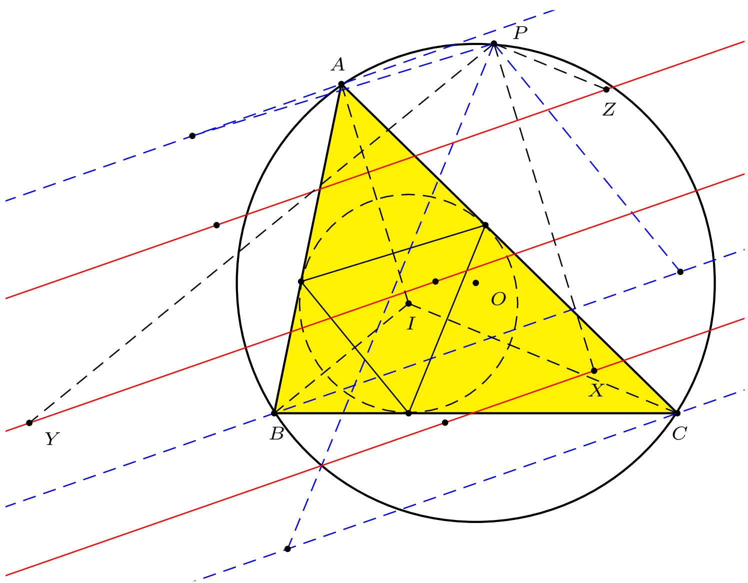


Figure 49. Reflections of cevians of P in the sidelines of the intouch triangle

(b) *If P is on the line OI, the point of concurrency traverses the conic*

$$\sum_{\text{cyclic}} (b-c)(b+c-a)^2x^2 + (b-c)(c+a-b)(a+b-c)yz = 0,$$

which is the Jerabek hyperbola of the intouch triangle (see Figure 50). *It has center*

$$(a(c+a-b)(a+b-c)(a^2(b+c) - 2a(b^2+c^2) + (b^3+c^3)) : \cdots : \cdots).$$

Finally, if we take $Q = \left(\frac{u}{a^2} : \frac{v}{b^2} : \frac{w}{c^2}\right)$ in the proof of Proposition 38, we obtain the following result.

Proposition 40. *Let $P_a^*P_b^*P_c^*$ be the cevian triangle of the isogonal conjugate of P. The reflections of AP in $P_b^*P_c^*$, BP in $P_c^*P_a^*$, CP in $P_a^*P_b^*$ are concurrent* (see Figure 51).

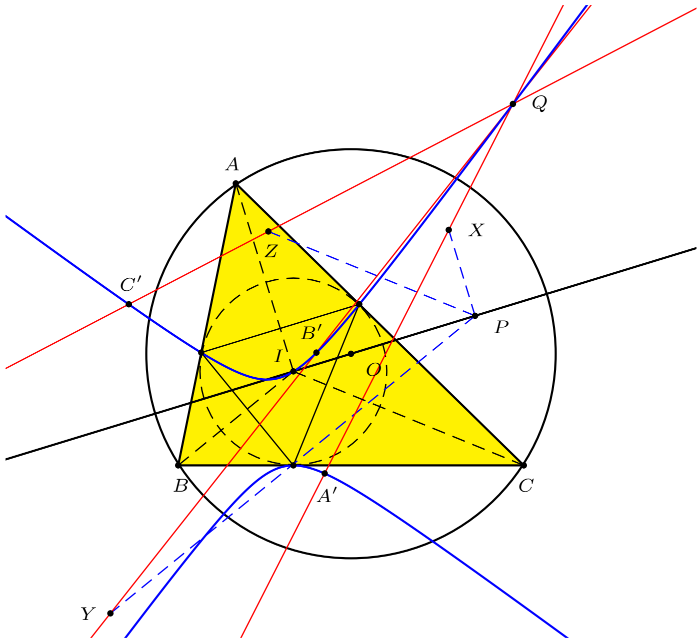


Figure 50. Reflections of cevians of P in the sidelines of the intouch triangle

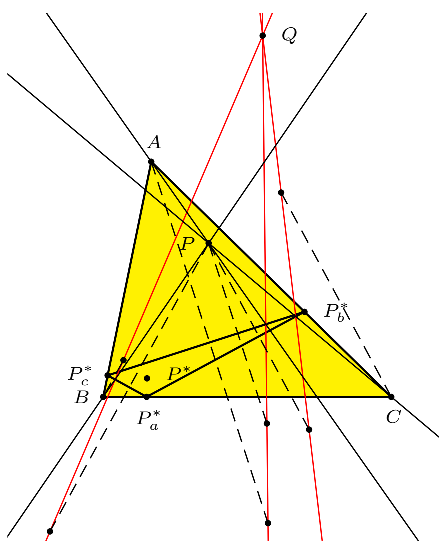


Figure 51. Reflections of cevians of P in cevian triangle of P^*

A special case is Proposition 37 above. For $P = X_3 = O$, the common point is $X_3 = O$. This is because the cevian triangle of $O^* = H$ is the orthic triangle, and the radii OA, OB, OC are perpendicular to the respective sides of the orthic triangle. Another example is $(P, Q) = (K,\ X_{427})$.

