Forum Geometricorum
Volume 10 (2010) 55–77.

FORUM GEOM
ISSN 1534-1178

Simple Relations Regarding the Steiner Inellipse of a Triangle

Benedetto Scimemi

Abstract. By applying analytic geometry within a special Cartesian reference, based on the Kiepert hyperbola, we prove a great number of relations (collinearities, similarities, inversions etc.) regarding central points, central lines and central conics of a triangle. Most - not all - of these statements are well-known, but somehow dispersed throughout the literature. Some relations turn out to be easy consequences of the action of a conjugation - an involutory Möbius transformation - whose fixed points are the foci of the Steiner inellipse.

1. Introduction

With the aim of making proofs simpler and more uniform, we applied analytic geometry to revisit a number of theorems regarding the triangle centers. The choice of an intrinsic Cartesian frame, which we call the *Kiepert reference*, turned out to be very effective in dealing with a good part of the standard results on central points and related conics: along with several well-known statements, a number of simple relations which seem to be new have emerged. Here is, perhaps, the most surprising example:

Theorem 1. *Let G, F_+, F_- denote, respectively, the centroid, the first and second Fermat points of a triangle. The major axis of its Steiner inellipse is the inner bisector of the angle $\angle F_+GF_-$. The lengths of the axes are $|GF_-| \pm |GF_+|$, the sum and difference of the distances of the Fermat points from the centroid.*

As a consequence, if $2c$ denotes the focal distance, c is the geometric mean between $|GF_-|$ and $|GF_+|$. This means that F_+ and F_- are interchanged under the action of an involutory Möbius transformation μ, the product of the reflection in the major axis by the inversion in the circle whose diameter is defined by the foci. This conjugation plays an interesting role in the geometry of the triangle. In fact, one easily discovers the existence of many other μ-coupled objects: the isodynamic points; the circumcenter and the focus of the Kiepert parabola; the orthocenter and the center of the Jerabek hyperbola; the Lemoine point and the Parry point; the circumcircle and the Brocard circle; the Brocard axis and the Parry circle, etc. By applying standard properties of homographies, one can then recognize various sets

Publication Date: June 1, 2010. Communicating Editor: Paul Yiu.

of concyclic central points, parallel central lines, orthogonal central circles, similar central triangles etc.

Notation and terminology. If $A,\ B$ are points, AB will indicate both the segment and the line through A, B, $|AB|$ is the length of the segment AB, $\overline{AB}$ is a vector; sometimes we also write $\overline{AB} = B - A$. The angle between $\overline{BA}$ and $\overline{BC}$ is $\angle ABC$. $\overline{AB} \cdot \overline{CD}$ is the scalar product. $A^B = C$ means that a half-turn about B maps A onto C; equivalently, we write $B = \frac{A+C}{2}$.

For the Cartesian coordinates of a point A we write $A = [x_A, y_A]$; for a vector, $\overline{AB}$=$[x_B - x_A, y_B - y_A]$.

In order to identify central points of a triangle $\mathbf{T} = A_1A_2A_3$ we shall use both capital letters and numbers, as listed by Clark Kimberling in [3, 2]; for example, $G = X_2$, $O = X_3$, $H = X_4$, etc.

2. The Kiepert reference

The Kiepert hyperbola $\mathcal{K}$ of a (non equilateral) triangle $\mathbf{T} = A_1A_2A_3$ is the (unique) rectangular hyperbola which is circumscribed to $\mathbf{T}$ and passes through its centroid G. We shall adopt an orthogonal Cartesian reference such that the equation for $\mathcal{K}$ is $xy = 1$. This is always possible unless $\mathcal{K}$ reduces to a pair of perpendicular lines; and this only happens if $\mathbf{T}$ is isosceles, an easy case that we shall treat separately in §14. How to choose between x and y, as well as orientations, will be soon treated. Within this *Kiepert reference*, for the vertices of $\mathbf{T}$ we write $A_1 = [x_1,\ \frac{1}{x_1}]$, $A_2 = [x_2,\ \frac{1}{x_2}]$, $A_3 = [x_3,\ \frac{1}{x_3}]$.

Since central points are symmetric functions of A_1, A_2, A_3, for their coordinates we expect to find symmetric functions of x_1, x_2, x_3 and hopefully algebraic functions of the elementary symmetric polynomials

$$s_1 := x_1 + x_2 + x_3, \quad s_2 := x_1x_2 + x_2x_3 + x_3x_1, \quad s_3 := x_1x_2x_3.$$

This is true for many, but not all of the classical central points. For example, for the centroid G we obviously have

$$G = \left[\frac{1}{3}(x_1 + x_2 + x_3),\ \frac{1}{3}\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right)\right] = \left[\frac{s_1}{3},\ \frac{s_2}{3s_3}\right].$$

However, for points like the incenter or the Feuerbach point we cannot avoid encountering functions like $\sqrt{1 + x_i^2x_j^2}$ which cannot be expressed explicitly in terms of s_1, s_2, s_3. Therefore this paper will only deal with a part of the standard geometry of central points, which nevertheless is of importance. Going back to the centroid, since G, by definition, lies on $\mathcal{K}$, we must have $\dfrac{s_2}{3s_3} = \dfrac{3}{s_1}$, so that $s_2 = \dfrac{9s_3}{s_1}$ can be eliminated and we are only left with functions of s_1, s_3. (Note that, under our assumptions, we always have s_1 and s_3 nonzero).

From now on, it will be understood that this reduction has been made, and we shall write, for short,

$$s_1 = x_1 + x_2 + x_3 = s \quad \text{and} \quad s_3 = x_1x_2x_3 = p.$$

The location of $G = \left[\frac{s}{3},\ \frac{3}{s}\right]$ will determine what was still ambiguous about the reference: without loss of generality, we shall assume that its coordinates are positive: $s > 0$. We claim that this implies $p < 0$. In fact the square of the Vandermonde product $V := (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ is symmetric:

$$V^2 = \frac{-4p(s^3 - 27p)^2}{s^3}.$$

Since we want to deal with proper triangles only, we assume $s^3 - 27p \neq 0$ and therefore $p < 0$, as we wanted. These inequalities will be essential when dealing with square roots as $\sqrt{-sp}$, $\sqrt{\frac{-s}{p}}$ etc., that we want to be (positive) real numbers. In fact, our calculations will take place within the field $F = Q(s, p)$ and its real quadratic extension $F(u)$, where $u = \sqrt{\dfrac{-sp}{3}}$.

As we shall see, the advantage of operating within the Kiepert reference can be summarized as follows: once coordinates and equations have been derived, which may require a moderate amount of accurate geometric and algebraic work, many statements will look evident at a glance, without any computing effort.

3. Central points

The center of the Kiepert hyperbola $\mathcal{K}$ is the origin of our reference: the Kiepert center

$$K = X_{115} = [0, 0],$$

By reflecting the centroid

$$G = X_2 = \left[\frac{s}{3},\ \frac{3}{s}\right] = \frac{1}{3s}[s^2,\ 9]$$

upon K, we obviously find another point of $\mathcal{K}$:

$$G^K = X_{671} = \left[\frac{-s}{3},\ \frac{-3}{s}\right].$$

As the hyperbola is rectangular, we also find on $\mathcal{K}$ the orthocenter

$$H = X_4 = \left[\frac{-1}{p},\ -p\right] = \frac{-1}{p}[1, p^2]$$

and the Tarry point:

$$T = H^K = X_{98} = \left[\frac{1}{p},\ p\right].$$

For the circumcenter we calculate

$$O = X_3 = \left[\frac{1 + sp}{2p},\ \frac{9 + sp}{2s}\right] = \frac{1}{2sp}[s(sp + 1), p(sp + 9)],$$

and check the collinearity of O, G, H and Euler's equation: $\overline{GH} = 2\overline{OG}$. By comparing coordinates, we notice that $sp = -3$ would imply $G = H = O$. Since this only holds for equilateral triangles (a case we have excluded), we may assume $t = sp + 3 \neq 0$. We shall soon find an important interpretation for the sign of t.

