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Constructions with Inscribed Ellipses in a Triangle

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Abstract. We give a simple construction of the axes and foci of an inscribed ellipse with prescribed center, and as an application, a simple solution of the problem of construction of a triangle with prescribed circumcevian triangle of the centroid.

1. Construction of the axes and foci of an inscribed ellipse

Given a triangle ABC, we give a simple construction of the axes and foci of an inscribed ellipse with given center or perspector. If a conic touches the sides BC, CA, AB at X, Y, Z respectively, then the lines AX, BY, CZ are concurrent at a point P called the perspector of the conic. The center M of the conic is the complement of the isotomic conjugate of P. In homogeneous barycentric coordinates, if $P = (p : q : r)$, then

$$M = \left(\frac{1}{q} + \frac{1}{r} : \frac{1}{r} + \frac{1}{p} : \frac{1}{p} + \frac{1}{q}\right) = (p(q + r) : q(r + p) : r(p + q)).$$

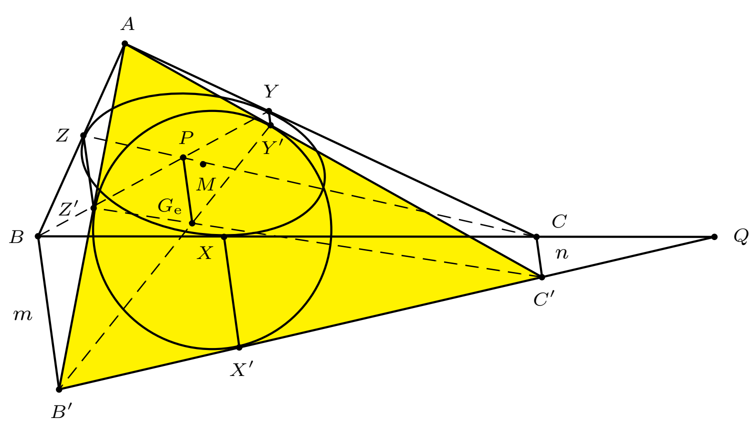

Figure 1

Consider triangle ABC and its inscribed ellipse with center $M = (u : v : w)$ as the orthogonal projections of a triangle $AB'C'$ and its incircle. We may take $A' = A$ and assume $BB' = m$, $CC' = n$. If the incircle of $AB'C'$ touches $B'C'$, $C'A$ and AB' at X', Y', Z' respectively, then X, Y, Z are the orthogonal projections of X', Y', Z'. It follows that the perspector P is the projection of

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the Gergonne point G_e of $AB'C'$. Suppose triangle $AB'C'$ has sides $B'C' = a'$, $C'A = b'$ and $AB' = c'$, then

$$b' + c' - a' : c' + a' - b' : a' + b' - c' = \frac{1}{p} : \frac{1}{q} : \frac{1}{r}.$$

It follows that

$$a' : b' : c' = \frac{1}{q} + \frac{1}{r} : \frac{1}{r} + \frac{1}{p} : \frac{1}{p} + \frac{1}{q} = u : v : w.$$

From this we draw a remarkable conclusion.

Proposition 1. *If a triangle and an inscribed ellipse are the orthogonal projections of a triangle and its incircle, the sidelengths of the latter triangle are proportional to the barycentric coordinates of the center of the ellipse.*

Since a', b', c' satisfy the triangle inequality, the center $M = (u : v : w)$ of the ellipse is an interior point of the medial triangle of BAC.

We determine the relative positions of ABC and $AB'C'$ leading to a simple construction of the axes and foci of the inscribed ellipse of given center M. We first construct two triangles A_+BC and A_-BC with sidelengths proportional to the barycentric coordinates of M.

Construction 2. *Let $M = (u : v : w)$ be a point in the interior of the medial triangle of ABC, with cevian triangle $A_0B_0C_0$. Construct*
(i) *the parallels of BB_0 and CC_0 through A to intersect BC at D and E respectively,*
(ii) *the circles $B(D)$ and $C(E)$ to intersect at two points A_+ and A_- symmetric in the line BC.* Label A_+ the one on the opposite side of BC as A.

