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Integer Triangles with $R/r = N$

Allan J. MacLeod

Abstract. We consider the problem of finding integer-sided triangles with R/r an integer, where R and r are the radii of the circumcircle and incircle respectively. We find that such triangles are relatively rare.

1. Introduction

Let ABC be a triangle with sides of integer length a, b, c. Possibly the two most fundamental circles associated with the triangle are the circumcircle which passes through the three vertices, and the incircle which has the three sides as tangents. The radii of these circles are denoted R and r respectively. If we look at the basic equilateral triangle with sides of length 1, it is clear that both circles will have their centers at the centroid of the triangle. Simple trigonometry gives $R = \frac{1}{\sqrt{3}}$ and $r = \frac{1}{\sqrt{12}}$ so that $\frac{R}{r} = 2$. It is interesting to speculate whether $\frac{R}{r}$ (a positive dimensionless quantity) can be an integer value for larger integers.

If we denote the distance between the centers of these two circles as d, then it is a standard result in triangle geometry that $d^2 = R(R-2r)$, so that $\frac{R}{r} \geq 2$. For the equilateral triangle, $d = 0$, and it is easy to prove the converse – if $d = 0$ then the triangle must be equilateral.

Basic trigonometry gives the formulae

$$R = \frac{abc}{4\Delta}, \qquad r = \frac{\Delta}{s} \tag{1}$$

with s being the semi-perimeter $\frac{1}{2}(a+b+c)$ and Δ the area. Thus,

$$\frac{R}{r} = \frac{abcs}{4\Delta^2}. \tag{2}$$

Now, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, so that

$$\frac{R}{r} = \frac{2abc}{(a+b-c)(a+c-b)(b+c-a)}. \tag{3}$$

If $\frac{R}{r} = N$, with N a strictly positive integer, we must find integers a, b, c such that

$$\frac{2abc}{(a+b-c)(a+c-b)(b+c-a)} = N, \tag{4}$$

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which bears a very strong resemblance to the integer representation problems in [1, 2]. In all cases, we look to express N as a ratio of two homogeneous cubics in 3 variables.

Expressing equation (4) as a single fraction, we derive the cubic

$$Na^3 - N(b+c)a^2 - (b^2N - 2bc(N+1) + c^2N)a + N(b+c)(b-c)^2 = 0. \quad (5)$$

We cannot get very far with this form, but we can proceed quickly if we replace c by $2s - a - b$.

$$2(4Ns - b(4N+1))a^2 - 2(b^2(4N+1) - 2bs(6N+1) + 8Ns^2)a + 8Ns(b-s)^2 = 0, \quad (6)$$

and it should be noted that we only need to consider rational a, b, c and scale to integer values, since this would not affect the value of $\frac{R}{r}$.

Since we want rational values for a, this quadratic must have a discriminant which is a rational square, so that there must be a rational d with

$$d^2 = (4N+1)^2b^4 - 4(2N+1)(4N+1)b^3s + 4(4N^2+8N+1)b^2s^2 - 16Nbs^3. \quad (7)$$

Define $d = \frac{s^2y}{4N+1}$ and $g = \frac{sx}{4N+1}$, giving the quartic

$$y^2 = x^4 - 4(2N+1)x^3 + 4(4N^2+8N+1)x^2 - 16N(4N+1)x, \quad (8)$$

which can be transformed to an equivalent elliptic curve by a birational transformation.

We find the curve to be

$$E_N : \; v^2 = u^3 + 2(2N^2 - 2N - 1)u^2 + (4N+1)u \quad (9)$$

with the transformation

$$\frac{b}{s} = \frac{v - (4N + 1 - (2N+1)u)}{(u-1)(4N+1)}. \quad (10)$$

As an example, consider the case of $N = 7$, so that E_7 is the curve $v^2 = u^3 + 166u^2 + 29u$. It is moderately easy to find the rational point $(u, v) = \left(\frac{29}{169}, \frac{6902}{2197}\right)$ which lies on the curve. This gives $\frac{b}{s} = \frac{63}{65}$, and the equation for a is $-14a^2 + 938a + 14560 = 0$, from which we find the representation $(a, b, c) = (-13, 63, 80)$, clearly not giving a real triangle.

2. Elliptic Curve Properties

We have shown that integer solutions to equation (4) are related to rational points on the curves E_N defined in equation (9). The problem is that equation (4) can be satisfied by integers which could be negative as in the representation problems of [1, 2].

To find triangles for the original form of the problem, we must enforce extra constraints on a, b, c. To investigate the effect of this, we must examine the properties of the curves.

We first note that the discriminant of the curve is given by

$$D = 256N^3(N-2)(4N+1)^2$$

so that the curve is singular for $N = 2$, which we now exclude from possible values, having seen before that the equilateral triangle gives $N = 2$.

Given the standard method of addition of rational points on an elliptic curve, see [3], the set of rational points forms a finitely-generated group. The points of finite-order are called torsion points, and we look for these first. The point at infinity is considered the identity of the group. The form of E_N implies, by the Nagell-Lutz theorem, that the coordinates of the torsion points are integers.

