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On a Triad of Similar Triangles Associated With the Perpendicular Bisectors of the Sides of a Triangle

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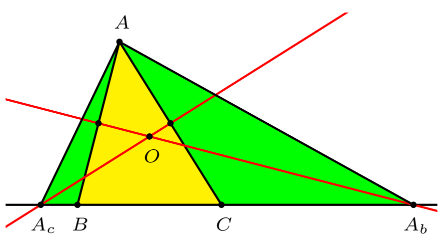
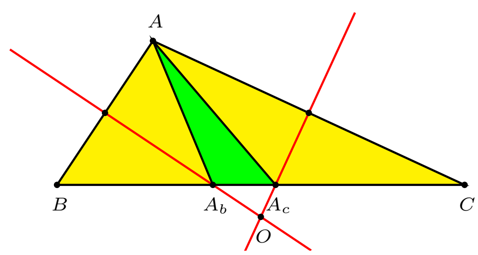
Abstract. We prove some perspectivity and orthology results related to a triad of similar triangles associated with the perpendiculars bisectors of the sides of a given triangle.

1. A triad of triangles from the perpendicular bisectors

Given a triangle ABC, let the perpendicular bisectors of the sides AB and AC intersect the sideline BC at A_b and A_c respectively. The angles of triangle AA_bA_c have measures $\pi - 2A$, $\pi - 2B$, $\pi - 2C$ if ABC is acute angled, and $2A - \pi$, $2B$, $2C$ if angle A is obtuse (see Figure 1).



(a) Acute angle case



(b) Obtuse angle case

Figure 1.

Similarly, we consider the intersections B_a and B_c of CA with the perpendicular bisectors of BA and BC, yielding triangle B_aBB_c, and the intersections C_a and C_b of AB with the perpendicular bisectors of CA, CB yielding triangle C_aC_bC. It is clear that the triangles AA_bA_c, B_aBB_c and C_aC_bC are all similar (see Figure 2). We record the homogeneous barycentric coordinates of these points.

$$\begin{aligned}
A_b &= (0 : -S_A + S_B : S_A + S_B), & A_c &= (0 : S_C + S_A : S_C - S_A);\\
B_c &= (S_B + S_C : 0 : -S_B + S_C), & B_a &= (S_A - S_B : 0 : S_A + S_B);\\
C_a &= (-S_C + S_A : S_C + S_A : 0), & C_b &= (S_B + S_C : S_B - S_C : 0).
\end{aligned}$$

Given a finite point P with homogeneous barycentric coordinates $(x : y : z)$ relative to ABC, we consider the points with the same coordinates in the triangles AA_bA_c, B_aBB_c and C_aC_bC. These are the points

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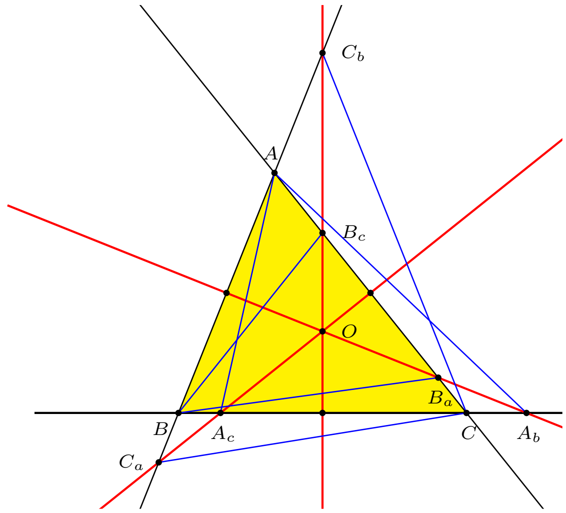

Figure 2.

$$\begin{aligned}
A_P &= (2S_{BC}x : \ -S_C(S_A - S_B)y + S_B(S_C + S_A)z : \ S_C(S_A + S_B)y - S_B(S_C - S_A)z), \\
B_P &= (S_A(S_B + S_C)z - S_C(S_A - S_B)x : \ 2S_{CA}y : \ -S_A(S_B - S_C)z + S_C(S_A + S_B)x), \\
C_P &= (-S_B(S_C - S_A)x + S_A(S_B + S_C)y : S_B(S_C + S_A)x - S_A(S_B - S_C)y : \ 2S_{AB}z).
\end{aligned}$$

with coordinates with respect to triangle ABC (see Figure 3).