Each of the triangles A_+BC and A_-BC have sidelengths $u : v : w$.

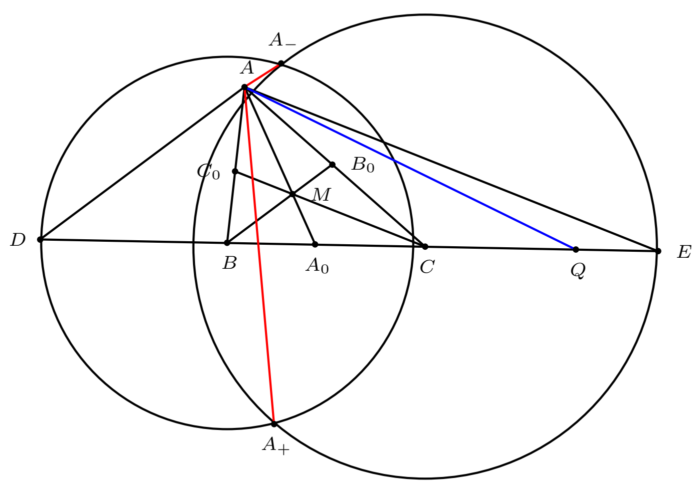

Figure 2

Proof. Note that

$$\frac{BC}{CA_-} = \frac{BC}{CE} = \frac{BC_0}{C_0A} = \frac{u}{v},$$
$$\frac{BC}{A_-B} = \frac{BC}{DB} = \frac{B_0C}{AB_0} = \frac{u}{w}.$$

From these, $BC : CA_- : A_-B = u : v : w$; similarly for $BC : CA_+ : A_+B$. □

Lemma 3. *The lengths of AA_+ and AA_- are given by*

$$AA_+ = \frac{\sqrt{\mathcal{Q} + 16\Delta\Delta'}}{\sqrt{2}u} \quad \text{and} \quad AA_- = \frac{\sqrt{\mathcal{Q} - 16\Delta\Delta'}}{\sqrt{2}u},$$

where

$$\mathcal{Q} = (b^2 + c^2 - a^2)u^2 + (c^2 + a^2 - b^2)v^2 + (a^2 + b^2 - c^2)w^2, \tag{1}$$

and Δ' is the area of the triangle with sidelengths u, v, w.

Proof. Applying the law of cosines to triangle ABA_-, we have

$$\begin{aligned} AA_-^2 =& \; AB^2 + BA_-^2 - 2AB \cdot BA_- \cos(ABC - A_-BC) \\ =& \; c^2 + \left(\frac{aw}{u}\right)^2 - 2c \cdot \frac{aw}{u}(\cos B \cos A_-BC + \sin B \sin A_-BC) \\ =& \; \frac{c^2u^2 + a^2w^2 - 2ca \cos B \cdot wu \cos A_-BC - 2ca \sin B \cdot wu \sin A_-BC}{u^2} \\ =& \; \frac{2c^2u^2 + 2a^2w^2 - (c^2 + a^2 - b^2)(w^2 + u^2 - v^2) - 16\Delta\Delta'}{2u^2} \\ =& \; \frac{\mathcal{Q} - 16\Delta\Delta'}{2u^2}. \end{aligned}$$

The case of AA_+ is similar. □

Let Q be the intersection of the lines BC and $B'C'$. The line AQ is the intersection of the planes ABC and $AB'C'$. The diameters of the incircle of $AB'C'$ parallel and orthogonal to AQ project onto the major and minor axes of the inscribed ellipse.