The points of order 2 are integer solutions of

$$u^3 + 2(2N^2 - 2N - 1)u^2 + (4N + 1)u = 0$$

and it is easy to see that the roots are $u = 0$ and $u = 2N+1-2N^2 \pm \sqrt{N(N-2)}$. The latter two are clearly irrational and negative for N a positive integer, so there is only one point of order 2.

The fact that there are 3 real roots implies that the curve consists of two components - an infinite component for $u \geq 0$ and a closed finite component (usually called the “egg”).

Points of order 2 allow one to look for points of order 4, since if $P = (j, k)$ has order 4, $2P$ must have order 2. For a curve of the form given by (9), the u-coordinate of $2P$ must be of the form $\frac{(j^2-4N-1)^2}{4k^2}$. Thus, we must have $j^2 = 4N + 1 = (2t + 1)^2$, so $N = t^2 + t$. Substituting these values into (9), we see that we get a rational point only if $t(t + 2)$ is an integer square, which never happens. There are thus no points of order 4.

Points of order 3 are points of inflexion of the curve. Simple analysis shows that there are points of inflexion at $(1,\ \pm 2N)$. We can also use the doubling formula to show that there are 2 points of order 6, namely $(4N + 1,\ \pm 2N(4N + 1))$.

The presence of points of orders 2, 3, 6, together with Mazur’s theorem on possible torsion structures, shows that the torsion subgroup must be isomorphic to $\mathbb{Z}_6$ or $\mathbb{Z}_{12}$.

The latter possibility would need a point P of order 12, with $2P$ of order 6, and thus an integer solution of

$$\frac{(j^2 - (4N + 1))^2}{4k^2} = 4N + 1$$

implying that $N = t^2+t$. Substituting into this equation, we get an integer solution if either $t^2 - 1$ or $t(t + 2)$ are integer squares - which they are not, unless $N = 2$ which has been excluded previously.

Thus the torsion subgroup is isomorphic to $\mathbb{Z}_6$, with finite points $(0, 0)$, $(1, \pm 2N)$, $(4N + 1,\ \pm 2N(4N + 1))$. Substituting these points into the $\frac{b}{s}$ transformation formula leads to $\frac{b}{s}$ being 0, 1 or undefined, none of which lead to a practical solution of the problem.

Thus, we must look at the second type of rational point - those of infinite order.

3. Practical Solutions

If there are rational points of infinite order, Mordell's theorem implies that there are r generator points $G_1, \ldots, G_r$, such that any rational point P can be written

$$P = T + n_1 G_1 + \cdots + n_r G_r \tag{11}$$

with T one of the torsion points, and $n_1, \ldots, n_r$ integers. The value r is called the *rank* of the curve.

Unfortunately, there is no simple method of determining firstly the rank, and then the generators. We used a computational approach to estimate the rank using the Birch and Swinnerton-Dyer conjecture. A useful summary of the computations involved can be found in the paper of Silverman [4].

Applying the calculations to a range of values of N, we find several examples of curves with rank greater than zero, mostly with rank 1. A useful byproduct of the calculations in the rank 1 case is an estimate of the height of the generator, where the height gives an indication of number of digits in the rational coordinates. For curves of rank greater than 1 and rank 1 curves with small height, we can reasonably easily find generators. However, when we backtrack the calculations to solutions of the original problem, we hit a significant problem.

The elliptic curve generators all give solutions to equation (4), but for the vast majority of N values, these include at least one negative value of a, b, c. Thus we find extreme difficulty in finding real triangles with strictly positive sides. In fact, for $3 \leq N \leq 99$, there are only 2 values of N where this occurs, at N = 26 and N = 74.

To investigate this problem, consider the quadratic equation (6), but written as

$$a^2 + (b - 2s)a + \frac{4Ns(b-s)^2}{4Ns - b(4N+1)}. \tag{12}$$

The sum of the roots of this is clearly $2s - b$, but since $a + b + c = 2s$, this means that the roots of this quadratic are in fact a and c. Positive triangles thus require $s > 0$, $b > 0$, $2s - b > 0$ and $4Ns - b(4N+1) > 0$, all of which reduce to

$$0 < \frac{b}{s} < \frac{4N}{4N+1}. \tag{13}$$

Looking at equation (10), we see that the analysis splits first according as $u > 1$ or $u < 1$. Consider first $u > 1$, so that, for $\frac{b}{s} > 0$ we need $v > 4N + 1 - (2N+1)u$. The line $v = 4N + 1 - (2N+1)u$ meets E_N in only two points, $(1,\ 2N)$ and $(4N+1,\ -2N(4N+1))$ with the line actually being a tangent to the curve at the latter point. Thus, if $u > 1$ we need $v > 0$ to give points on the curve with $\frac{b}{s} > 0$.