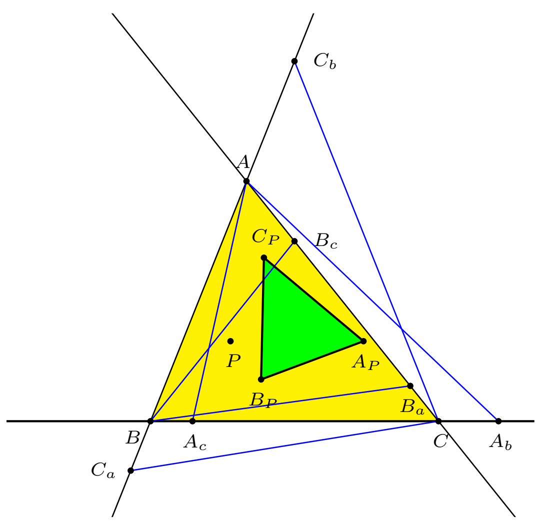

Figure 3.

Proposition 1. *Triangle $A_PB_PC_P$ is oppositely similar to ABC.*

Proof. The square lengths of the sides are

$$(B_PC_P)^2 = k \cdot a^2, \quad (C_PA_P)^2 = k \cdot b^2, \quad (A_PB_P)^2 = k \cdot c^2,$$

where

$$k = \frac{S^2\left(\sum_{\text{cyclic}} a^4 S_{AA} yz\right) - (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2 S_{BB}S_{CC}x\right)}{(2S_{ABC}(x+y+z))^2}.$$

From this the similarity is clear. Note that the multiplier k is nonnegative. Now, the areas are related by

$$\Delta A_PB_PC_P = -k \cdot \Delta ABC.$$

It follows that the two triangles are oppositely oriented. □

2. Perspectivity

Proposition 2. *Triangle $A_PB_PC_P$ is perspective with the medial triangle at*

$$\left(\frac{x}{S_A}\left(-\frac{x}{S_A} + \frac{y}{S_B} + \frac{z}{S_C}\right) : \cdots : \cdots\right). \tag{1}$$

Proof. Let DEF be the medial triangle. The line joining DA_P has equation

$$S_A(S_Cy - S_Bz)\mathbb{X} + S_{BC}x\mathbb{Y} - S_{BC}x\mathbb{Z} = 0,$$

or

$$\left(\frac{y}{S_B} - \frac{z}{S_C}\right)\mathbb{X} + \frac{x}{S_A}(\mathbb{Y} - \mathbb{Z}) = 0.$$

This clearly contains the point $\left(-\frac{x}{S_A} : \frac{y}{S_B} : \frac{z}{S_C}\right)$, which is a vertex of the anticevian triangle of $Q = \left(\frac{x}{S_A} : \frac{y}{S_B} : \frac{z}{S_C}\right)$. Similarly, the lines EB_P and FC_P contain the corresponding vertices of the same anticevian triangle. It follows that $A_PB_PC_P$ and DEF are perspective at the cevian quotient G/Q whose coordinates are given by (1) above. □

Proposition 3. *Triangle $A_PB_PC_P$ is perspective with ABC if and only if P lies on the Jerabek hyperbola of ABC. In this case, the perspector traverses the Euler line.*

Proof. The lines AA_P, BB_P, CC_P have equations

$$\begin{aligned}
(S_C(S_A+S_B)y + S_B(S_C-S_A)z)\mathbb{Y} + (S_C(S_A-S_B)y - S_B(S_C+S_A)z)\mathbb{Z} &= 0,\\
(S_A(S_B-S_C)z - S_C(S_A+S_B)x)\mathbb{X} + (S_A(S_B+S_C)z + S_C(S_A-S_B)x)\mathbb{Z} &= 0,\\
(S_B(S_C+S_A)x + S_A(S_B-S_C)y)\mathbb{X} + (S_B(S_C-S_A)x - S_A(S_B+S_C)y)\mathbb{Y} &= 0.
\end{aligned}$$

They are concurrent if and only if

$$2S_{ABC}(x+y+z)\left(\sum_{\text{cyclic}} S_A(S_{BB} - S_{CC})yz\right) = 0.$$

Since P is a finite point, this is the case when P lies on the conic

$$\sum_{\text{cyclic}} S_A(S_{BB}-S_{CC})yz=0,$$

which is the Jerabek hyperbola, the isogonal conjugate of the Euler line.