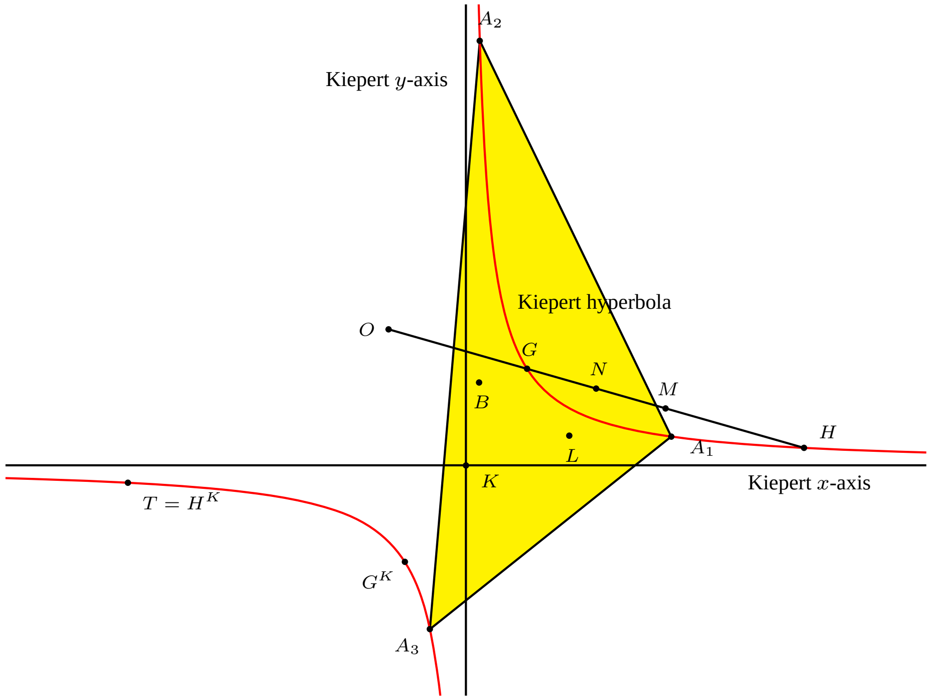

Figure 1.

We also want to calculate the nine-point center N and the center M of the orthocentroidal circle (see Figure 1):

$$N = \frac{O+H}{2} = X_5 = \left[\frac{sp-1}{4p}, \frac{9-sp}{4s}\right] = \frac{1}{-4sp}[s(1-sp), -p(9-sp)],$$
$$M = \frac{G+H}{2} = O^G = X_{381} = \frac{3-sp}{-6sp}[s, -3p].$$

We now want the symmedian or Lemoine point L, the isogonal conjugate of G. To find its coordinates, we can use a definition of isogonal conjugation which is based on reflections: if we reflect G in the three sides of $\mathbf{T}$ and take the circumcenter of the resulting triangle, we find

$$L = X_6 = \frac{2}{3-sp}[s,\ -3p].$$

It appears, at a glance, that K, M, L are collinear. Another central point we want is the Brocard point

$$B = \frac{O+L}{2} = X_{182} = \frac{1}{-4sp(3-sp)}[s(s^2p^2-6sp-3),\ p(s^2p^2+18sp-27)].$$

Notice that G, N, M, H, L (unlike O) always have positive coordinates, and this gives interesting information about their location.

4. Central lines

The central points O, G, N, M, H are collinear on the

Euler line: $$-3px + sy + sp - 3 = 0.$$

The line through M, L, K is known as the

Fermat axis: $$3px + sy = 0.$$

We shall often apply reflections in the xy-axes or parallel lines and map a vector $[X, Y]$ onto $[X, -Y]$ or $[-X, Y]$. For example, looking at coefficients in the equations above, we notice that such a reflection maps $[3p,\ -s]$ onto $\pm[3p,\ s]$. This proves that the asymptotic directions of $\mathcal{K}$ bisect the angles between the Euler line and the Fermat axis.

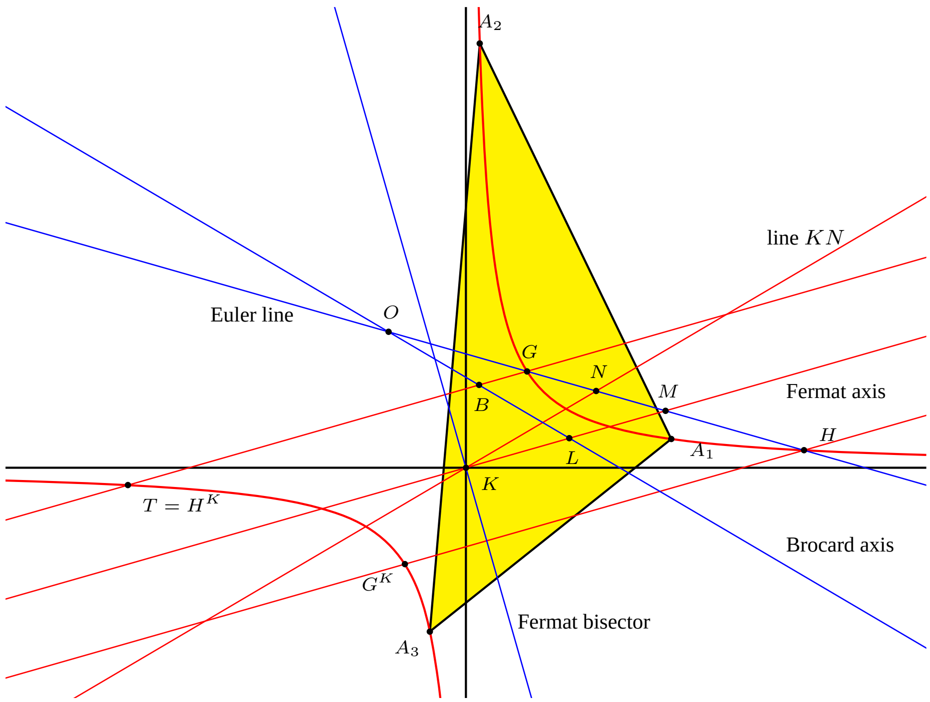

Figure 2.

The line through K, perpendicular to the Fermat axis, will be called the

Fermat bisector: $$sx - 3py = 0.$$

These names clearly anticipate the location of the Fermat points. The line through O, B, L is known as the

Brocard axis: $$-p(9 - sp)x + s(1 - sp)y + 8sp = 0.$$

A comparison with the line

KN : $$p(9 - sp)x + s(1 - sp)y = 0$$

shows that, by the same argument, the asymptotic directions of $\mathcal{K}$ bisect also the angles formed by the Brocard axis and the line KN.

5. Central circles

The circumcircle has equation

$$\mathcal{O}: \qquad x^2+y^2-\frac{sp+1}{p}x-\frac{sp+9}{s}y+\frac{s^2+9p^2}{sp}=0$$

and for the circumradius r_O we find

$$r_O^2=\frac{s^2-2s^3p+81p^2+s^4p^2-18sp^3+s^2p^4}{4s^2p^2}.$$

By direct substitution, it is easy to check that the Tarry point T lies on $\mathcal{O}$. Indeed, T is the fourth intersection (after the three vertices) of $\mathcal{K}$ and $\mathcal{O}$. The antipode of T on $\mathcal{O}$ is the

Steiner point: $$S=T^O=X_{99}=\left[s,\frac{9}{s}\right]=\frac{1}{s}[s^2,9].$$

S is the Kiepert center of the complementary triangle of **T**, and therefore can be equivalently defined by the relation $\overline{GS}=2\overline{KG}$.

The second intersection (after S) of $\mathcal{O}$ with the line GK is the

Parry point: $$P=X_{111}=\frac{s^2+9p^2}{p(s^4+81)}[s^2,9].$$

The nine-point circle, with center in N and radius $r_N=\frac{r_O}{2}$, has equation

$$\mathcal{N}: \qquad x^2+y^2+\frac{1-sp}{2p}x-\frac{9-sp}{2s}y=0,$$

This shows that K lies on $\mathcal{N}$, as expected for the center of a circumscribed rectangular hyperbola. This is also equivalent to stating that S lies on $\mathcal{O}$. Since KN and SO are parallel, the last remark of §4 reads: the angle $\angle SOL$ is bisected by the asymptotic directions of $\mathcal{K}$.

The orthocentroidal circle, centered in M, is defined by its diameter GH and has equation:

$$\mathcal{M}: \qquad x^2+y^2+\frac{3-sp}{3p}x-\frac{3-sp}{s}y-\frac{s^2+9p^2}{3sp}=0.$$

We now introduce a central circle $\mathcal{D}$ whose role in the geometry of the triangle has been perhaps underestimated (although it is mentioned in [2, p.230]). First define a central point D as the intersection of the Fermat bisector with the line through G, normal to the Euler line:

$$D:=\frac{s^3+27p}{18s^2p}[3p,s].$$

Then consider the circle centered in D , passing through G:

$$\mathcal{D}: \qquad x^2+y^2-\frac{s^3+27p}{3s^2}x-\frac{s^3+27p}{9sp}y+\frac{s^2+9p^2}{3sp}=0.$$

