Forum Geometricorum
Volume 10 (2010) 119–130.

FORUM GEOM
ISSN 1534-1178

Calculations Concerning the Tangent Lengths and Tangency Chords of a Tangential Quadrilateral

Martin Josefsson

Abstract. We derive formulas for the length of the tangency chords and some other quantities in a tangential quadrilateral in terms of the tangent lengths. Three formulas for the area of a bicentric quadrilateral are also proved.

1. Introduction

A *tangential quadrilateral* is a quadrilateral with an incircle, *i.e.*, a circle tangent to its four sides. We will call the distances from the four vertices to the points of tangency the *tangent lengths*, and denote these by e, f, g and h, as indicated in Figure 1.

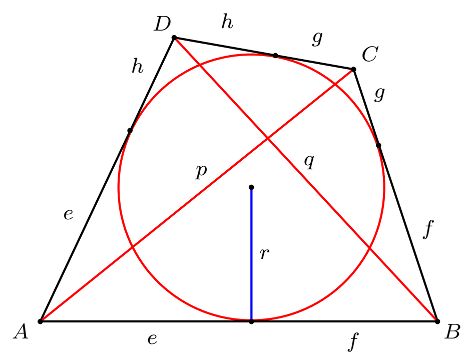

Figure 1. The tangent lengths

What is so interesting about the tangent lengths is that they alone can be used to calculate for instance the inradius r, the area of the quadrilateral K and the length of the diagonals p and q. The formula for r is

$$r = \sqrt{\frac{efg + fgh + ghe + hef}{e + f + g + h}} \tag{1}$$

and its derivation can be found in [5, p.26], [6, pp.187-188] and [13, 15]. Using the well known formula $K = rs = r(e + f + g + h)$, where s is the semiperimeter, we get the area of the tangential quadrilateral [6, p.188]

$$K = \sqrt{(e + f + g + h)(efg + fgh + ghe + hef)}. \tag{2}$$

Publication Date: October 4, 2010. Communicating Editor: Paul Yiu.

Hajja [13] has also derived formulas for the length of the diagonals $p = AC$ and $q = BD$. They are given by

$$p = \sqrt{\frac{e+g}{f+h}((e+g)(f+h)+4fh)},$$
$$q = \sqrt{\frac{f+h}{e+g}((e+g)(f+h)+4eg)}. \qquad (3)$$

In this paper we prove some formulas that express a few other quantities in a tangential quadrilateral in terms of the tangent lengths.

2. The length of the tangency chords

If the incircle in a tangential quadrilateral $ABCD$ is tangent to the sides AB, BC, CD and DA at W, X, Y and Z respectively, then the segments WY and XZ are called the *tangency chords* according to Dörrie [10, pp.188-189]. One interesting property of the tangency chords is that their intersection is also the intersection of the diagonals AC and BD (see [12, 20] and [24, pp.156-157]; the paper by Tan contains nine different proofs).

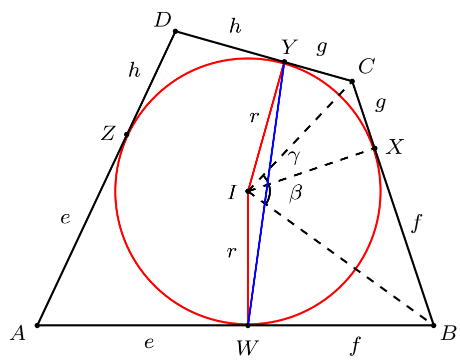

Figure 2. The tangency chord $k = WY$

Theorem 1. *The lengths of the tangency chords WY and XZ in a tangential quadrilateral are respectively*

$$k = \frac{2(efg+fgh+ghe+hef)}{\sqrt{(e+f)(g+h)(e+g)(f+h)}},$$
$$l = \frac{2(efg+fgh+ghe+hef)}{\sqrt{(e+h)(f+g)(e+g)(f+h)}}.$$