Synopsis

Triangle centers	References	**Triangle centers**	References
$F_{\rm e}$	Table following Thm. 4	X_{79}	Rmk (1) following Prop. 7
	Table following Prop. 10	X_{80}	Rmk (2) following Prop. 7
$F_{\pm}$	End of §2.2		Rmk following Thm. 10
	Table following Prop. 10		Table following Prop. 22
$J_{\pm}$	Prop. 1(c); end of §2.2	X_{81}	Prop. 35
Ω, Ω'	Table in §6.3	$X_{95} = N^{\bullet}$	Table following Prop. 31
E	Rmk (3) following Thm. 3; Figure 4	X_{98}	Table in Rmk (1) following Prop. 28
	Table following Thm. 4	X_{99}	Table following Prop. 22
	Rmk (1) following Prop. 9		Table in §8.2
	Table following Prop. 22	X_{100}	Table following Prop. 22
	Table in §8.2		Table in §8.2
	Table following Prop. 31	X_{104}	Table in Rmk (1) following Prop. 28
E_{∞}	Rmk following Prop. 7	X_{108}	Table following Thm. 4
	Rmk (2) following Prop. 9	X_{109}	Rmk (3) following Thm. 3
$W = X_{484}$	Rmk (2) following Prop. 7	X_{112}	Rmk (3) following Thm. 3
	Rmk following Prop. 16		Table following Thm. 4
N^*	Rmk (1) following Prop. 5	X_{115}	Table following Thm. 4
	Rmk (2) following Prop. 9		Table following Prop. 10
	Prop. 14		§§6.2, 6.3
X_{19}	Rmk (1) following Prop. 31	X_{125}	Table following Thm. 4
X_{21}	Rmk (1) following Prop. 17		Table following Prop. 10;
	Table following Prop. 31		§6.3
X_{24}	Rmk (2) following Prop. 17		Rmk (6) following Prop. 23
	Rmk (3) following Prop. 25	X_{141}	Table in §8.2
	Table in Rmk (2) following Prop. 12	X_{143}	Rmk (3) Prop. following 17
	Table in §8.2	$X_{155} = H/O$	Table following Prop. 6
$X_{25} = H/K$	Table following Prop. 6	X_{186}	Rmk (2) followingProp. 12
	Rmk (4) following Prop. 23		Rmk (2) following Prop. 25
X_{35}	Rmk (2) at the end of §8.2		Table in Rmk following Prop. 27
X_{40}	§6.1	$X_{193} = H/G$	Table following Prop. 6
$X_{46} = H/I$	Table following Prop. 6	X_{195}	Rmk (1) following Prop. 5
$X_{52} = H/N$	Table following Prop. 6		Rmk (2) following Prop. 9
$X_{55} = G_{\rm e}^*$	Prop. 36	X_{214}	Table following Prop. 22
X_{57}	Table in §8.2	$X_{249} = X_{115}^*$	Prop. 17(b); Table in §8.2
$X_{59} = F_{\rm e}^*$	Table in §8.2	$X_{250} = X_{125}^*$	Prop. 17(b); Table in §8.2
X_{60}	Rmk (4) following Prop. 17	$X_{265} = r_1(O)$	Table following Prop. 10
X_{65}	Table in Rmk (1) following Prop. 23		Tables following Prop. 22, 23, 27
X_{66}	Table in Rmk (1) following Prop. 23	X_{354}	Table in §8.2
X_{67}	Table following Prop. 10	X_{393}	Rmk (2) following Prop. 31
X_{68}	Table in Rmk (1) following Prop. 23	X_{399}	Rmk (1) following Prop. 9
	Table in Rmk (2) following Prop. 12		§5.1.2; §5.1.3
$X_{69} = H^{\bullet}$	Table in Rmk (1) following Prop. 23	X_{403}	Rmk (2) following Prop. 12
	Table in §8.2		Table in Rmk (1) Prop. 23
	Table following Prop. 31	X_{427}	Rmk (5) following Prop. 23
X_{72}	Table in Rmk (1) following Prop. 23		Rmk following Prop. 40
X_{74}	Table in Rmk (1) following Prop. 23	X_{429}	Table in Rmk (1) Prop. 23
	Table in Rmk (1) following Prop. 28	X_{442}	Table in Rmk (1) Prop. 23