Proposition 4. *The line AQ is the internal bisector of angle A_+AA_-.*

Proof. The sidelengths of triangle A_-BC are $BC = a$, $CA_- = \frac{av}{u} = \frac{ab'}{a'}$ and $A_-B = \frac{aw}{u} = \frac{ac'}{a'}$. Set up a Cartesian coordinate system so that $A = (x_1, y_1)$, $B = (-a, 0)$, $C = (0, 0)$, $A_+ = (x_0, -y_0)$ and $A_- = (x_0, y_0)$, where

$$x_0 = \; -CA_- \cos A_-CB = -\frac{a'^2 + b'^2 - c'^2}{2a'^2} \cdot a, \tag{2}$$

$$x_1 = \; -b \cos C = -\frac{a^2 + b^2 - c^2}{2a}. \tag{3}$$

Since the lines AA_+ and AA_- have equations

$$(y_1 + y_0)x - (x_1 - x_0)y - (x_1y_0 + x_0y_1) = \; 0,$$
$$(y_1 - y_0)x - (x_1 - x_0)y + (x_1y_0 - x_0y_1) = \; 0,$$

a bisector of angle A_+AA_- meets BC at a point with coordinates $(x,0)$ satisfying

$$\frac{((y_1-y_0)x+(x_1y_0-x_0y_1))^2}{(y_1-y_0)^2+(x_1-x_0)^2}=\frac{((y_1+y_0)x-(x_1y_0+x_0y_1))^2}{(y_1+y_0)^2+(x_1-x_0)^2}.$$

Simplifying this into

$$(x_1-x_0)x^2+(x_0^2+y_0^2-x_1^2-y_1^2)x+x_0(x_1^2+y_1^2)-x_1(x_0^2+y_0^2)=0,$$

and making use of (2) and (3), we obtain

$$\left(\frac{a'^2+b'^2-c'^2}{2a'^2}-\frac{a^2+b^2-c^2}{2a^2}\right)x^2+a\left(\frac{b'^2}{a'^2}-\frac{b^2}{a^2}\right)x$$
$$-\left(b^2\cdot\frac{a'^2+b'^2-c'^2}{2a'^2}-b'^2\cdot\frac{a^2+b^2-c^2}{2a^2}\right)=0. \qquad (4)$$

Now, since

$$a'^2=(m-n)^2+a^2,\quad b'^2=n^2+b^2,\quad c'^2=m^2+c^2, \qquad (5)$$

we reorganize (4) into the form

$$((m-n)x-na)((m(a^2+b^2-c^2)+n(c^2+a^2-b^2))x-(n(b^2+c^2-a^2)-2mb^2)a)=0.$$

Note that the two roots

$$X_1=\frac{na}{m-n}\quad\text{and}\quad X_2=\frac{(n(b^2+c^2-a^2)-2mb^2)a}{m(a^2+b^2-c^2)+n(c^2+a^2-b^2)}$$

satisfy

$$(x_0-x_1)(X_1-X_2)=\frac{m^2b^2-mn(b^2+c^2-a^2)+n^2c^2}{a^2+(m-n)^2}>0,$$

since the discriminant of $m^2b^2-mn(b^2+c^2-a^2)+n^2c^2$ (as a quadratic form in m, n) is $(b^2+c^2-a^2)^2-4b^2c^2=-4b^2c^2\sin^2 A<0$. We conclude that if $x_1<x_0$, then $X_1>X_2$ and X_1 is the root that corresponds to the internal bisector. Since $\frac{X_1}{X_1+a}=\frac{n}{m}$, this intercept is the intersection Q of the lines BC and $B'C'$. □

Theorem 5. *The inscribed ellipse of triangle ABC with center $M=(u:v:w)$* (inside the medial triangle) *has semiaxes given by*

$$a_{\max}=\frac{u(AA_++AA_-)}{2(u+v+w)}\quad\text{and}\quad a_{\min}=\frac{u(AA_+-AA_-)}{2(u+v+w)}.$$

Proof. Making use of (5), we have

$$2mn=m^2+n^2-(m-n)^2=(b'^2+c'^2-a'^2)-(b^2+c^2-a^2).$$

It follows that

$$4(b'^2-b^2)(c'^2-c^2)=((b'^2+c'^2-a'^2)-(b^2+c^2-a^2))^2,$$

and

$$\begin{aligned}
&4b'^2c'^2 - (b'^2 + c'^2 - a'^2)^2 + 4b^2c^2 - (b^2 + c^2 - a^2)^2 \\
=& 4b^2c'^2 + 4c^2b'^2 - 2(b^2 + c^2 - a^2)(b'^2 + c'^2 - a'^2) \\
=& 2((b^2 + c^2 - a^2)a'^2 + (c^2 + a^2 - b^2)b'^2 + (a^2 + b^2 - c^2)c'^2). \qquad (6)
\end{aligned}$$

