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Diophantine Steiner Triples and Pythagorean-Type Triangles

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Abstract. We present a connection between Diophantine Steiner triples (integer triples related to configurations of two circles, the larger containing the smaller, in which the Steiner chain closes) and integer-sided triangles with an angle of 60°, 90° or 120°. We introduce an explicit formula and provide a geometrical interpretation.

1. Introduction

In [3] we described all integer triples (R, r, d), $R > r + d$, for which a configuration of two circles of radii R and r with the centers d apart possesses a closed Steiner chain. This means that there exists a cyclic sequence of n circles $L_1, \dots, L_n$ each tangent to the two circles of radii R and r, and to its two neighbors in the sequence. Such triples are called *Diophantine Steiner (DS) triples*. For obvious reasons the consideration can be limited to *primitive* DS triples, *i.e.*, DS triples with $\gcd(R, r, d) = 1$. We also proved in [3] that the only possible length of a Steiner chain in a DS triple is 3, 4 or 6. Therefore, the set of primitive DS triples can be divided into three disjoint sets DS_n for $n = 3, 4, 6$. The elements of these sets are solutions of the following Diophantine equations:

n	relation
3	$R^2 - 14Rr + r^2 - d^2 = 0,$
4	$R^2 - 6Rr + r^2 - d^2 = 0,$
6	$3R^2 - 10Rr + 3r^2 - 3d^2 = 0.$

The sequence of R in DS_4 is

$$6, 15, 20, 28, \dots.$$

In the ENCYCLOPEDIA OF INTEGER SEQUENCES (EIS) [6], this is the sequence A020886 of semi-perimeters of Pythagorean triangles. This suggested to us that DS_4 might be closely connected with the Pythagorean triangles. It turns out that in the same manner the sets DS_3 and DS_6 are connected with integer sided triangles having an angle of 120° or of 60°, respectively. Such triangles were considered in

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papers [1, 4, 5]. Together with Pythagorean triangles, these form a set of triangles that we will call *Pythagorean-type triangles*.

It is surprising that bijective correspondences between three pairs of triples sets are given by the same formula (Theorem 1 below). It is the purpose of this paper to present this formula, provide a geometrical interpretation and derive some further curiosities.

2. Bijective correspondence between the sets Q_φ and DS_n

The sides of Pythagorean-type triangles form three sets of triples, which we denote by Q_{60}, Q_{90} and Q_{120} respectively. The set Q_φ contains all primitive integer triples (a, b, c) such that a triangle with the sides a, b, c contains the angle φ degrees opposite to side c. We also require $b > a$. (This excludes the triple $(1, 1, 1)$ from Q_{60} and avoids duplication of triples with the roles of a and b interchanged.)

It is also convenient to slightly modify the sets Q_{60} and Q_{120} to sets Q'_{60} and Q'_{120} as follows: triple (a, b, c) with three odd numbers a, b, c is replaced with a triple $(2a, 2b, 2c)$. Other triples remain unchanged. Modification in Q_{90} is not necessary, since primitive Pythagorean triples always include exactly one even number.

Theorem 1. *The correspondences $DS_4 \longleftrightarrow Q_{90}$, $DS_3 \longleftrightarrow Q'_{120}$ and $DS_6 \longleftrightarrow Q'_{60}$ given by*

$$(R, r, d) \mapsto \left(\tfrac{1}{2}(R + r - d), \tfrac{1}{2}(R + r + d), R - r\right) \tag{1}$$

$$(a, b, c) \mapsto \left(\tfrac{1}{2}(a + b + c), \tfrac{1}{2}(a + b - c), b - a\right) \tag{2}$$

are bijective and inverse to each other.

Proof. It is straightforward that the above maps are mutually inverse and that they map the solution (R, r, d) of the equation $R^2 - 6Rr + r^2 - d^2 = 0$ into the solution (a, b, c) of the equation $a^2 + b^2 - c^2 = 0$, and vice versa. The same could be proved for the pair $R^2 - 14Rr + r^2 - d^2 = 0$ and $a^2 + b^2 + ab - c^2 = 0$, as well as for the pair $3R^2 - 10Rr + 3r^2 - 3d^2 = 0$ and $a^2 + b^2 - ab - c^2 = 0$.

Using standard arguments, we also prove that the given primitive triple of DS_4 corresponds to the primitive triple of Q_{90}, and vice versa. In the other two cases, consideration is similar but with a slight difference: the triples from Q_{60} and Q_{120} can have three odd components; therefore, the multiplication by 2 was needed. Now we prove that triples (R, r, d) from DS_3 and DS_6 with an even d correspond to the modified triples of Q'_{60} and Q'_{120} of the form $(2a, 2b, 2c)$, a, b, c being odd; and triples with odd d correspond to the untouched triples of Q'_{60} and Q'_{120}. In each case, the primitiveness of the triples from Q_{60} and Q_{120} implies the primitiveness of those from DS_3 and DS_6, and vice versa. □

Remark. Without restriction to integer values, these correspondences extend to the configurations (R, r, d) with Steiner chains of length $n = 3, 4, 6$ and triangles containing an angle $\frac{180°}{n}$.