Proof. If I is the incenter and angles β and γ are defined as in Figure 2, by the law of cosines in triangle WYI we get

$$k^2 = 2r^2 - 2r^2\cos(2\beta+2\gamma) = 2r^2(1-\cos(2\beta+2\gamma)).$$

Hence, using the addition formula

$$\frac{k^2}{2r^2} = 1 - \cos 2\beta \cos 2\gamma + \sin 2\beta \sin 2\gamma.$$

From the double angle formulas, we have

$$\cos 2\beta = \frac{1-\tan^2\beta}{1+\tan^2\beta} = \frac{r^2 - r^2\tan^2\beta}{r^2+r^2\tan^2\beta} = \frac{r^2-f^2}{r^2+f^2} \tag{4}$$

and

$$\sin 2\beta = \frac{2\tan\beta}{1+\tan^2\beta} = \frac{2rf}{r^2+f^2}.$$

Similar formulas holds for γ, with g instead of f. Thus, we have

$$\frac{k^2}{2r^2} = 1 - \frac{r^2-f^2}{r^2+f^2}\cdot\frac{r^2-g^2}{r^2+g^2} + \frac{2rf}{r^2+f^2}\cdot\frac{2rg}{r^2+g^2} = 2r^2\cdot\frac{(f+g)^2}{(r^2+f^2)(r^2+g^2)}$$

so

$$k^2 = (2r^2)^2\cdot\frac{(f+g)^2}{(r^2+f^2)(r^2+g^2)}.$$

Now we factor r^2+f^2, where r is given by (1). We get

$$\begin{aligned} r^2+f^2 &= \frac{efg+fgh+ghe+hef+f^2(e+f+g+h)}{e+f+g+h} \\ &= \frac{e(fg+fh+gh+f^2)+f(gh+f^2+fg+fh)}{e+f+g+h} \\ &= \frac{(e+f)(g(f+h)+f(h+f))}{e+f+g+h} \\ &= \frac{(e+f)(f+g)(f+h)}{e+f+g+h}. \end{aligned}$$

In the same way

$$r^2+g^2 = \frac{(e+g)(f+g)(g+h)}{e+f+g+h} \tag{5}$$

so

$$k^2 = (2r^2)^2\cdot\frac{(f+g)^2(e+f+g+h)^2}{(e+f)(f+g)(f+h)(e+g)(f+g)(g+h)}.$$

After simplification

$$k = 2r^2\cdot\frac{e+f+g+h}{\sqrt{(e+f)(f+h)(h+g)(g+e)}}$$

and using (1) we finally get

$$k = \frac{2(efg+fgh+ghe+hef)}{\sqrt{(e+f)(f+h)(h+g)(g+e)}}.$$

The formula for l can either be derived the same way, or we can use the symmetry in the tangential quadrilateral and need only to make the change $f \leftrightarrow h$ in the formula for k. □

From Theorem 1 we get the following result, which was Problem 1298 in the MATHEMATICS MAGAZINE [8].

Corollary 2. *In a tangential quadrilateral with sides a, b, c and d, the quotient of the tangency chords satisfy*

$$\left(\frac{k}{l}\right)^2 = \frac{bd}{ac}.$$

Proof. Taking the quotient of k and l from Theorem 1, after simplification we get

$$\frac{k}{l} = \sqrt{\frac{(e+h)(f+g)}{(e+f)(h+g)}} = \sqrt{\frac{db}{ac}},$$

and the result follows. □

Corollary 3. *The tangency chords in a tangential quadrilateral are of equal length if and only if it is a kite.*

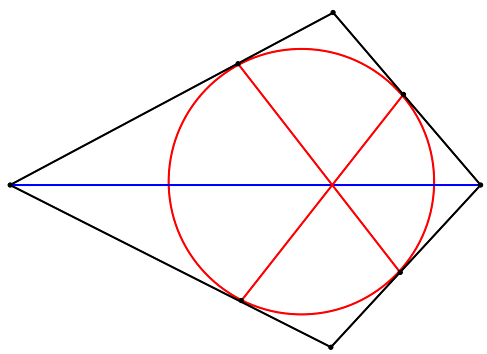

Figure 3. The tangency chords in a kite

Proof. ($\Rightarrow$) If the quadrilateral is a kite it directly follows that the tangency chords are of equal length because of the mirrow symmetry in the longest diagonal (see Figure 3).