Note that $4b^2c^2 - (b^2+c^2-a^2)^2 = 4b^2c^2(1-\cos^2 A) = 4b^2c^2\sin^2 A = 16\Delta^2$ and similarly, $4b'^2c'^2 - (b'^2 + c'^2 - a'^2)^2 = 16\Delta(AB'C')^2$. Let ρ and ρ' be the inradii of triangle $AB'C'$ and the one with sidelengths u, v, w. These two triangles have ratio of similarity $\frac{\rho}{\rho'} = \frac{\rho(u+v+w)}{2\Delta'}$, we have

$$\Delta(AB'C') = \Delta'\left(\frac{\rho(u+v+w)}{2\Delta}\right)^2 = \frac{\rho^2(u+v+w)^2}{4\Delta'}.$$

With this, (6) can be rewritten as

$$2(u+v+w)^4\rho^4 - (u+v+w)^2\mathcal{Q}\rho^2 + 32\Delta^2\Delta'^2 = 0.$$

This has roots $\pm\rho_1$ and $\pm\rho_2$, where

$$\rho_1 = \frac{\sqrt{\mathcal{Q}+16\Delta\Delta'} + \sqrt{\mathcal{Q}-16\Delta\Delta'}}{2\sqrt{2}(u+v+w)}, \qquad \rho_2 = \frac{\sqrt{\mathcal{Q}+16\Delta\Delta'} - \sqrt{\mathcal{Q}-16\Delta\Delta'}}{2\sqrt{2}(u+v+w)}.$$

Note that $\rho_1\rho_2 = \frac{4\Delta\Delta'}{(u+v+w)^2}$, and

$$\frac{a_{\min}}{a_{\max}} = \frac{\Delta}{\Delta(AB'C')} = \frac{4\Delta\Delta'}{\rho^2(u+v+w)^2} = \frac{\rho_1\rho_2}{\rho^2}.$$

Since $\rho_1 > \rho_2$, it follows that $a_{\max} = \rho = \rho_1$ and $a_{\min} = \rho_2$. □

Now we construct the axes and foci of an inscribed ellipse.

Construction 6. *Let M be a point in the interior of the medial triangle of ABC, and A_+BC, A_-BC triangles with sidelengths proportional to the barycentric coordinates of M* (see Construction 2). *Construct*
(1) *the internal bisector of angle A_+AA_- to intersect the line BC at Q,*
(2) *the parallel of AQ through M.* This is the major axis of the ellipse.
Further construct
(3) *the orthogonal projection S of Q on AA_+,*
(4) *the parallel through M to AA_+ to intersect A_1S at T,*
(5) *the circle $M(T)$.* This is the auxiliary circle of the ellipse.
Finally, construct
(6) *the perpendiculars to the sides at the intersections with the circle $M(T)$ to intersect the major axis at F and F'.* These are the foci of the ellipse. See Figure 3.

