Forum Geometricorum
Volume 11 (2011) 27–51.

FORUM GEOM
ISSN 1534-1178

Triangles with Given Incircle and Centroid

Paris Pamfilos

Abstract. In this article we study the set of triangles sharing a common incircle and a common centroid. It is shown that these triangles have their vertices on a conic easily constructible from the given data. It is also shown that the circumcircles of all these triangles are simultaneously tangent to two fixed circles, hence their centers lie also on a conic.

1. Introduction

The object of study here is on triangles sharing a common incircle and centroid. It belongs to the wider subject of (one-parameter) families of triangles, initiated by the notion of *poristic* triangles, which are triangles sharing a common incircle and also a common circumcircle [7, p.22]. There is a considerable literature on poristic triangles and their variations, which include triangles sharing the same circumcircle and Euler circle [16], triangles sharing the same incircle and Euler circle [6], or incircle and orthocenter [11, vol.I, p.15], or circumcircle and centroid [8, p.6]. The starting point of the discussion here, and the content of §2, is the exploration of a key configuration consisting of a circle c and a homothety f. This configuration generates in a natural way a homography H and the conic $c^* = H(c)$. §3 looks a bit closer at homography H and explores its properties and the properties of the conic c^*. §4 uses the results obtained previously to the case of the incircle and the homothety with ratio $k = -2$ centered at the centroid G of the triangle of reference to explore the kind of the conic $c^{**} = f(c^*)$ on which lie the vertices of all triangles of the poristic system. §5 discusses the locus of circumcenters of the triangles of the poristic system locating, besides the assumed fixed incircle c, also another fixed circle c_0, called *secondary*, to which are tangent all Euler circles of the system. Finally, §6 contains miscellaneous facts and remarks concerning the problem.

2. The basic configuration

The basic configuration of this study is a circle c and a homothety f centered at a point G and having a ratio $k \neq 1$. In the following the symbol p_X throughout denotes the polar line of point X with respect to c, this line becoming tangent to c when $X \in c$. Lemma 1 handles a simple property of this configuration.

Lemma 1. *Let c be a circle and G a point. Let further D be a point on the circle and L a line parallel to the tangent p_D at D and not intersecting the circle. Then there is exactly one point A on L such that the tangents $\{t, t'\}$ to c from A and line AG intersect on p_D equal segments $BM = MC$.*

Publication Date: February 25, 2011. Communicating Editor: Paul Yiu.

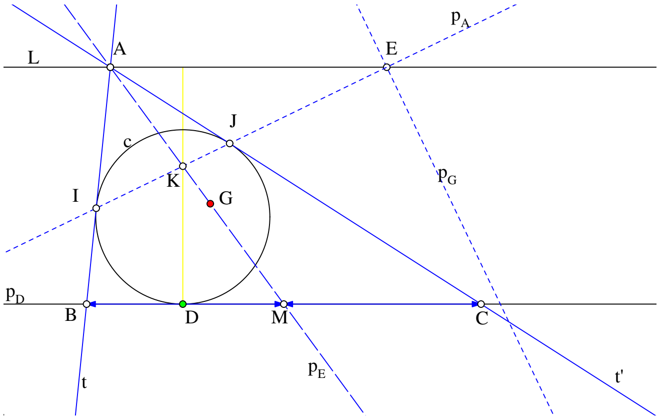

Figure 1. A basic lemma

Figure 1 suggests a proof. Let $\{E, A\}$ be the intersections of lines $E = L \cap p_G$ and $A = L \cap p_E$. Draw from A the tangents $\{t = AI, t' = AJ\}$ to c intersecting line p_D respectively at B and C. Let also M be the intersection $M = p_D \cap AG$. Then $BM = MC$. This follows immediately from the construction, since by the definitions made the pencil of lines $A(I, J, K, E)$ at A is harmonic. Thus, this pencil defines on p_D a harmonic division and, since the intersection of lines $\{p_D, L\}$ is at infinity, M is the middle of BC. Conversely, the equality $BM = MC$ implies that the pencil of four lines AB, AC, AM, L is harmonic. If E is the intersection $E = L \cap p_A$ then p_E coincides with AM and $G \in p_E \Rightarrow E \in p_G$. Thus A is uniquely determined as $A = L \cap p_E$, where $E = L \cap p_G$.