($\Leftarrow$) Conversely, if the tangency chords are of equal length in a tangential quadrilateral, from Corollary 2 we get $ac = bd$. In all tangential quadrilaterals the consecutive sides a, b, c and d satisfy $a+c = b+d$ ($= e+f+g+h$; see also [1, p.135], [2, pp.65-67] and [23]). Squaring, this implies $a^2 + 2ac + c^2 = b^2 + 2bd + d^2$ and using $ac = bd$ it follows that $a^2 + c^2 = b^2 + d^2$. This is the characterization for orthodiagonal quadrilaterals[1] [24, p.158]. The only tangential quadrilateral with perpendicular diagonals is the kite. We give an algebraic proof of this claim. Rewriting two of the equations above, we have

$$a - b = d - c, \tag{6}$$
$$a^2 - b^2 = d^2 - c^2 \tag{7}$$

[1]A quadrilateral with perpendicular diagonals.

Factorizing the second, we get

$$(a-b)(a+b) = (d-c)(d+c). \tag{8}$$

Case 1. If $a = b$ we also have $d = c$ using (6).

Case 2. If $a \neq b$, then we get $a + b = d + c$ after division in (8) by $a - b$ and $d - c$ on respective sides (which by (6) are equal). Now adding $a + b = d + c$ and $a - b = d - c$, we get $2a = 2d$. Hence $a = d$ and also $b = c$ using (6).

In both cases two pairs of adjacent sides are equal, so the quadrilateral is a kite. □

3. The angle between the tangency chords

In the proof of the next theorem we will use the following simple lemma.

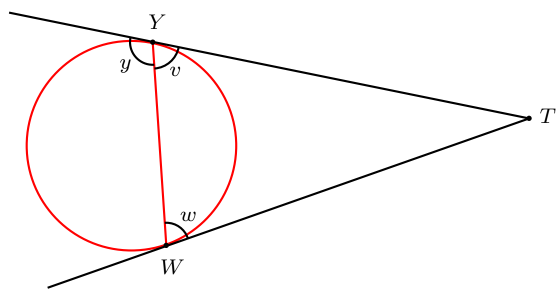

Figure 4. Alternate angles w and y

Lemma 4. *The alternate angles between a chord and two tangents to a circle are supplementary angles, i.e.,* $w + y = \pi$ *in* Figure 4.

Proof. Extend the tangents at W and Y to intersect at T, see Figure 4. Triangle TWY is isosceles according to the two tangent theorem, so the angles at the base are equal, $w = v$. Also, $v + y = \pi$ since they are angles on a straight line. Hence $w + y = \pi$. □

Now we derive a formula for the angle between the two tangency chords.

Theorem 5. *If* e, f, g *and* h *are the tangent lengths in a tangential quadrilateral, the angle* φ *between the tangency chords is given by*

$$\sin\varphi = \sqrt{\frac{(e+f+g+h)(efg+fgh+ghe+hef)}{(e+f)(f+g)(g+h)(h+e)}}.$$

Proof. We start by relating the angle φ to two opposite angles in the tangential quadrilateral (see Figure 5).

From the sum of angles in quadrilaterals $BWPX$ and $DYPZ$ we have $w + x + \varphi + B = 2\pi$ and $y + z + \varphi + D = 2\pi$. Adding these,

$$w + x + y + z + 2\varphi + B + D = 4\pi. \tag{9}$$

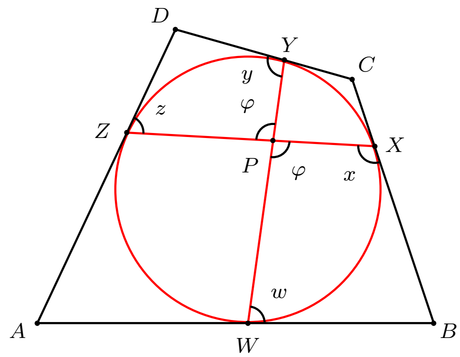

Figure 5. The angle φ between the tangency chords

Using the lemma, $w + y = \pi$ and $x + z = \pi$. Inserting these into (9), we get

$$2\pi + 2\varphi + B + D = 4\pi \quad \Leftrightarrow \quad B + D = 2\pi - 2\varphi. \tag{10}$$