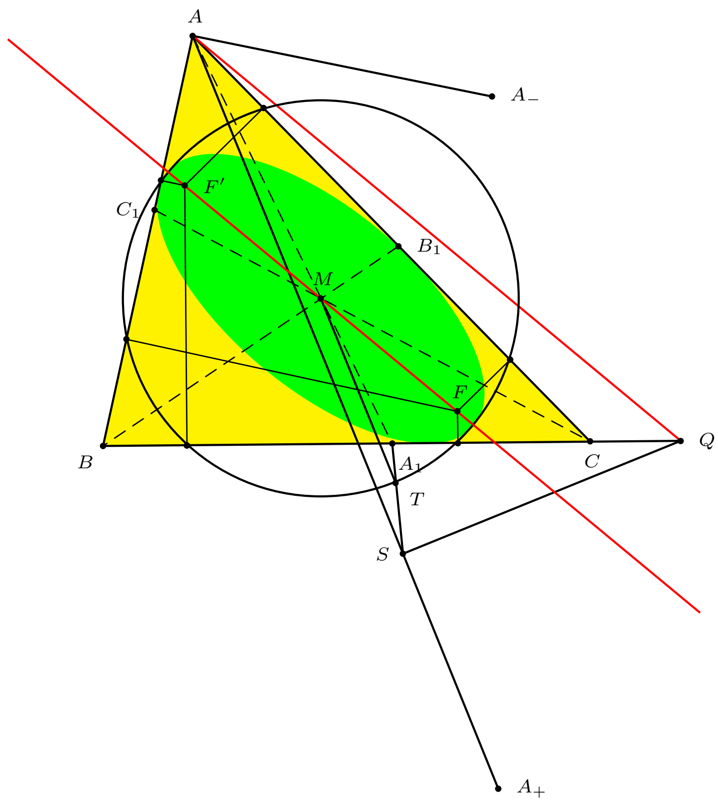

Figure 3.

2. The circumcevian triangle of the centroid

For the Steiner inellipse with center $G = (1 : 1 : 1)$, the triangles A_+BC and A_-BC are equilateral triangles, and

$$AA_+ = \sqrt{\frac{a^2 + b^2 + c^2 + 4\sqrt{3}\Delta}{2}} \quad \text{and} \quad AA_- = \sqrt{\frac{a^2 + b^2 + c^2 - 4\sqrt{3}\Delta}{2}}.$$

By Theorem 5,

$$a_{\max} = \frac{AA_+ + AA_-}{6} \quad \text{and} \quad a_{\min} = \frac{AA_+ - AA_-}{6}.$$

Theorem 7. *Let G be the centroid of ABC, with circumcevian triangle $A_1B_1C_1$. The pedal circle of G relative to $A_1B_1C_1$ has center the centroid G_1 of $A_1B_1C_1$. Hence, G is a focus of the Steiner inellipse of triangle $A_1B_1C_1$.*

Proof. Let a, b, c and a_1, b_1, c_1 be the sidelengths of triangles ABC and $A_1B_1C_1$ respectively. Let $x = AG$, $y = BG$ and $z = CG$. By Apollonius' theorem,

$$x^2 = \frac{2b^2 + 2c^2 - a^2}{9}, \quad y^2 = \frac{2c^2 + 2a^2 - b^2}{9}, \quad z^2 = \frac{2a^2 + 2b^2 - c^2}{9}.$$

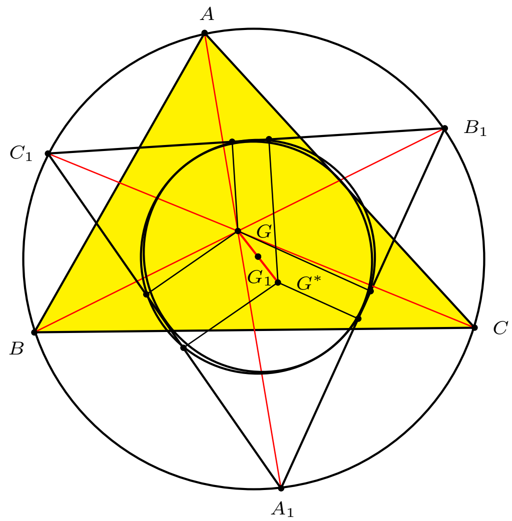

Figure 4

If $\mathscr{P}$ is the power of G relative to circumcircle of ABC, then

$$GA_1 = \frac{\mathscr{P}}{x}, \quad GB_1 = \frac{\mathscr{P}}{y}, \quad GC_1 = \frac{\mathscr{P}}{z}.$$

From the similarity of triangles GBC and GC_1B_1, we have

$$\frac{a_1}{a} = \frac{GC_1}{BG} = \frac{CG \cdot GC_1}{BG \cdot GC} = \frac{\mathscr{P}}{yz}.$$