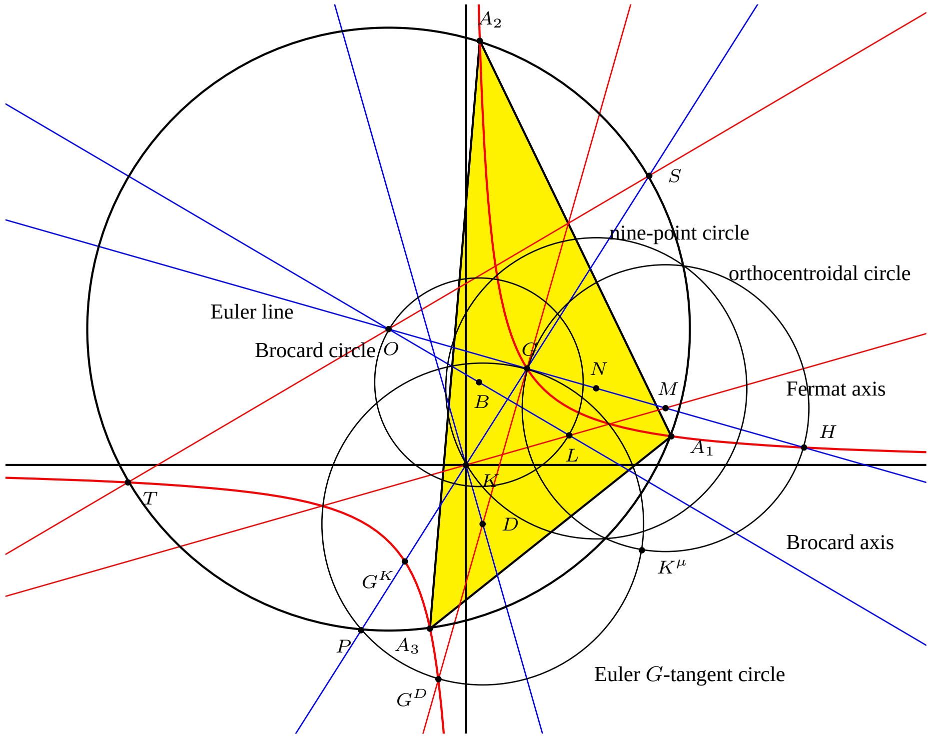

Figure 3.

We shall call $\mathcal{D}$ the Euler G-tangent circle because, by construction, it is tangent to the Euler line in G. This circle will turn out to contain several interesting central points besides G. For example: the Parry point P is the second intersection (after G) of $\mathcal{D}$ with the line GK; the antipode of G on $\mathcal{D}$ is

$$G^D = \left[\frac{9p}{s^2}, \frac{s^2}{9p}\right]$$

which is clearly a point of $\mathcal{K}$. Another point on $\mathcal{D}$ that we shall meet later is the reflection of P in the Fermat bisector:

$$K^\mu = \frac{1}{p(s^4+81)}[-s(s^3-54p-9sp^2),\ 3(3s^2+2s^3p-27p^2)].$$

This point (the symbol K^μ will be clear later) is collinear with G and L. In fact K^μ is the second intersection (after G) of the circles $\mathcal{M}$ and $\mathcal{D}$, whose radical axis is therefore the line GL.

Most importantly, the Fermat points will also be shown to lie on $\mathcal{D}$.

Lastly, let us consider the *Brocard circle* $\mathcal{B}$, centered at B and defined by its diameter OL; rather than writing down its equation, we shall just calculate its radius r_B. The resulting formula looks rather complicated:

$$r_B^2 = \frac{1}{4}\overline{OL}\cdot\overline{OL} = \frac{(sp+3)^2(s^2-2s^3p+81p^2+s^4p^2-18sp^3+s^2p^4)}{16(3-sp)^2s^2p^2}.$$

But we notice the appearance of the same polynomial which we have found for the circumradius. In fact, we have a very simple ratio of the radii of the Brocard and the nine-point circles:

$$\frac{r_B}{r_N} = \left|\frac{3+sp}{3-sp}\right|.$$

We shall find a surprisingly simple geometrical meaning of this ratio in the next section.

6. The Steiner inellipse

The Steiner inellipse $\mathcal{S}$ of a triangle $\mathbf{T}$ is the unique conic section which is centered at G and tangent to the sides of $\mathbf{T}$. From this definition one calculates the equation:

$$\mathcal{S}: \qquad 3x^2 - spy^2 - 2sx + 6py = 0.$$

Since the term in xy is missing, the axes of $\mathcal{S}$ (the Steiner axes) are parallel to the asymptotes of the Kiepert hyperbola $\mathcal{K}$. Just looking at the equation, we also notice that K lies on $\mathcal{S}$ and the line tangent to $\mathcal{S}$ at K is parallel to the Fermat bisector (see Figure 4).

By introducing the traditional parameters a, b for the lengths of the semi-axes, the equation for $\mathcal{S}$ can be rewritten as

$$\frac{(x-\frac{s}{3})^2}{a^2} + \frac{(y-\frac{3}{s})^2}{b^2} = 1,$$

where $a^2 = \frac{s^3-27p}{9s}$ and $b^2 = \frac{s^3-27p}{-3s^2p}$.

We cannot distinguish between the major and the minor axis unless we take into account the sign of $a^2 - b^2 = \frac{(s^3-27p)(sp+3)}{9s^2p}$. This gives a meaning to the sign of $t = sp + 3$, with respect to our reference. In fact we must distinguish two cases:
Case 1: $t < 0$, $a > b$: the major Steiner axis is parallel to the x-axis.
Case 2: $t > 0$, $a < b$: the major Steiner axis is parallel to the y-axis.

Notice that the possibility that $\mathcal{S}$ is a circle ($t = 0, a = b$) has been excluded, as the triangle $\mathbf{T}$ would be equilateral.

This reduction to cases will appear frequently. For example, we can use a single formula $2c = 2\sqrt{|a^2-b^2|}$ for the focal distance, but for the foci U_+, U_- we must write, respectively,

$$U_\pm = \begin{cases} \frac{1}{3sp}[s^2p \pm \sqrt{p(3+sp)(s^3-27p)},\ 9p], & \text{if } sp+3<0, \\ \frac{1}{3sp}[s^2p,\ 9p \pm \sqrt{-p(3+sp)(s^3-27p)}], & \text{if } sp+3>0. \end{cases}$$

The number $u = \sqrt{\frac{-sp}{3}} = \frac{a}{b}$ is the tangent of an angle $\frac{\alpha}{2}$ which measures the eccentricity e of $\mathcal{S}$. Notice, however, that either $e^2 = 1 - u^{-2}$ or $e^2 = 1 - u^2$ according as $sp + 3 < 0$ or > 0. What we do not expect is for the number

$$|\cos\alpha| = \frac{|1-u^2|}{1+u^2} = \frac{|a^2-b^2|}{a^2+b^2} = \left|\frac{3+sp}{3-sp}\right|$$

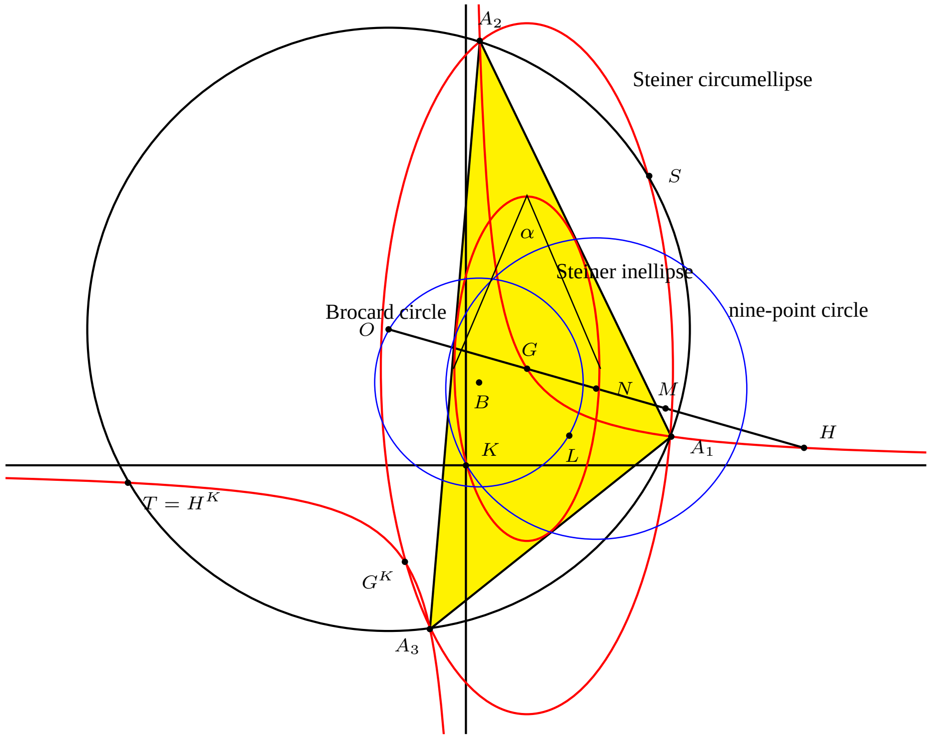

Figure 4.

to be precisely the ratio of the radii that we have found at the end of §5. Taking into account the meaning of $t = sp + 3$ we conclude that, in any case,

Theorem 2. *The ratio between the radii of the Brocard circle and the nine-point circle equals the cosine of the angle under which the minor axis of the Steiner ellipse is viewed from an extreme of the major axis:* $\frac{r_B}{r_N} = |\cos\alpha|$.