Remarks. (1) Point K, defined as the intersection $K = AG \cap p_A$, is on the diameter of c through D. This follows from the fact that K is on the polar of A and also on the polar of E. Hence its polar is $p_K = AE$.

(2) In other words the lemma says that there is exactly one triangle with vertex $A \in L$ such that the circumscribed to c triangle ABC has AG as median for side BC. Consequently if this happens also for another vertex and BG is the median also of CA then G coincides with the centroid of ABC and the third line CG is also a median of this triangle.

The next proposition explores the locus of point $M = M(D)$ uniquely determined from D in the previous lemma by letting D vary on the circle and taking the parallel L to p_D to be identical with the homothetic to p_D line $L_D = f(p_D)$ (see Figure 2).

Proposition 2. *Consider a circle c and a homothety f with ratio $k \neq 1$ and center at a point G. For each point $D \in c$ let $L_D = f(p_D)$ be the homothetic image of p_D and $E = L_D \cap p_G$, $M = p_E \cap p_D$. Then M describes a conic c^* as D moves on the circle.*

The proof of the proposition is given using homogeneous cartesian coordinates with the origin set at G and the x-axis identified with line IG, where I is the center

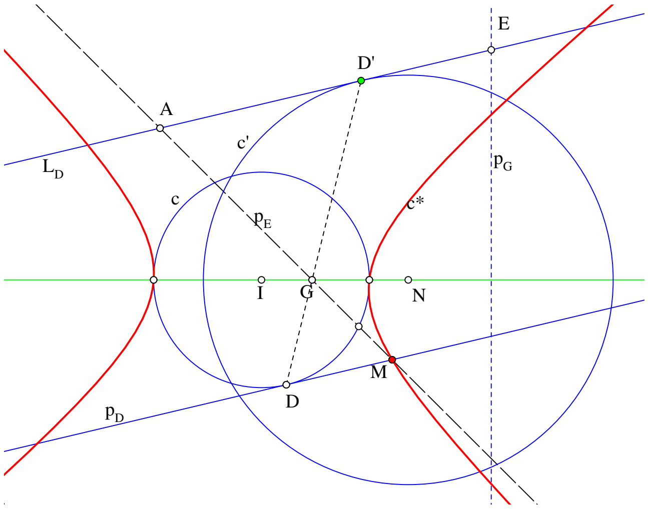

Figure 2. Locus of M=$p_D \cap p_E$

of the given circle c. In such a system circle c is represented by the equation

$$x^2+y^2-2uxz+(u^2-r^2)z^2=0 \Longleftrightarrow \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Here $(u,0)$ are the corresponding cartesian coordinates of the center and r is the radius of the circle. The polar p_G of point $G(0,0,1)$ is computed from the matrix and is represented by equation

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Longleftrightarrow -ux+(u^2-r^2)z=0.$$

The homothety f is represented by the matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Hence the tangent p_D at $D(x,y,z)$ and its image L_D under f have correspondingly coefficients

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & -u \\ 0 & 1 & 0 \\ -u & 0 & u^2-r^2 \end{pmatrix} = \begin{pmatrix} x-uz & y & -ux+(u^2-r^2)z \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{1}{k}(x-uz) & \frac{1}{k}y & -ux+(u^2-r^2)z \end{pmatrix}.$$

The homogeneous coordinates of $E = p_G \cap L_D$ are then computed through the vector product of the coefficients of these lines which is:

$$\left(-\frac{1}{k}y(u^2-r^2), \quad u(-ux+(u^2-r^2)z)+\frac{1}{k}(x-uz)(u^2-r^2), \quad -\frac{u}{k}y\right).$$