From this we obtain a_1, and similarly b_1 and c_1:

$$a_1 = \frac{a\mathscr{P}}{yz}, \quad b_1 = \frac{b\mathscr{P}}{zx}, \quad c_1 = \frac{c\mathscr{P}}{xy}. \tag{7}$$

The homogeneous barycentric coordinates of G relative to $A_1B_1C_1$ are

$$\begin{aligned}
&\Delta GB_1C_1 : \Delta GC_1A_1 : \Delta GA_1B_1\\
=&\frac{\Delta GB_1C_1}{\Delta GBC} : \frac{\Delta GC_1A_1}{\Delta GCA} : \frac{\Delta GA_1B_1}{\Delta GAB}\\
=&\frac{GB_1 \cdot GC_1}{GB \cdot GC} : \frac{GC_1 \cdot GA_1}{GC \cdot GA} : \frac{GA_1 \cdot GB_1}{GA \cdot GB}\\
=&\frac{(BG \cdot GB_1)(CG \cdot GC_1)}{BG^2 \cdot CG^2} : \frac{(CG \cdot GC_1)(AG \cdot GA_1)}{CG^2 \cdot AG^2} : \frac{(AG \cdot GA_1)(BG \cdot GB_1)}{AG^2 \cdot BG^2}\\
=&\frac{\mathscr{P}^2}{y^2z^2} : \frac{\mathscr{P}^2}{z^2x^2} : \frac{\mathscr{P}^2}{x^2y^2}\\
=&x^2 : y^2 : z^2.
\end{aligned}$$

The isogonal conjugate of G (relative to triangle $A_1B_1C_1$) is the point

$$G^* = \frac{a_1^2}{x^2} : \frac{b_1^2}{y^2} : \frac{c_1^2}{z^2} = a^2 : b^2 : c^2.$$

We find the midpoint of GG^* by working with absolute barycentric coordinates. Since $x^2+y^2+z^2=\frac{1}{3}(a^2+b^2+c^2)$, and

$$3x^2+a^2=3y^2+b^2=3z^2+c^2=\frac{2(a^2+b^2+c^2)}{3},$$

we have for the first component,

$$\frac{1}{2}\left(\frac{x^2}{x^2+y^2+z^2}+\frac{a^2}{a^2+b^2+c^2}\right)=\frac{3x^2+a^2}{2(a^2+b^2+c^2)}=\frac{1}{3}.$$

Similarly, the other two components are also equal to $\frac{1}{3}$. It follows that the midpoint of GG^* is the centroid G_1 of $A_1B_1C_1$. As such, it is the center of the pedal circle of the points G and G^*, which are the foci of an inconic that has center and perspector G_1. This inconic is the Steiner inellipse of triangle $A_1B_1C_1$. □

3. A construction problem

Theorem 7 gives an elementary solution to a challenging construction problem in Altshiller-Court [1, p.292, Exercise 11]. The interest on this problem was rejuvenated by a recent message on the Hyacinthos group [2].

Problem 8. *Construct a triangle given, in position, the traces of its medians on the circumcircle.*

Solution. Let a given triangle $A_1B_1C_1$ be the circumcevian triangle of the (unknown) centroid G of the required triangle ABC. We construct the equilateral triangles $A_+B_1C_1$ and $A_-B_1C_1$ on the segment B_1C_1. Let G_1 be the centroid of $A_1B_1C_1$ and $r=\frac{1}{6}(A_1A_++A_1A_-)$. The circle $G_1(r)$ is the auxiliary circle of the Steiner inellipse, hence the pedal circle of G (one of the foci) relative to $A_1B_1C_1$. From the intersections of this circle with the sides of $A_1B_1C_1$ and the parallel from G_1 to the internal bisector of angle $A_+A_1A_-$ we determine the point G (two solutions). The second intersections of the circle with the lines A_1G, B_1G, C_1G give the points A, B, C.

References

[1] N. Altshiller-Court, *College Geometry*, second edition, Barnes and Noble, 1952; Dover reprint, 2007.

[2] Hyacinthos message 19264, September 14, 2010.

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