By applying a homothety of coefficient 2 and fixed point G, the Steiner inellipse $\mathcal{S}$ is transformed into the Steiner circumellipse. This conic is in fact circumscribed to $\mathbf{T}$ and passes through the Steiner point S, which is therefore the fourth intersection (after the triangle vertices) of the Steiner circumellipse with the circumcircle $\mathcal{O}$. The fourth intersection with $\mathcal{K}$ is $S^G = G^K = X_{671}$.

7. The Kiepert parabola and its focus

The Kiepert parabola of a triangle $\mathbf{T}$ is the (unique) parabola $\mathcal{P}$ which is tangent to the sides of the triangle, and has the Euler line as directrix. By applying this definition one finds for $\mathcal{P}$ a rather complicated equation:

$$\mathcal{P}: \; s^2\left(s\left(x+\frac{s}{3}\right)+3p\left(y+\frac{3}{s}\right)\right)^2+\frac{8}{9}((s^3-27p)^2-3s(s^4x+81p^2y))=0$$

or

$$s^4x^2+9s^2p^2y^2+6s^3pxy-2s^2x(s^3-9p)+2spy((s^3-81p)+s^6-42s^3p+729p^3=0.$$

From this formula one can check that the tangency points E_i (on the sides A_jA_h of $\mathbf{T}$) and the vertices A_i are perspective; the center of perspective (sometimes called the Brianchon point) is the Steiner point $S = X_{99}$. Less well-known, but not difficult to prove, is the fact that the orthocenter of the Steiner triangle $E_1E_2E_3$ is O, the circumcenter of $\mathbf{T}$.

Direct calculations show that the focus of $\mathcal{P}$ is

$$E = X_{110} = \frac{1}{s(s^2+9p^2)}[s(s^3-18p+3sp^2), 3s^2-2s^3p+81p^2].$$

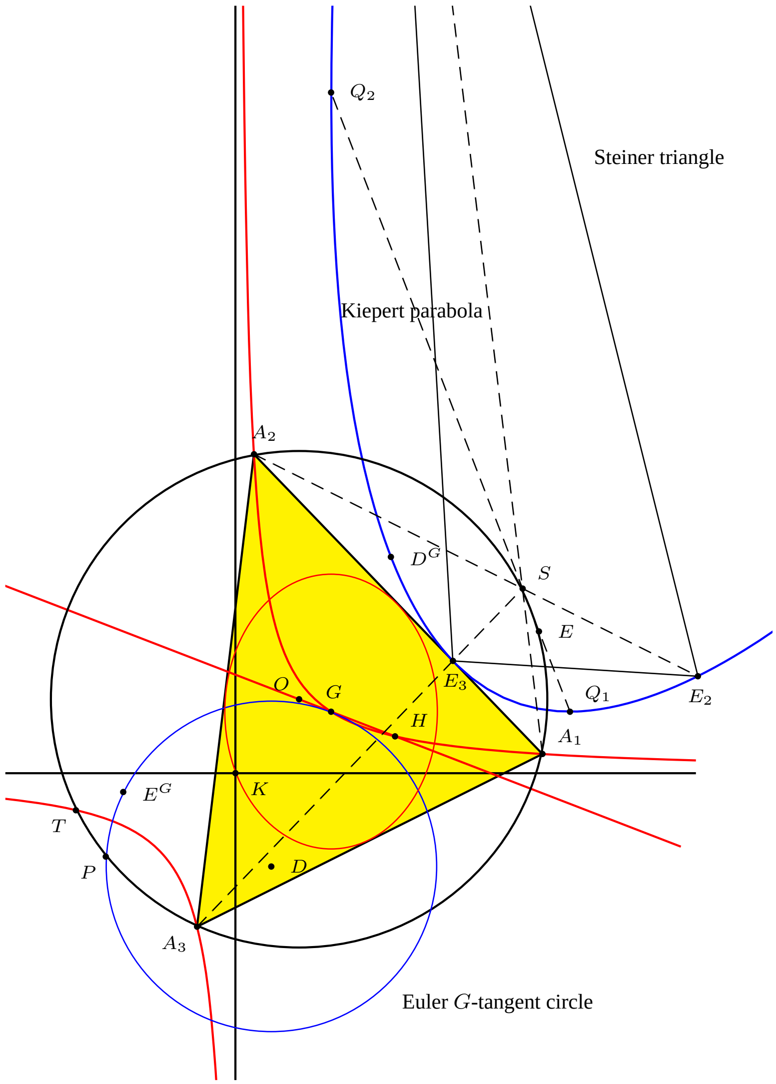

Figure 5.

One can verify that E is a point of the circumcircle $\mathcal{O}$; this also follows from the well-known fact that, when reflecting the Euler line in the sides of the triangle, these three lines intersect at E. Thus E is the isogonal conjugate of the point at infinity, normal to the Euler line. Further calculations show that G, E, T are collinear on the line

GE : $$3px + sy - 3sp - 3 = 0$$

which is clearly parallel to the Fermat axis. Another well-known collinearity regards the points E, L, and P. The proof requires less easy calculations and the equation for this line will not be reported. On the other hand, the line

ES : $$sx - 3py - \frac{s^3 - 27p}{s} = 0$$

is parallel to the Fermat bisector. This line meets $\mathcal{P}$ at the points

$$Q_1 = \left[\frac{s^3 - 18p}{s^2}, \frac{3}{s}\right] \quad \text{and} \quad Q_2 = \left[\frac{s}{3}, \frac{81p - 2s^3}{9sp}\right],$$

each of which lies on a Steiner axis. When substituting the values $y = \frac{3}{s}$ or $x = \frac{s}{3}$ in the equation of $\mathcal{P}$ one discovers a property that we have not found in literature:

Theorem 3. *The axes of the Steiner ellipse of a triangle are tangent to its Kiepert parabola. The tangency points are collinear with the focus and the Steiner point* (see Figure 5).

As a consequence, the images of E under reflections in the Steiner axes both lie on the Euler line. The relatively poor list of central points which are known to lie on $\mathcal{P}$ may be enriched, besides by Q_1 and Q_2, by the addition of

$$D^G = \frac{1}{-18s^2p}[-9p(s^3 - 9p),\ s(s^3 - 81p)].$$

The tangent to $\mathcal{P}$ at D^G is the perpendicular bisector of GE. We recall that D was defined in §5 as the center of the G-tangent circle $\mathcal{D}$. The close relation between $\mathcal{P}$ and $\mathcal{D}$ is confirmed by the fact that, somehow symmetrically, the point

$$E^G = \frac{1}{3s(s^2 + 9p^2)}[-s(s^3 - 54p - 9sp^2), 3(3s^2 + 2s^3p - 27p^2)]$$

lies on $\mathcal{D}$.

8. Reflections and angle bisectors

In what follows we shall make frequent use of reflections of vectors in lines parallel to the xy-axes and write the new coordinates by just changing signs, as explained in §4. Consider, for example,

$$\overline{MG} = G - M = [\frac{s}{3}, \frac{3}{s}] - [\frac{sp - 3}{6p}, -\frac{sp - 3}{2s}] = \frac{sp + 3}{6sp}[s, 3p]$$

and its reflection in the x-axis: $\frac{sp+3}{6sp}[s, -3p]$. By comparing coordinates, we see that the latter is parallel to the vector

$$\overline{ML} = L - M = [\frac{2s}{3 - sp}, -\frac{6p}{3 - sp}] - [\frac{sp - 3}{6p}, -\frac{sp - 3}{2s}] = \frac{(3 + sp)^2}{6sp(3 - sp)}[s, -3p].$$

Note that orientations depend on the sign of the factor $t = 3 + sp$. But we are aware of the meaning of this sign (compare §6) and therefore we know that, in any case, a reflection in the minor Steiner axis maps $\overline{MG}$ into a vector which is parallel and has the same orientation as $\overline{ML}$.