For the area K of a tangential quadrilateral we have the formula

$$K = \sqrt{abcd}\,\sin\frac{B + D}{2} \tag{11}$$

where a, b, c and d are the sides of the tangential quadrilateral [9, p.28]. Inserting (10), we get

$$K = \sqrt{abcd}\,\sin(\pi - \varphi) = \sqrt{abcd}\,\sin\varphi,$$

hence

$$\sin\varphi = \frac{K}{\sqrt{abcd}} = \frac{\sqrt{(e+f+g+h)(efg+fgh+ghe+hef)}}{\sqrt{(e+f)(f+g)(g+h)(h+e)}}$$

where we used (2). □

From equation (10) we also get the following well known characterization for a quadrilateral to be *bicentric*, *i.e.*, both tangential and cyclic. We will however formulate it as a characterization for the tangency chords to be perpendicular. Our proof is similar to that given in [10, pp.188-189] (if we include the derivation of (10) from the last theorem). Other proofs are given in [4, 11].

Corollary 6. *The tangency chords in a tangential quadrilateral are perpendicular if and only if it is a bicentric quadrilateral.*

Proof. In any tangential quadrilateral, $B + D = 2\pi - 2\varphi$ by (10). The tangency chords are perpendicular if and only if

$$\varphi = \frac{\pi}{2} \quad \Leftrightarrow \quad B + D = \pi$$

which is a well known characterization for a quadrilateral to be cyclic. Hence this is a characterization for the quadrilateral to be bicentric. □

4. The area of the contact quadrilateral

If the incircle in a tangential quadrilateral $ABCD$ is tangent to the sides AB, BC, CD and DA at W, X, Y and Z respectively, then in [11] Yetti[2] calls the quadrilateral $WXYZ$ the *contact quadrilateral* (see Figure 6). Here we shall derive a formula for its area in terms of the tangent lengths.

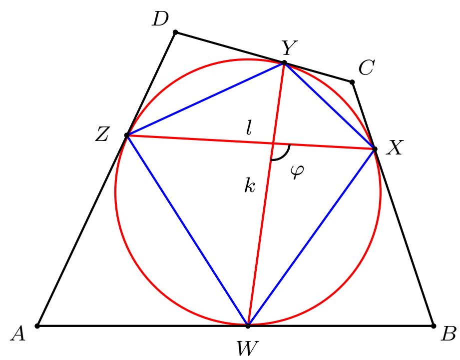

Figure 6. The contact quadrilateral $WXYZ$

Theorem 7. *If e, f, g and h are the tangent lengths in a tangential quadrilateral, then the contact quadrilateral has area*

$$K_{\rm c} = \frac{2\sqrt{(e+f+g+h)(efg+fgh+ghe+hef)^5}}{(e+f)(e+g)(e+h)(f+g)(f+h)(g+h)}.$$

Proof. The area of any convex quadrilateral is

$$K = \tfrac{1}{2}pq\sin\theta \tag{12}$$

where p and q are the length of the diagonals and θ is the angle between them (see [21, p.213] and [22]). Hence for the area of the contact quadrilateral we have

$$K_c = \tfrac{1}{2}kl\sin\varphi$$

where k and l are the length of the tangency chords and φ is the angle between them. Using Theorems 1 and 5, the formula for K_c follows at once after simplification. □

5. The angles of the tangential quadrilateral

The next theorem gives formulas for the sines of the half angles of a tangential quadrilateral in terms of the tangent lengths.

[2]Yetti is the username of an American physicist at the website *Art of Problem Solving* [3].