Triangle centers	References
X_{476}	Table following Thm. 4
X_{477}	Table in Rmk (1) following Prop. 28
X_{571}	Rmk (2) following Prop. 25
X_{671}	Tables following Prop. 10, 22
X_{895}	Table following Prop. 22
X_{942}	Rmk (3) following Prop. 31
X_{925}	Table Prop. 31
X_{953}	Table in Rmk (1) following Prop. 28
X_{1105}	Table Prop. 31
X_{1141}	Rmk (3) following Prop. 7
X_{1145}	Table following Prop. 22
X_{1156}	Tables following Prop. 10, 22
X_{1157}	Rmk (3) following Prop. 7
$= (N^*)^{-1}$	Table following Prop. 9
	Corollary 15; §5.1.1
X_{1320}	Table following Prop. 10
	Table following Prop. 22
X_{1444}	Rmk (1) following Prop. 31
X_{1618}	Table in §8.2

Triangle centers	References
X_{1986}	Rmk (1) following Prop. 12
	Table following Prop. 17
	Rmk (3) following Prop. 25
X_{2698}	Table in Rmk (1) following Prop. 28
X_{2715}	Table following Thm. 4
X_{2720}	Table following Thm. 4
X_{2482}	Table following Prop. 22
X_{3003}	Rmk (1) following Prop. 25
X_{3025}	Rmk (2) following Prop. 29
X_{3528}	Rmk (3) following Prop. 29
superiors of Fermat points	§5.1.4
new	End of §2.2
	Rmk (2) following Prop. 12
	§5.1.3
	Rmk following Prop. 27
	Rmk (2) following Prop. 29
	Rmk (2) following Prop. 30
	Prop. 37, 39

Reflection triangles	References
O	§1
H	Rmk (1) following Prop. 12; Rmk following Prop. 27
N	§1, Prop. 5
K	Rmk (4) following Prop. 23

Cevian triangles	References
G (medial)	§7.1
I (incentral)	Rmk (1) following Prop. 7; §9
H (orthic)	Prop. 6; §5.1.1, §7.2; Rmk (5) following Prop. 23

Anticevian triangles	References
I (excentral)	Figure 11; §5.1.3
K (tangential)	Prop. 1(a); §5.1.2; Rmk (4) following Prop. 23
N^*	§5.1.1

Lines	References
Euler line	Figure 4; Prop. 17, 23, 24, 33; Rmk (2) following Prop. 29
OI	Prop. 39

Circles	References
Circumcircle	Prop. 1(d); Thm. 3; Prop. 17, 33, 39
Incircle	Rmk (2) following Prop. 29
Nine-point circle	Rmk 2 Prop. 2; §6.3; Prop. 23
Apollonian circles	Prop. 1(b)
Brocard circle	§6.2
Pedal circle of G	§6.3
$P^{(a)}P^{(b)}P^{(c)}$	Prop. 2; Rmk following Prop. 10
Circles containing $A^{(a)}$, $B^{(b)}$, $C^{(c)}$	§5 passim

Conics	References
Steiner circum-ellipse	§6.4
Jerabek hyperbola	Prop. 23, 24, 33
bicevian conic $\mathscr{C}(G, Q)$	Prop. 22
bicevian conic $\mathscr{C}(X_{115}^*,\ X_{125}^*)$	Prop. 17
Jerabek hyperbola of intouch triangle	Prop. 39
circumconic with center P	Prop. 22
Inscribed parabola with focus E	Rmk (2) following Prop. 29
rectangular circum-hyperbola through P	Rmk following Prop. 10; Rmk following Prop. 32
Inscribed conic with a given focus P	Prop. 29

Cubics	References
Neuberg cubic K001	Prop. 7, 8, 9, 16, 26, 27
Macay cubic K003	Rmk (3) following Prop. 29
Napoleon cubic K005	Prop. 9, 27
Orthocubic K006	Prop. 26
p$K(X_{1989},\ X_{265})$ = K060	Prop. 7, 8
p$K(X_{3003},\ H)$ = K339	Prop. 25
p$K(X_{186},\ X_{571})$	Prop. 25
Reflection conjugate of Euler line	§8.6

Quartics	References
Isogonal conjugate of nine-point circle	Prop. 17
Isogonal conjugate of Brocard circle	§6.4

Constructions	References
H/P	Prop. 6
$r_0(P)$	Rmk (3) following Thm. 3; Thm. 4; Prop. 20
$r_1(P)$	Prop. 10, Prop. 11
$r_2(P)$	Prop. 12
$r_3(P)$	Prop. 22
$r_4(P)$	Prop. 22
$r_5(P)$	Prop. 30
$r_6(P)$	§8.2
$r_7(P)$	Prop. 31
$r_8(P)$	Prop. 31
$r_9(P)$	Prop. 32

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