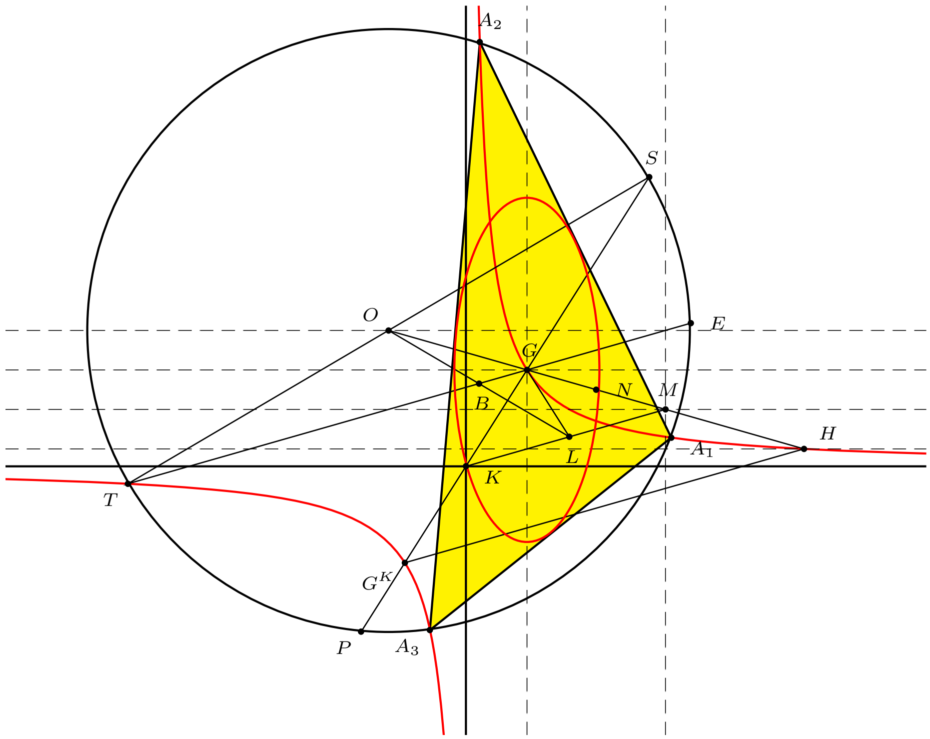

Figure 6.

If we apply the same argument to other pairs of vectors, as

$$\overline{GE} = \frac{2(s^3 - 27p)}{3s(s^2 + 9p^2)}[s,\ -3p],$$

$$\overline{GO} = \frac{3 + sp}{6sp}[-s,\ -3p],$$

$$\overline{OL} = \frac{3 + sp}{-2sp(3 - sp)}[s(1 - sp),\ p(9 - sp)],$$

$$\overline{OS} = \frac{1}{-2sp}[s(1 - sp),\ -p(9 - sp)],$$

we can conclude similarly:

Theorem 4. *The inner bisectors of the angles $\angle GML$ and $\angle SOL$ are parallel to the minor Steiner axis. The inner bisector of $\angle EGO$ is the major Steiner axis* (see Figure 6).

These are refinements of well-known statements. Similar results regarding other angles will appear later.

9. The Fermat points

We now turn our attention to the Fermat points. For their coordinates we cannot expect to find symmetric polynomials in x_1, x_2, x_3, as these points are interchanged by an odd permutation of the triangle vertices. In fact, by applying the traditional constructions (through equilateral triangles constructed on the sides of $\mathbf{T}$) one ends up with the twin points

$$\frac{s \cdot V}{2\sqrt{3}p(s^3 - 27p)}\left[\frac{s}{3}, \frac{-1}{p}\right] \quad \text{and} \quad \frac{s \cdot V}{2\sqrt{3}p(s^3 - 27p)}\left[-\frac{s}{3}, \frac{1}{p}\right],$$

where V is the Vandermonde determinant (see §2). We cannot yet tell which is which, but we already see that they both lie on the Fermat axis and their midpoint is the Kiepert center K. Less obvious, but easy to check analytically, is the fact that both the Fermat points lie on the G-tangent circle $\mathcal{D}$. (Incidentally, this permits a non traditional construction of the Fermat points from the central points G, O, K via M). By squaring and substituting for V^2, the expressions above can be rewritten as

$$F_+ = \left[\sqrt{\frac{-s}{3p}}, \sqrt{\frac{-3p}{s}}\right], \quad F_- = \left[-\sqrt{\frac{-s}{3p}}, -\sqrt{\frac{-3p}{s}}\right].$$

This shows that F_+ and F_- belong to the Kiepert hyperbola $\mathcal{K}$ (see Figure 7).

In the next formulas we want to avoid the symbol $\sqrt{}$ and use instead the (positive) parameter $u = \sqrt{\frac{-sp}{3}} = \frac{a}{b}$, which was introduced in connection with the Steiner inellipse in §6. We know that $u \neq 1$. Moreover, $u > 1$ or < 1 according as $sp + 3 < 0$ or > 0. We now want to distinguish between the two Fermat points and claim that

$$F_+ = -\frac{u}{sp}[s, -3p], \quad \text{and} \quad F_- = -\frac{u}{sp}[-s, 3p].$$

Note that F_+ and F_- are always in the first and third quadrants respectively. This follows by applying the distance inequality $|GX_{13}| < |GX_{14}|$, a consequence of their traditional definitions, and only checking the inequality:

$$|GF_-|^2 - |GF_+|^2 = \frac{4u(s^3 - 27p)}{3s^2p^2} > 0.$$

Let us now apply the reflection argument, as described in section 8, to the vectors $\overline{GF_+}$ and $\overline{GF_-}$. We claim that the major Steiner axis is the inner bisector of $\angle F_+GF_-$. We shall show, equivalently, that the reflection τ in the major axis maps $\overline{GF_+}$ onto a vector $\overline{GF_+^\tau}$ that has the same direction and orientation as $\overline{GF_-}$. Again, we must treat two cases separately.

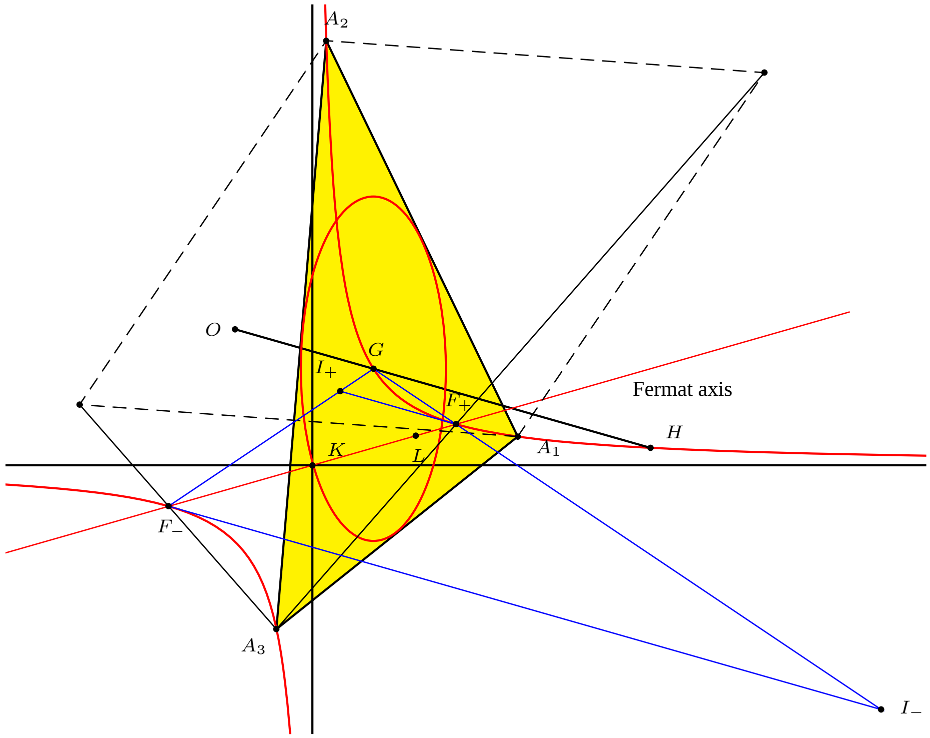

Figure 7.