Theorem 8. *If e, f, g and h are the tangent lengths in a tangential quadrilateral $ABCD$, then its angles satisfy*

$$\sin\frac{A}{2} = \sqrt{\frac{efg+fgh+ghe+hef}{(e+f)(e+g)(e+h)}},$$

$$\sin\frac{B}{2} = \sqrt{\frac{efg+fgh+ghe+hef}{(f+e)(f+g)(f+h)}},$$

$$\sin\frac{C}{2} = \sqrt{\frac{efg+fgh+ghe+hef}{(g+e)(g+f)(g+h)}},$$

$$\sin\frac{D}{2} = \sqrt{\frac{efg+fgh+ghe+hef}{(h+e)(h+f)(h+g)}}.$$

Proof. If the incircle has center I and is tangent to sides AB and AD at W and Z (see Figure 7), then by the law of cosines in triangle WZI

$$WZ^2 = 2r^2(1-\cos 2\alpha) = \frac{4e^2r^2}{r^2+e^2}$$

where we used

$$\cos 2\alpha = \frac{r^2-e^2}{r^2+e^2}$$

which we get from (4) when making the change $f \leftrightarrow e$.

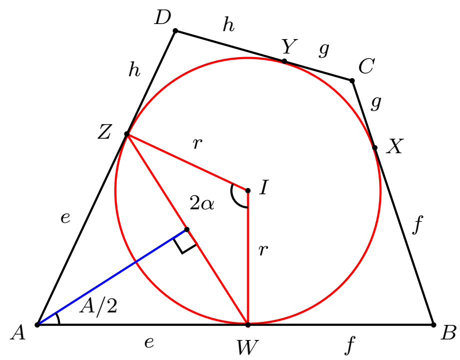

Figure 7. Half the angle of A

Now using (1) and

$$r^2+e^2 = \frac{(e+f)(e+g)(e+h)}{e+f+g+h}$$

which by symmetry follows from (5) when making the change $g \leftrightarrow e$, we have

$$WZ^2 = 4e^2 \cdot \frac{efg+fgh+ghe+hef}{e+f+g+h} \cdot \frac{e+f+g+h}{(e+f)(e+g)(e+h)}$$

hence

$$WZ = 2e\sqrt{\frac{efg+fgh+ghe+hef}{(e+f)(e+g)(e+h)}}.$$

Finally, from the definition of sine, we get (see Figure 7)

$$\sin\frac{A}{2} = \frac{\frac{1}{2}WZ}{e} = \sqrt{\frac{efg+fgh+ghe+hef}{(e+f)(e+g)(e+h)}}.$$

The other formulas can be derived in the same way, or we get them at once using symmetry. □

6. The area of a bicentric quadrilateral

The formula for the area of a *bicentric quadrilateral* (see Figure 8) is almost always derived in one of two ways.[3] Either by inserting $B + D = \pi$ into formula (11) or by using $a + c = b + d$ in Brahmagupta's formula[4]

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

for the area of a cyclic quadrilateral, where s is the semiperimeter. A third derivation was given by Stapp as a solution to a problem[5] by Rosenbaum in an old number of the MONTHLY [18]. Another possibility is to use the formula[6]

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac-bd)^2}$$

for the area of a tangential quadrilateral [9, p.29], inserting Ptolemy's theorem $pq = ac + bd$ (derived in [1, pp.128-129], [9, p.25] and [24, pp.148-150]) and factorize the radicand.

Here we shall give a fifth proof, using the tangent lengths in a way different from what Stapp did in [18].

Theorem 9. *A bicentric quadrilateral with sides a, b, c and d has area*

$$K = \sqrt{abcd}.$$

Proof. From formula (2) we get

$$\begin{aligned}
K^2 &= (efg+fgh+ghe+hef)(e+f+g+h)\\
&= ef(g+h)(e+f) + ef(g+h)^2 + gh(e+f)^2 + gh(e+f)(g+h)\\
&= (e+f)(g+h)(ef+gh+eg+hf-eg-hf) + ef(g+h)^2 + gh(e+f)^2\\
&= (e+f)(g+h)(f+g)(e+h) - (eg-fh)^2
\end{aligned}$$

where we used the factorizations $ef + gh + eg + hf = (f+g)(e+h)$ and

$$(e+f)(g+h)(-eg-hf) + ef(g+h)^2 + gh(e+f)^2 = -(eg-fh)^2,$$

---

[3]Or intended to be derived so; in many books [1, 7, 16, 24] this is an exercise rather than a theorem.

[4]For a derivation, see [7, pp.57-58] or [9, p.24].