Now, if P is the isogonal conjugate of the point $(S_{BC}+t:S_{CA}+t:S_{AB}+t)$, then the perspector is the point

$$(a^2((2S_{ABC}+t(S_A+S_B+S_C))\cdot S_A-t\cdot S_{BC}):\cdots:\cdots).$$

Since $(a^2S_{BC}:b^2S_{CA}:c^2S_{AB})$ is the point X_{25} (the homothetic center of the tangential and orthic triangles) on the Euler line, the locus of the perspector is the Euler line. □

3. Orthology

Proposition 4. *The triangles ABC and $A_PB_PC_P$ are orthologic.*
(a) *The perpendiculars from A to B_PC_P, B to C_PA_P, and C to A_PB_P intersect at the point*

$$Q=\left(\frac{a^2S_A}{-\frac{x}{S_{ABC}}+\frac{y}{b^2S_{BB}}+\frac{z}{c^2S_{CC}}}:\frac{b^2S_B}{\frac{x}{a^2S_{AA}}-\frac{y}{S_{ABC}}+\frac{z}{c^2S_{CC}}}:\frac{c^2S_C}{\frac{x}{a^2S_{AA}}+\frac{y}{b^2S_{BB}}-\frac{z}{S_{ABC}}}\right).$$

(b) *The perpendiculars from A_P to BC, B_P to CA, and C_P to AB intersect at the point*

$$Q'=\left(\left(\sum_{\text{cyclic}}\frac{x}{a^2S_{AA}}\right)\frac{1}{S_A}-\frac{S^2}{a^2b^2c^2S_{ABC}}\cdot\frac{(S^2-S_{AA})x}{S_{AA}}:\cdots:\cdots\right).$$

Remarks. (1) The orthology center Q lies on the circumcircle of triangle ABC. The orthology centers $Q(P)$ and $Q(P')$ are the same if and only if the line PP' contains the circumcenter O. Here are some examples.

P	G	I	K	N_a	X_{110}	X_{69}	X_{184}
Q	X_{1294}	X_{1295}	X_{1297}	X_{2734}	X_{2693}	X_{98}	X_{74}

(2) The coordinates of Q' with respect to $A_PB_PC_P$ are the same as those of Q with respect to ABC.

(3) Therefore, Q' lies on the circumcircle of triangle $A_PB_PC_P$.

4. An example: Gergonne points of the triad

We assume ABC to be acute angled, so that the triangles AA_bA_c, B_aBB_c and C_aC_bC have angles $\pi-2A$, $\pi-2B$ and $\pi-2C$ and sidelengths in the proportion of $a^2S_A:b^2S_B:c^2S_C$. In this case, the Gergonne point A_{g} of AA_bA_c has homogeneous barycentric coordinates $(S_A:S_B:S_C)$ with respect to the triangle. Therefore, with $P=(S_A:S_B:S_C)$ (the triangle center X_{69}, we have $A_{\text{g}}=A_P=(2S_A:S_B+S_C:S_B+S_C)$, and likewise $B_{\text{g}}=(S_C+S_A:2S_B:S_C+S_A)$ and $C_{\text{g}}=(S_A+S_B:S_A+S_B:2S_C)$. From the table above, the orthology center