Case 1: $a < b$. In this case $u < 1$. The major Steiner axis is parallel to the y-axis.

$$\overline{GF_+} = F_+ - G = \left[\frac{-u}{p} - \frac{s}{3}, \frac{3u}{s} - \frac{3}{s}\right],$$
$$\overline{GF_+^\tau} = \left[\frac{u}{p} + \frac{s}{3}, \frac{3u}{s} - \frac{3}{s}\right];$$
$$\overline{GF_-} = \left[\frac{u}{p} - \frac{s}{3}, \frac{-3u}{s} - \frac{3}{s}\right].$$

We now calculate both the vector and the scalar products of the last two vectors:

$$-(\frac{u}{p}+\frac{s}{3})(\frac{-3u}{s}-\frac{3}{s})+(\frac{3u}{s}-\frac{3}{s})(\frac{u}{p}-\frac{s}{3}) = \frac{3u^2}{sp}+\frac{3u}{sp}+u+1+\frac{3u^2}{sp}-u-\frac{3u}{sp}+1=0,$$
$$(\frac{u}{p}+\frac{s}{3})(\frac{u}{p}-\frac{s}{3})+(\frac{3u}{s}-\frac{3}{s})(\frac{-3u}{s}-\frac{3}{s}) = \frac{u^2}{p^2}-\frac{s^2}{9}+\frac{9}{s^2}-\frac{9u^2}{s^2} = \frac{(s^3-27p)(sp+3)}{9s^2p} > 0.$$

Since the last fraction equals $b^2 - a^2 > 0$, this is what we wanted. Moreover, since the vectors $\overline{GF_+}^\tau$ and $\overline{GF_-}$ share both directions and orientations, we can easily calculate absolute values as follows

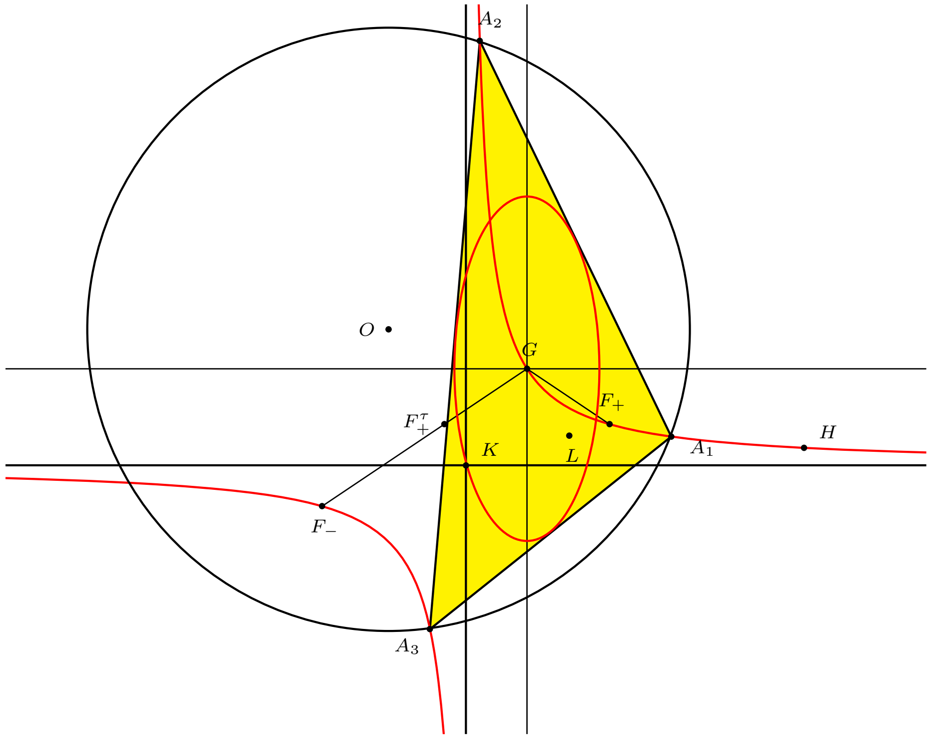

Figure 8.

$$\begin{aligned}
|GF_+|+|GF_-| =& \ |\overline{GF_+^\tau}|+|\overline{GF_-}| = |\overline{GF_+^\tau}+\overline{GF_-}| \\
=& \ \left|\left[\frac{u}{p}+\frac{s}{3},\ \frac{3u}{s}-\frac{3}{s}\right]+\left[\frac{u}{p}-\frac{s}{3},\ \frac{-3u}{s}-\frac{3}{s}\right]\right| = \left|\left[\frac{2u}{p},\ \frac{-6}{s}\right]\right| \\
=& \ \sqrt{\frac{4u^2}{p^2}+\frac{36}{s^2}} = 2\sqrt{-\frac{s^3-27p}{3s^2p}} = 2b; \\
|GF_+|-|GF_-| =& \ |\overline{GF_+^\tau}|-|\overline{GF_-}| = |\overline{GF_+^\tau}-\overline{GF_-}| \\
=& \ \left|\left[\frac{u}{p}+\frac{s}{3},\ \frac{3u}{s}-\frac{3}{s}\right]-\left[\frac{u}{p}-\frac{s}{3},\ \frac{-3u}{s}-\frac{3}{s}\right]\right| = \left|\left[\frac{2s}{3},\ \frac{6u}{s}\right]\right| \\
=& \ \sqrt{\frac{4s^2}{9}+\frac{36u^2}{s^2}} = 2\sqrt{\frac{s^3-27p}{9s}} = 2a.
\end{aligned}$$

The last computation could be spared by deriving the difference from the sum and the product

$$|GF_+||GF_-| = \overline{GF_+^\tau}\cdot\overline{GF_-} = \frac{(s^3-27p)(3+sp)}{-9s^2p} = b^2-a^2.$$

Case 2: $a > b$. In this case $u > 1$. The major Steiner axis is parallel to the x-axis and $\overline{GF_+^\tau} = [\frac{-u}{p} - \frac{s}{3},\ \frac{-3u}{s} + \frac{3}{s}]$. Taking products of $\overline{GF_+^\tau}$ and $\overline{GF_-}$ leads to similar expressions: the vector product vanishes, while the scalar product equals $a^2 - b^2 > 0$. As for absolute values, the results are $|GF_+| + |GF_-| = 2a$ and $|GF_-| - |GF_+| = 2b$, and $|GF_+||GF_-| = a^2 - b^2$.

All these results together prove Theorem 1.

10. Relations regarding areas

There are some well-known relations regarding areas, which could somehow anticipate the close relation between the Steiner ellipse and the Fermat point, as described in Theorem 1.

The area Δ of the triangle $\mathbf{T} = A_1A_2A_3$ can be calculated from the coordinates of the vertices $A_i = \left[x_i, \frac{1}{x_i}\right]$ as a determinant which reduces to Vandermonde (see §2):

$$\Delta = \left|\frac{V}{2p}\right| = \frac{1}{2}\sqrt{\frac{(s^3 - 27p)^2}{-s^3p}}.$$

If we compare this area with that of the Steiner inellipse $\mathcal{S}$, we find that the ratio is invariant:

$$\Delta(\mathcal{S}) = \pi ab = \pi\sqrt{\frac{(s^3 - 27p)}{9s}} \cdot \sqrt{\frac{(s^3 - 27p)}{-s^2p}} = \frac{\pi}{3\sqrt{3}}\sqrt{\frac{V^2}{4p^2}} = \frac{\pi}{3\sqrt{3}}\Delta.$$

There actually exists a more elegant argument to prove this result, based on invariance under affine transformations.

Another famous area relation has to do with the Napoleon triangles $\mathbf{N}_{\text{ap}+}$ and $\mathbf{N}_{\text{ap}-}$. It is well-known that these equilateral triangles are both centered in G and their circumcircles pass through F_- and F_+ respectively. Their areas are easily calculated in terms of their radius:

$$\Delta(\mathbf{N}_{\text{ap}+}) = \frac{3\sqrt{3}}{4}|GF_-|^2, \qquad \Delta(\mathbf{N}_{\text{ap}-}) = \frac{3\sqrt{3}}{4}|GF_+|^2.$$

The difference turns out to be precisely the area of $\mathbf{T}$:

$$\Delta(\mathbf{N}_{\text{ap}+}) - \Delta(\mathbf{N}_{\text{ap}-}) = \frac{3\sqrt{3}}{4}(|GF_-|^2 - |GF_+|^2) = 3\sqrt{3}ab = \sqrt{\frac{V^2}{4p^2}} = \Delta.$$

11. An involutory Möbius transformation

Let U_+ and U_- be the foci of the Steiner inellipse. The focal distance is

$$|U_+U_-| = 2c = 2\sqrt{|a^2 - b^2|} = 2\sqrt{|GF_+||GF_-|}.$$

If we introduce the circle $\mathcal{U}$, centered at G, with U_+U_- as diameter, we know from §9 that the reflection τ in the major axis maps $\overline{GF_+}$ onto $\overline{GF_+^\tau}$, a vector which has the same direction and orientation as $\overline{GF_-}$. Furthermore, we know from Theorem 1 that c is the geometric mean between $|GF_+|$ and $|GF_-|$. This means that the inversion in the circle $\mathcal{U}$ maps $\tau(F_+)$ onto F_-. Thus F_- is the inverse in $\mathcal{U}$ of F_+^τ, the reflection of F_+ in the major axis. We shall denote by μ the composite of the reflection τ in the major axis of $\mathcal{S}$ and the inversion in the circle $\mathcal{U}$ whose diameter is given by the foci of $\mathcal{S}$. Note that this composite is independent of the order of the reflection and the inversion.