[5]The problem was to prove our Theorem 9. Stapp used the tangent lengths in his calculation.

[6]This formula can be derived independently from (11) and Brahmagupta's formula.

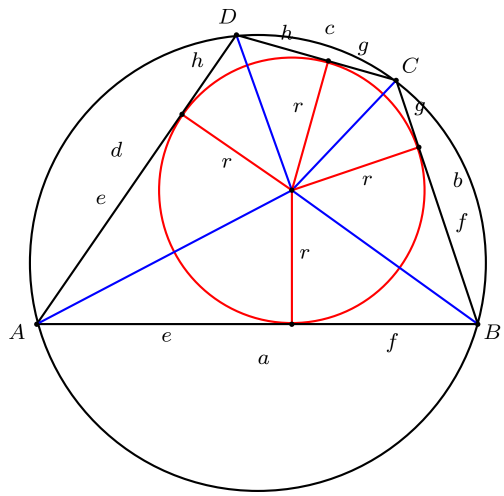

Figure 8. A bicentric quadrilateral $ABCD$

which are easy to check. Hence

$$K^2 = acbd - (eg - fh)^2$$

and we have $K = \sqrt{abcd}$ if and only if $eg = fh$, which according to Hajja[7] [13] is a characterization for a tangential quadrilateral to be cyclic, *i.e.*, bicentric. □

In a bicentric quadrilateral there is a simpler formula for the area in terms of the tangent lengths than (2), according to the next theorem.

Theorem 10. *A bicentric quadrilateral with tangent lengths e, f, g and h has area*

$$K = \sqrt[4]{efgh}(e + f + g + h).$$

Proof. The quadrilateral has an incircle, so (see Figure 8)

$$r = e\tan\frac{A}{2} = f\tan\frac{B}{2} = g\tan\frac{C}{2} = h\tan\frac{D}{2},$$

hence

$$r^4 = efgh\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}\tan\frac{D}{2}. \tag{13}$$

It also has a circumcircle, so $A + C = \pi$. Hence $\frac{A}{2} = \frac{\pi}{2} - \frac{C}{2}$ and it follows that

$$\tan\frac{A}{2} = \cot\frac{C}{2} \quad\Leftrightarrow\quad \tan\frac{A}{2}\tan\frac{C}{2} = 1.$$

In the same way

$$\tan\frac{B}{2}\tan\frac{D}{2} = 1.$$

[7]Note that Hajja uses a, b, c and d for the tangent lengths.

Thus, in a bicentric quadrilateral we get[8]

$$r^4 = efgh.$$

Finally, the area[9] is given by

$$K = rs = \sqrt[4]{efgh}(e + f + g + h)$$

where s is the semiperimeter. □

We conclude with another interesting and possibly new formula for the area of a bicentric quadrilateral in terms of the lengths of the tangency chords and the diagonals.

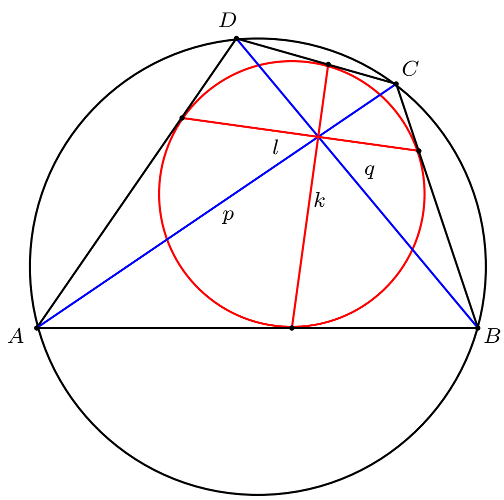

Figure 9. The tangency chords and diagonals

Theorem 11. *A bicentric quadrilateral with tangency chords k and l, and diagonals p and q has area*

$$K = \frac{klpq}{k^2 + l^2}.$$

Proof. Using (12), Theorem 9 and Ptolemy's theorem, we have

$$K = \tfrac{1}{2}pq\sin\theta \quad \Leftrightarrow \quad \sqrt{abcd} = \tfrac{1}{2}(ac + bd)\sin\theta.$$

Hence

$$\frac{2}{\sin\theta} = \frac{ac + bd}{\sqrt{abcd}} = \sqrt{\frac{ac}{bd}} + \sqrt{\frac{bd}{ac}} = \frac{l}{k} + \frac{k}{l} = \frac{k^2 + l^2}{kl}$$

[8] This derivation was done by Yetti in [19], where there are also some proofs of formula (1).