For the second half of the inequality with $u > 1$, we need $v < 1 + (2N-1)u$. The line $v = 1 + (2N-1)u$ meets E_N only at $(1,\ 2N)$, so none of the points with $u > 1$, $v > 0$ are satisfactory. Thus to satisfy (13) we must look in the range $u < 1$. Firstly, in the interval $[0,\ 1)$, we have $\frac{b}{s} > 0$, since the numerator and denominator of (10) are negative. For the second half, however, we need $v > 1 + (2N-1)u$, but the previous analysis shows this cannot happen.

Thus, the only possible way of achieving real-world triangles is to have points on the egg component. From the previous analysis it is clear that any point on the egg leads to $\frac{b}{s} > 0$. For the other part of (10), we must consider where the egg lies in relation to the line $v = 1 + (2N - 1)u$. Since the line only meets E_N at $u = 1$, the entire egg either lies above or below the line. The line meets the u-axis at $u = \frac{-1}{2N-1}$, and the extreme left-hand end of the egg is at $u = 2N + 1 - 2N^2 - \sqrt{N(N-2)}$ which is less than -1 for $N \geq 3$. Thus the entire egg lies above the line so $v > 1 + (2N - 1)u$ and so (10) is satisfied.

Thus, we get a real triangle if and only if (u, v) is a rational point on E_N with $u < 0$.

Consider now the effect of the addition $P + T = Q$ where P lies on the egg and T is one of the torsion points. All of the five finite torsion points lie on the infinite component. Since E_N is symmetrical about the u-axis, P, T, $-Q$ all lie on a straight line, and since the egg is a closed convex region, simple geometry implies that $-Q$ and hence Q must lie on the egg. Similarly, if P lies on the infinite component, then Q must also lie on the infinite component.

Geometry also shows that $2P$ must lie on the infinite component irrespective of where P lies.

TABLE 1
Integer sided triangles with $\frac{R}{r} = N$

N	a	b	c
2	1	1	1
26	11	39	49
74	259	475	729
218	115	5239	5341
250	97	10051	10125
314	177487799	55017780825	55036428301
386	1449346321141	2477091825117	3921344505997
394	12017	2356695	2365193
458	395	100989	101251
586	3809	18411	22201
602	833	14703	15523
634	10553413	1234267713	1243789375
674	535	170471	170859
746	47867463	6738962807	6782043733
778	1224233861981	91266858701995	92430153628659
866	3025	5629	8649

This shows that if the generators of E_N all lie on the infinite component then there is NO rational point on the egg, and hence no real triangle.

We have, for $1 \leq N \leq 999$, found 16 examples of integer triangles, which are given in Table 1. There are probably more to be found, but these almost certainly come from rank 1 curves with generators having a large height and therefore difficult to find. I am not sure that the effort to find more examples is worthwhile.

A close look at the values of N shows that they all satisfy $N \equiv 2 \pmod 8$. Is this always true? If so, why?

4. Nearly-equilateral Triangles

As mentioned in the introduction, if we have an equilateral triangle with side 1 then $N = 2$. This suggests investigating how close we can get to $N = 2$ with non-equilateral integer triangles. We thus investigate

$$\frac{R}{r} = 2 + \frac{1}{M}$$

with M a positive integer.

We can proceed in an almost identical manner to before, so the precise details are left out, but we use the same names for the lengths and semi-perimeter. The problem is equivalent to finding rational points on the elliptic curve

$$F_M: \; v^2 = u^3 + (6M^2 + 12M + 4)u^2 + (9M^4 + 4M^3)u \tag{14}$$

with

$$\frac{b}{s} = \frac{v - (9M^3 + 4M^2 - (5M + 2)u)}{(u - M^2)(9M + 4)}.$$

The curves F_M have an obvious point of order 2 at $(0,\ 0)$, and can be shown to have points of order 3 at $(M^2,\ \pm 2M^2(2M + 1))$ and order 6 at $(9M^2 + 4M,\ \pm 2M(2M + 1)(9M + 4))$. In general these are the only torsion points, none of which lead to a practical solution.

For $M = 2k^2 + 2k$, however, with $k > 0$, the elliptic curve has 3 points of order 2, which lead to the isosceles triangles with $a = 2k$, $b = c = 2k + 1$. This shows that we can get as close to $N = 2$ as we like with an isosceles triangle. If we reverse the process and start with an isosceles triangle, we can show that M must be of the form $2k^2 + 2k$.

The curves F_M have two components, the infinite one and the egg, and, as before, we can show that real triangles can only come from rational points on the egg. Numerical experiments show that these are much more common than for E_N.

As an example, the analysis for $M = 2009$ leads to an elliptic curve of rank 1, with generator of height 33.94, and finally to sides

$$\begin{aligned} a &= 89378043697968459026749303724134010455925561 6 \\ b &= 877646641306278516279522874129152375921514449 \\ c &= 885805950860882235231118974654122876065715369 \end{aligned}$$

with angles $A = 60.9045^\circ$, $B = 59.0969^\circ$ and $C = 59.9987^\circ$.

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Allan J. MacLeod: Department of Mathematics and Statistics, University of the West of Scotland, High St., Paisley, Scotland. PA1 2BE.

E-mail address: allan.macleod@uws.ac.uk