The mapping μ is clearly an involutory Möbius transformation. Its fixed points are the foci, its fixed lines are the Steiner axes. The properties of the mapping μ become evident after introducing in the plane a complex coordinate z such that the foci are the points $z = 1, z = -1$. Then μ is the complex inversion: $\mu(z) = \frac{1}{z}$. What we have proved so far is that μ interchanges the Fermat points. But μ acts similarly on other pairs of central points. For example, if we go back to the end of §8, we realize that we have partially proved that μ interchanges the circumcenter O with the focus E of the Kiepert parabola; what we still miss is the equality $|GE||GO| = c^2$, which only requires a routine check. In order to describe more examples, let us consider the isodynamic points I_+ and I_-, namely, the isogonal conjugates of F_+ and F_- respectively. A straightforward calculation gives for these points:

$$I_+ = \frac{1}{sp(sp+3)}[4s^2p + 3us(1-sp),\ 12sp^2 + 3up(9-sp)],$$
$$I_- = \frac{1}{sp(sp+3)}[4s^2p - 3us(1-sp),\ 12sp^2 - 3up(9-sp)].$$

We claim that μ interchanges I_+ and I_-. One can proceed as before: discuss the cases $u > 1$ and $u < 1$ separately, reflect $\overline{GI_+}$ to get $\overline{GI_+^\tau}$, then calculate the vanishing of the vector product of $\overline{GI_+^\tau}$ and $\overline{GI_-}$, and finally check that the scalar product is $|GI_+||GI_-| = c^2$. In the present case, however, one may use an alternative argument. In fact, from the above formulas it is possible to derive several well-known properties such as:
(i) I_+ and I_- both lie on the Brocard axis;
(ii) the lines F_+I_+ and F_-I_- are both parallel to the Euler line;
(iii) G, F_+, I_- are collinear;
(iv) G, F_-, I_+ are collinear.

These statements imply, in particular, that there is a homothety which has G as a fixed point and maps F_+ onto I_-, F_- onto I_+ (see Figure 7). Combined with what we know about the Fermat points, this proves that μ interchanges I_+ and I_-, as we wanted.

The next theorem gives a list of μ-conjugated objects.

Theorem 5. *The mapping μ interchanges the following pairs of central points*

circumcenter $O = X_3$	Focus of Kiepert parabola $E = X_{110}$
orthocenter $H = X_4$	Jerabek center $J = X_{125}$
Lemoine point $L = X_6$	Parry point $P = X_{111}$
Fermat point $F_+ = X_{13}$	Fermat point— $F_- = X_{14}$
isodynamic point $I_+ = X_{15}$	isodynamic point $I_- = X_{16}$
center of Brocard circle X_{182}	inverse of centroid in circumcircle $U = X_{23}$

and the following pairs of central lines and circles

Euler line: G, O, H, U, T^μ	line: G, E, J, B, T
Fermat axis: *M, L, K,* F_+, F_-	Euler G-tangent circle: $G,\ P,\ E^G = G^J,\ K^\mu,\ F_-,\ F_+$
circumcircle: *E, P, S, T*	Brocard circle: *O, L,* S^μ, T^μ
Fermat bisector: *K, D, J*	orthocentroidal circle: *G,* K^μ, D^μ, *H*
Brocard axis: *O, B, L,* I_+, I_-	Parry circle: *G, E, U, P,* I_-, I_+
line: $D^\mu,\ K^\mu,\ M^\mu = E^G,\ K$	circle: $G,\ D,\ K,\ M,\ K^\mu$

All the μ-coupling of points can be proved through the argument of §8. The μ-coupling of lines and circles follows from properties of Möbius transformations. Some central points mentioned in Theorem 5 are not listed in [3] but can be rather naturally characterized:
(i) K^μ (whose coordinates have been calculated in §5) is the reflection of P in the Fermat bisector and also the second intersection (after G) of $\mathcal{M}$ and $\mathcal{D}$;
(ii) T^μ is the second intersection (after O) of the Euler line with the Brocard circle;
(iii) S^μ is the second intersection (after L) of the line GL with the Brocard circle;
(iv) D^μ is the reflection of G in the Fermat axis;
(v) M^μ =E^G= G^J.

Further well-known properties of homographies can be usefully applied, such as the conservation of orthogonality between lines or circles and the *reflection principle*: if a point P is reflected (inverted) onto Q in a line (circle) $\mathcal{L}$, then P^μ is reflected (inverted) onto Q^μ by the line (circle) $\mathcal{L}^\mu$. A great number of statements are therefore automatically proved. Here are some examples:
(i) inversion in the orthocentroidal circle interchanges the Fermat points; it also interchanges L and K;
(ii) inversion in the Brocard circle interchanges the isodynamic points;
(iii) $\mathcal{M}$ and $\mathcal{D}$ are orthogonal;
(iv) the Parry circle is orthogonal to both the circumcircle and the Brocard circle, etc.

These statements are surely present in literature but not so easily found.

Among relations which have probably passed unnoticed, we mention equalities of angles, deriving from similarities which also follow from general properties of homographies. Consider any two pairs of μ-coupled points, say $Z_+ \leftrightarrow Z_-$, $W^+ \leftrightarrow W^-$. Then there exists a direct similarity which fixes G and simultaneously transforms Z_+ onto W^- and W^+ onto Z_-. As a consequence, we are able to recognize a great number of direct similarities (dilative rotations around G) between triangles, such as the following pairs:

$$(GOF_-,\ GF_+E),\quad (GEF_-,\ GF_+O),\quad (GOI_-,\ GI_+E),$$
$$(GOI_+,\ GI_-E),\quad (GEL,\ GPO).$$

A special case regards the Steiner foci, which are fixed under the action of μ. In fact, any pair of μ-corresponding points, say Z_+, Z_-, belong to a circle passing through the foci U_+, U_-. This cyclic quadrangle $U_+Z_+U_-Z_-$ is therefore split into two pairs of directly similar triangles having G as a common vertex:

$GZ_+U_- \leftrightarrow GU_-Z_+$, $GZ_+U_+ \leftrightarrow GU_-Z_-$. All these similarities can be read in terms of geometric means.

12. Construction of the Steiner foci and a proof of Marden's theorem

Conversely, having at disposal the centroid G and any pair of μ-corresponding points, say Z_+, Z_-, the Steiner foci U_+, U_- can be easily constructed (by ruler and compass) through the following simple steps:

(1) Construct the major and minor Steiner axes, as inner and outer bisectors of $\angle Z_+GZ_-$.

(2) Construct the perpendicular bisector of Z_+Z_-.

(3) Find the intersection W of the line in (2) with the minor axis.

(4) Construct the circle centered in W, passing through Z_+, Z_-.

The foci U_+ and U_- are the intersections of the circle in (4) with the major axis (see Figure 9).

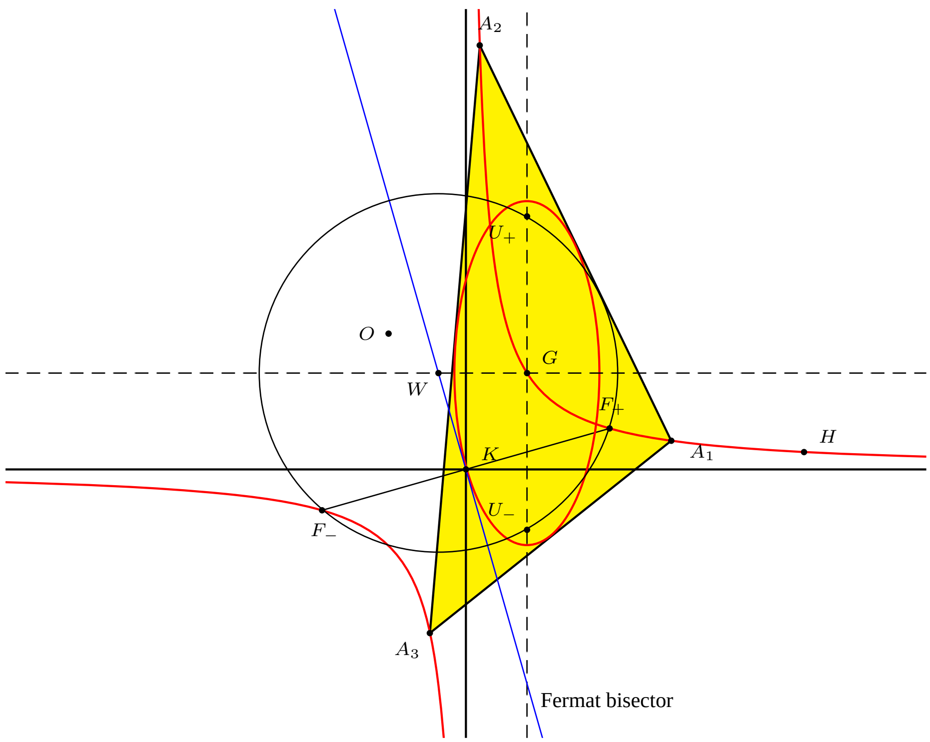

Figure 9.