[9] This formula also gives the area of a tangential trapezoid. Since it has two adjacent supplementary angles, $\tan\frac{A}{2}\tan\frac{D}{2} = \tan\frac{B}{2}\tan\frac{C}{2} = 1$ or $\tan\frac{A}{2}\tan\frac{B}{2} = \tan\frac{C}{2}\tan\frac{D}{2} = 1$; thus the formula for r is still valid.

where we used Corollary 2. Then we get the area of the bicentric quadrilateral using (12) again

$$K = \frac{\sin\theta}{2}pq = \frac{klpq}{k^2+l^2}$$

completing the proof. □

References

[1] N. Altshiller-Court, *College Geometry*, Dover reprint, 2007.
[2] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, Birkhäuser, Boston, 2004.
[3] *Art of problem Solving*, Olympiad geometry forum, http://www.artofproblemsolving.com/Forum/viewforum.php?f=4&
[4] M. Bataille, A Duality for Bicentric Quadrilaterals, *Crux Math.*, 35 (2009) 310–312.
[5] C. J. Bradley, *Challenges in Geometry*, Oxford University press, 2005.
[6] J. Casey, *A Treatise on Plane Trigonometry*, BiblioLife, 2009.
[7] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Math. Assoc. Amer., 1967.
[8] H. Demir and J. M. Stark, Problem 1298, *Math. Mag.* 61 (1988) 195; solution, ibid., 62 (1989) 200–201.
[9] C. V. Durell and A. Robson, *Advanced Trigonometry*, Dover reprint, 2003.
[10] H. Dörrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965.
[11] Fermat-Euler (username), Prove that cc_1 bisects, *Art of Problem Solving*, 2005, http://www.artofproblemsolving.com/Forum/viewtopic.php?t=55918
[12] D. Grinberg, *Circumscribed quadrilaterals revisited*, 2008, available at http://www.cip.ifi.lmu.de/∼grinberg/CircumRev.pdf
[13] M. Hajja, A condition for a circumscriptible quadrilateral to be cyclic, *Forum Geom.*, 8 (2008) 103–106.
[14] M. Josefsson, On the inradius of a tangential quadrilateral, *Forum Geom.*, 10 (2010) 27–34.
[15] M. S. Klamkin, Five Klamkin Quickies, *Crux Math.* 23 (1997) 70–71.
[16] Z. A. Melzak, *Invitation to Geometry*, Dover reprint, 2008.
[17] N. Minculete, Characterizations of a tangential quadrilateral, *Forum Geom.* 9 (2009) 113–118.
[18] J. Rosenbaum and M. C. Stapp, Problem E 851, *Amer. Math. Monthly*, 56 (1949) 104; solution, ibid., 56 (1949) 553.
[19] P. Shi, Looks so easy [find inradius of circumscribed quadrilateral], *Art of Problem Solving*, 2005, http://www.artofproblemsolving.com/Forum/viewtopic.php?t=28864
[20] K. Tan, Some proofs of a theorem on quadrilateral, *Math. Mag.*, 35 (1962) 289–294.
[21] J. A. Vince, *Geometry for Computer Graphics. Formulae, Examples and Proofs*, Springer, 2005.
[22] E. Weisstein, Quadrilateral at *MathWorld*, http://mathworld.wolfram.com/Quadrilateral.html
[23] C. Worrall, A journey with circumscribable quadrilaterals, *Mathematics Teacher*, 98 (2004) 192–199.
[24] P. Yiu, *Euclidean Geometry Notes*, 1998, Florida Atlantic University Lecture Notes, available at http://math.fau.edu/Yiu/EuclideanGeometryNotes.pdf

Martin Josefsson: Västergatan 25d, 285 37 Markaryd, Sweden
E-mail address: martin.markaryd@hotmail.com