3. Geometrical interpretation

We present a geometrical interpretation of the relations (1) and (2). Let (R, r, d) be a DS triple from $DS_n,\ n \in \{3, 4, 6\}$. Beginning with two points S_1, S_2 at a distance d apart, we construct two circles $S_1(R)$ and $S_2(r)$. Let the line S_1S_2 intersect the circle $S_1(R)$ at the points U, V and $S_2(r)$ at W and Z. On opposite sides of S_1S_2, construct two similar isosceles triangles VUI_c and WZI on the segments UV and WZ, with angle $\frac{180^\circ}{n}$ between the legs. Complete the triangle ABC with I as incenter I. Then I_c is the excenter on the side c along the line S_1S_2. This is the corresponding Pythagorean-type triangle (see Figure 1 for the case of $n = 4$). To prove this, it is enough to show that the sides of triangle ABC are $a = \frac{1}{2}(R + r - d),\ b = \frac{1}{2}(R + r + d),\ c = R - r$, i.e. the sides given in (1).

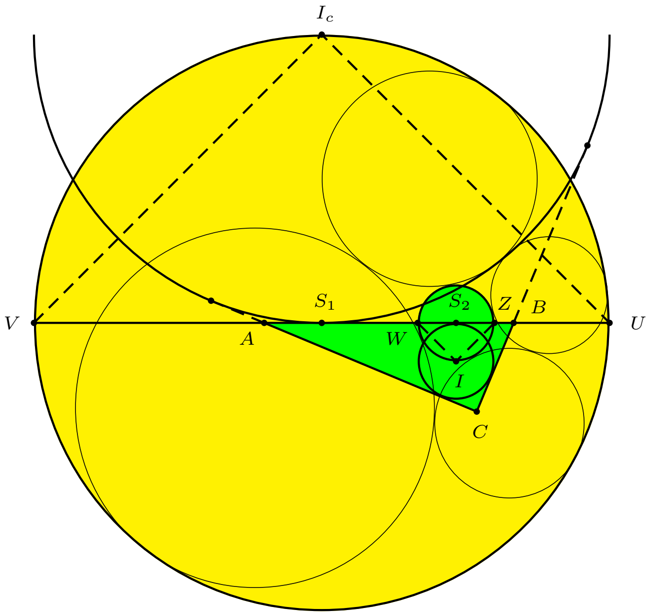

Figure 1.

This construction yields a triangle with given incircle, C-excircle and their touching points with side c. To calculate sides a, b and c, we make use of the following formulas, where r_{i}, r_c, and d are the inradius, C-exradius, and the distance between the touching points:

$$\begin{aligned}
a =&\ \tfrac{1}{2}\left(\sqrt{4r_{\mathrm{i}}r_c + d^2}\cdot\tfrac{r_c+r_{\mathrm{i}}}{r_c-r_{\mathrm{i}}} - d\right),\\
b =&\ \tfrac{1}{2}\left(\sqrt{4r_{\mathrm{i}}r_c + d^2}\cdot\tfrac{r_c+r_{\mathrm{i}}}{r_c-r_{\mathrm{i}}} + d\right),\\
c =&\ \sqrt{4r_{\mathrm{i}}r_c + d^2}.
\end{aligned}$$

Now let us consider different $n \in \{3,4,6\}$. In the case $n = 3$, according to the construction, $r_{\mathrm{i}} = r\sqrt{3}$, $r_c = R\sqrt{3}$ and $4r_{\mathrm{i}}r_c + d^2 = 12Rr + d^2$. Since triples from DS_3 satisfy $R^2 - 14Rr + r^2 - d^2 = 0$, we have $12Rr + d^2 = (R - r)^2$. Hence, $\sqrt{4r_{\mathrm{i}}r_c + d^2} = R - r$.

For $n = 4$ and $n = 6$, we have different r_{i} and r_c and apply different Diophantine equations, but end up with the same value of the square root. Applying all these to the formulas above, we get the desired sides a, b, c.