Q is $X_{98} = \left(\frac{1}{S_{AA}-S_{BC}} : \frac{1}{S_{BB}-S_{CA}} : \frac{1}{S_{CC}-S_{AB}}\right)$, the Tarry point (see Figure 4). The other orthology center Q', being the Tarry point of $A_{\mathrm{g}}B_{\mathrm{g}}C_{\mathrm{g}}$, is

$$\begin{aligned}
&(S_{BB}-S_{CA})(S_{CC}-S_{AB})(2S_A, S_B+S_C, S_B+S_C)\\
&+(S_{CC}-S_{AB})(S_{AA}-S_{BC})(S_C+S_A, 2S_B, S_C+S_A)\\
&+(S_{AA}-S_{AB})(S_{BB}-S_{CA})(S_A+S_B, S_A+S_B, 2S_C)\\
&=(\sum_{\text{cyclic}} S_{BC}-S_{AA})((S_{BC}+S_{CA}+S_{AB})(S_A,S_B,S_C)\\
&\qquad +(S_A+S_B+S_C)(S_{BC},S_{CA},S_{AB})).
\end{aligned}$$

This is the triangle center X_{1352}, which is the midpoint of H and X_{69}.

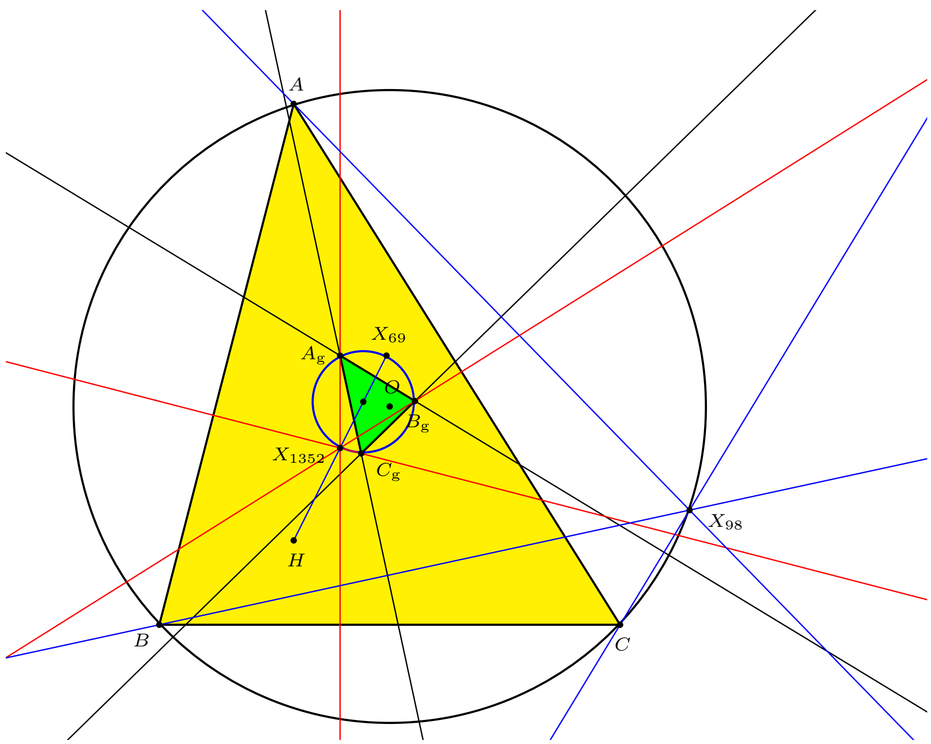

Figure 4.

Remarks. (1) The circumcenter of $A_{\mathrm{g}}B_{\mathrm{g}}C_{\mathrm{g}}$ is the point

$$\begin{aligned}
&S_A(S_B+S_C)(2S_A, S_B+S_C, S_B+S_C)\\
&+S_B(S_C+S_A)(S_C+S_A, 2S_B, S_C+S_A)\\
&+S_C(S_A+S_B)(S_A+S_B, S_A+S_B, 2S_C)\\
&=3(S_{BC}+S_{CA}+S_{AB})(S_A,S_B,S_C)+(S_A+S_B+S_C)(S_{BC},S_{CA},S_{AB}).
\end{aligned}$$

This is the midpoint of X_{69} and X_{1352}.

(2) Since the Tarry point and the Steiner point are antipodal on the circumcircle, we conclude that X_{69} is the Steiner point of triangle $A_\mathrm{g}B_\mathrm{g}C_\mathrm{g}$ (see [1]).

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