Avoiding all sorts of calculations, a short synthetic proof of this construction relies on considering the reflection of Z_+ in the major axis and the power of G with respect to the circle in (4). In particular, by choosing the Fermat points for Z_+ and Z_-, then we obtain

Theorem 6. *The foci of the Steiner inellipse of a triangle are the intersections of the major axis and the circle through the Fermat points and with center on the minor axis.*

A direct analytic proof of this statement is achieved by considering, as usual, separate cases as shown below:

Case 1: $a > b$. $W = [\frac{s}{3},\ \frac{s^2}{9p}]$. The circle in (4) has equation

$$x^2 + y^2 - x\frac{2s}{3} - y\frac{2s^2}{9p} + \frac{s^2 + 9p^2}{3sp} = 0,$$

and intersects the line $y = \frac{3}{s}$. As expected, we find the foci

$$U_\pm = \left[\frac{s}{3} \pm c,\ \frac{3}{s}\right],$$

where $c = \sqrt{a^2 - b^2}$.

Case 2: $a < b$. $W = [\frac{9p}{s^2},\ \frac{3}{p}]$. The circle in (4) has equation

$$x^2 + y^2 - x\frac{18p}{s^2} - y\frac{6}{s} + \frac{s^2 + 9p^2}{3sp} = 0$$

and intersects the line $x = \frac{s}{3}$. The foci are the points

$$U_\pm = \left[\frac{s}{3},\ \frac{3}{s} \pm c\right],$$

where $c = \sqrt{b^2 - a^2}$.

Regarding the Steiner foci, we mention a beautiful result often referred to as Marden's Theorem. If one adopts complex coordinates $z = x + iy$, the following curious property was proved by J. Sieberg in 1864 (for this reference and a different proof, see [1]; also [4]). Assume the triangle vertices are $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$, and let $F(z) = (z - z_1)(z - z_2)(z - z_3)$. Then the foci of the Steiner inellipse are the zeros of the derivative $F'(z)$.

We shall give a short proof of this statement by applying our Kiepert coordinates. Write

$$\begin{aligned} F(z) &= z^3 - \sigma_1 z^2 + \sigma_2 z - \sigma_3, \\ F'(z) &= 3z^2 - 2\sigma_1 z + \sigma_2, \end{aligned}$$

where

$$\begin{aligned} \sigma_1 &= z_1 + z_2 + z_3 = s_1 + i\frac{s_2}{s_3} = s + i\frac{9}{s}, \\ \sigma_2 &= z_1 z_2 + z_2 z_3 + z_3 z_1 = s_2 - \frac{s_1}{s_3} + 6i = \frac{9p^2 - s^2}{sp} + 6i. \end{aligned}$$

Again we have two cases. Assuming, for example, $a > b$, we have found for the Steiner foci $u^{\pm} = \pm c + \frac{s}{3} + i \cdot \frac{3}{s}$, where $c = \sqrt{a^2 - b^2}$. Therefore what we have to check is just

$$\begin{aligned} u_+ + u_- &= 2\frac{s}{3} + i\frac{6}{s} = \frac{2\sigma_1}{3}, \\ u_+ u_- &= (c + \frac{s}{3})(-c + \frac{s}{3}) - \frac{9}{s^2} + i\frac{3}{s} \cdot \frac{2s}{3} \\ &= \frac{s^2}{9} + 2i - \frac{9}{s^2} - \frac{(s^3 - 27p)(3 + sp)}{9s^2 p} = \frac{\sigma_2}{3}. \end{aligned}$$

When $a < b$, the foci are $u_{\pm} = \frac{s}{3} + i\left(\pm c + \frac{3}{s}\right)$, where $c = \sqrt{b^2 - a^2}$ and the values of $u_+ + u_-$ and $u_+ u_-$ turn out again to be what we wanted.

13. Further developments and possible obstacles

There are many other central points, lines and conics that can be conveniently treated analytically within the Kiepert reference, leading to coordinates and coefficients which still belong to the field $F = Q(s, p)$ or its quadratic extensions. This is the case, for example, for the Napoleon points X_{17} and X_{18} which are proved to lie on $\mathcal{K}$ and be collinear with L. Incidentally, this line

$$p(27 + sp)x - s(1 + 3sp)y - 16sp = 0$$

turns out to be the radical axis of the orthocentroidal circle and the Lester circle. The Jarabek hyperbola, centered at $J = X_{125}$, can also be treated within the Kiepert reference. These points, lines and conics, however, produce relatively complicated formulas. On the other hand, they do not seem to be strictly connected with the involution μ, whose action is the main subject of the present paper.

As we said in the Introduction, the Kiepert reference may be inconvenient in dealing with many other problems regarding central points: serious difficulties arise if one tries to treat the incenter, the excenters, the Gergonne and Nagel points, the Feuerbach points and, more generally, any central point whose definition involves the angle bisectors. The main obstacle is the fact that the corresponding coordinates are no longer elements of the fields $Q(s, q)$ nor of a quadratic extension. Typically, for this family of points one encounters rational functions of $\sqrt{1 + x_1^2 x_2^2}$, $\sqrt{1 + x_2^2 x_3^2}$, $\sqrt{1 + x_3^2 x_1^2}$, which can hardly be reduced to the basic parameters $s = s_1, p = s_3$.

14. Isosceles triangles

In all of the foregoing sections we have left out the possibility that the triangle $\mathbf{T}$ is isosceles, in which case the Kiepert hyperbola $\mathcal{K}$ degenerates into a pair of orthogonal lines and cannot be represented by the equation $xy = 1$. However, unless the triangle is equilateral - an irrelevant case - all results remain true, although most become trivial. To prove such results, instead of the Kiepert reference, one makes use of a Cartesian reference in which the vertices have coordinates

$$A_1 = [-1,\ 0], \qquad A_2 = [1,\ 0], \qquad A_3 = [0,\ h]$$

and assume $0 < h \neq \sqrt{3}$. Thanks to symmetry, all central points turn out to lie on the y-axis. Here are some examples.

$$\begin{array}{ll}
G = [0,\ \frac{h}{3}], & O = [0, \frac{h^2-1}{2h}], \\
H = [0, \frac{1}{h}], & M = [0, \frac{3+h^2}{6h}], \\
N = [0,\ \frac{h^2+1}{4h}], & L = [0, \frac{2h}{3+h^2}], \\
E = [0, h] = S, & P = [0, \frac{-1}{h}] = T, \\
F_+ = [0,\ \frac{\sqrt{3}}{3}], & F_- = [0,\ \frac{-\sqrt{3}}{3}], \\
I_+ = [0,\ \frac{\sqrt{3}(h^2+1)-4h}{h^2-3}], & I_- = [0,\ \frac{-\sqrt{3}(h^2+1)-4h}{h^2-3}].
\end{array}$$

Central lines are the reference axes: either $x = 0$ (Euler, Fermat, Brocard) or $y = 0$ (Fermat bisector). Here are the familiar central conics:

circumcircle:	$x^2 + y^2 - y\frac{h^2-1}{h} - 1 = 0$
nine-point circle:	$x^2 + y^2 - y\frac{h^2+1}{2h} = 0$
Kiepert hyperbola:	$xy = 0$
Kiepert parabola:	$y = h$
Steiner inellipse:	$\frac{x^2}{a} + \frac{(y-\frac{h}{3})^2}{b^2} = 1$, $a = \frac{\sqrt{3}}{3},\ b = \frac{h}{3}$

All the relations between the Fermat points and the ellipse $\mathcal{S}$ remain true, and proofs still require us to consider separate cases :

Case 1: $h < \sqrt{3}$. $a > b$; $c^2 = \frac{h^2-3}{9}$. The major axis is parallel to the x- axis.

$$|GF_-| = a + b = \frac{\sqrt{3}+h}{3}, \quad |GF_+| = a - b = \frac{\sqrt{3}-h}{3}.$$

The foci are cut on the line $y = \frac{h}{3}$ by the circle $x^2 + y^2 - \frac{1}{3} = 0$:

$$U_\pm = \frac{1}{3}[\pm\sqrt{3-h^2},\ h].$$

The mapping μ is the product of the reflection in the y-axis by the inversion in the circle

$$x^2 + (y - \frac{h}{3})^2 - \frac{3-h^2}{9} = 0.$$

Case 2: $h > \sqrt{3}$. $a < b$; $c^2 = \frac{3-h^2}{9}$. The major axis is parallel to the y- axis;

$$|GF_-| = a + b = \frac{\sqrt{3}+h}{3}, \qquad |GF_+| = b - a = \frac{h-\sqrt{3}}{3}.$$

The foci are cut by the same circle on the line $x = 0$:

$$U_\pm = \frac{1}{3}[0,\ h \pm \sqrt{h^2-3}].$$

The mapping μ is the product of the reflection in the line $y = \frac{h}{3}$ by the inversion in the circle

$$x^2 + (y - \frac{h}{3})^2 - \frac{h^2-3}{9} = 0.$$

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Benedetto Scimemi: Dip. Matematica Pura ed Applicata, Università degli studi di Padova, via Trieste 63, -35121, Padova, Italy
E-mail address: `scimemi@math.unipd.it`