4. The relation between sets DS_3 and DS_6

In [3] we found an injective (but not surjective) map from DS_3 to DS_6. In this section, we will explain the background and provide a geometrical interpretation of this relation. In §2, we have the bijections $DS_3 \leftrightarrow Q'_{120} \leftrightarrow Q_{120}$ and $DS_6 \leftrightarrow Q'_{60} \leftrightarrow Q_{60}$. Besides, it is clear from Figure 2 that the map $(a,b,c) \mapsto (a, a+b, c)$ represent an injective map from Q_{120} to Q_{60}.

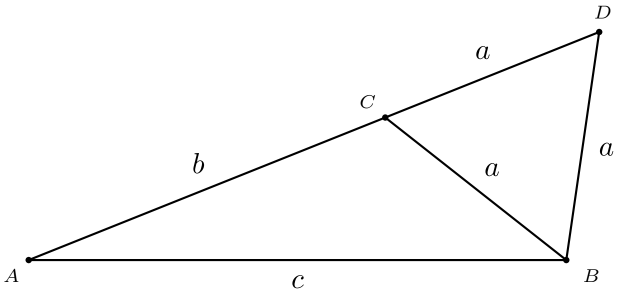

Figure 2.

The same is true for the map $(a,b,c) \mapsto (b, a+b, c)$. We therefore have two maps $Q_{120} \to Q_{60}$, the union of their disjoint images being the whole Q_{60}. Therefore, the sequence of maps $DS_3 \to Q'_{120} \to Q_{120} \to Q_{60} \to Q'_{60} \to DS_6$ defines two maps $DS_3 \to DS_6$. Following step by step, we can easily find both explicit formulas:

$$\begin{aligned} g_1(R,r,d) &= k \cdot \left(\tfrac{1}{4}(5R + r - d), \tfrac{1}{4}(R + 5r - d), \tfrac{1}{2}(R + r + d)\right) \\ g_2(R,r,d) &= k \cdot \left(\tfrac{1}{4}(5R + r + d), \tfrac{1}{4}(R + 5r + d), \tfrac{1}{2}(R + r - d)\right) \end{aligned}$$

with the appropriate factor $k \in \{2, 1, \frac{1}{2}\}$.

The correspondence noticed in [3] is, in fact, just g_2 with the chosen maximal possible factor $k = 2$ (in multiplying by a larger factor, we only lose primitiveness). The existence of two maps g_1 and g_2 whose images cover DS_6 explains why the image of g_2 alone covered only "one half" of DS_6.

Now we give a geometric interpretation of these maps. Let us start with a triple $(R,r,d) \in DS_3$ and construct the associated triangle ABC from Q'_{120}. According to (1), the sides a and b of this triangle are $a = \frac{R+r-d}{2}$ and $b = \frac{R+r+d}{2}$. Hence, $g_1(R,r,d) = k \cdot (R + \frac{a}{2}, r + \frac{a}{2}, b)$. To get the first possible configuration of two circles with the closed Steiner triple of the length $n = 6$, we draw circles with centers A and C with the radii $R' = R + \frac{a}{2}$ and $r' = r + \frac{a}{2}$ (see Figure 3). Similarly, drawing the circles with centers B and C with the radii $R' = R + \frac{b}{2}$ and $r' = r + \frac{b}{2}$,

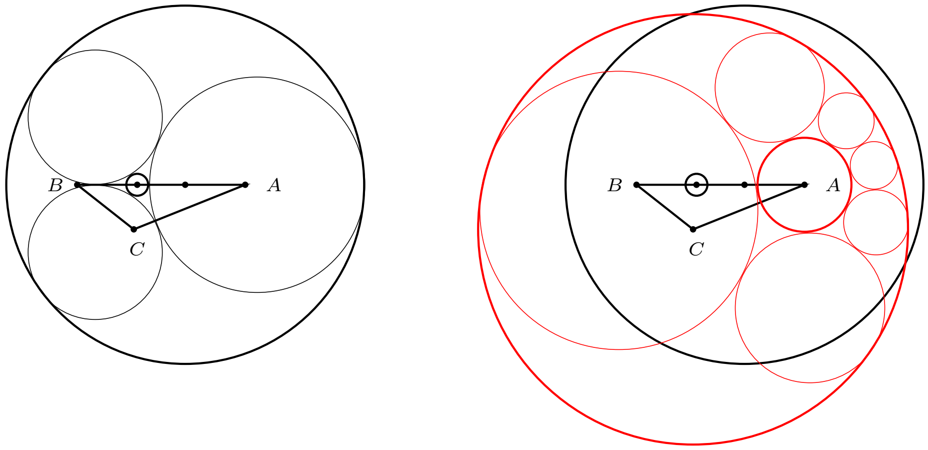

Figure 3.

we get the second possibility, arising from g_2. To obtain triples in DS_6, we must consider the effect of k: *i.e.*, it is possible that the elements (R', r', d') need to be multiplied or divided by 2.

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