The coefficients of the polar p_E are then seen to be

$$
\begin{aligned}
&\left(-\tfrac{1}{k}y(u^2-r^2) \quad u(-ux+(u^2-r^2)z)+\tfrac{1}{k}(x-uz)(u^2-r^2) \quad -\tfrac{u}{k}y\right)\begin{pmatrix}1 & 0 & -u\\ 0 & 1 & 0\\ -u & 0 & u^2-r^2\end{pmatrix}\\
&= \left(-\tfrac{1}{k}y(u^2-r^2)+\tfrac{u^2}{k}y \quad u(-ux+(u^2-r^2)z)+\tfrac{1}{k}(x-uz)(u^2-r^2) \quad 0\right)\\
&= \left(-\tfrac{1}{k}yr^2 \quad u(-ux+(u^2-r^2)z)+\tfrac{1}{k}(x-uz)(u^2-r^2) \quad 0\right)\\
&\cong \left(yr^2 \quad (u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z \quad 0\right).
\end{aligned}
$$

The homogeneous coordinates of $M = p_E \cap p_D$ are again computed through the vector product of the coefficients of the corresponding lines:

$$
\begin{aligned}
&\begin{pmatrix} x-uz \\ y \\ -ux+(u^2-r^2)z \end{pmatrix} \times \begin{pmatrix} yr^2 \\ (u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z \\ 0 \end{pmatrix}\\
&= \begin{pmatrix} ((u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z)(ux-(u^2-r^2)z) \\ (-yr^2)(ux-(u^2-r^2)z) \\ (-ux+(u^2-r^2)z)(r^2z+u(1-k)(uz-x)) \end{pmatrix}\\
&\cong \begin{pmatrix} (u^2(1-k)-r^2)x+u(u^2-r^2)(k-1)z \\ -yr^2 \\ u(1-k)x-(r^2+u^2(1-k))z \end{pmatrix}\\
&= \begin{pmatrix} u^2(1-k)-r^2 & 0 & u(u^2-r^2)(k-1) \\ 0 & -r^2 & 0 \\ u(1-k) & 0 & -(r^2+u^2(1-k)) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}
$$

The matrix product in the last equation defines a homography H and shows that the locus of points M is the image $c^* = H(c)$ of the circle c under this homography. This completes the proof of Proposition 2

Remarks. (3) The properties of the conic c^* are of course tightly connected to the properties of the homography H appearing at the end of the proposition and denoted by the same letter

$$
H = \begin{pmatrix} u^2(1-k)-r^2 & 0 & u(u^2-r^2)(k-1) \\ 0 & -r^2 & 0 \\ u(1-k) & 0 & -(r^2+u^2(1-k)) \end{pmatrix}.
$$

These properties will be the subject of study in the next section.

(4) Composing with the homothety f we obtain the locus of $A = f(M)$ which is the homothetic conic $c^{**} = f(c^*) = (f \circ H)(c)$. It is this conic rather, than c^*, that relates to our original problem. Since though the two conics are homothetic, work on either leads to properties for both of them.

(5) Figure 2 underlines the symmetry between these two conics. In it c' denotes the homothetic image $c' = f(c)$ of c. Using this circle instead of c and the inverse homothety $g = f^{-1}$ we obtain a basic configuration in which the roles of c^* and c^{**} are interchanged.

(6) If $D \in c^*$ and point $A = f(M)$ is outside the circle c then triangle ABC constructed by intersecting p_D with the tangents from A has, according to the lemma, line AG as median. Inversely, if a triangle ABC is circumscribed in c and has its median AM passing through G and divided by it in ratio $\frac{GA}{GM} = k$, then, again according to the lemma, it has M on the conic c^*. In particular if a triangle ABC is circumscribed in c and has two medians passing through G then it has all three of them passing through G, the ratio $k = -2$ and the middles of its sides are points of the conic c^*.

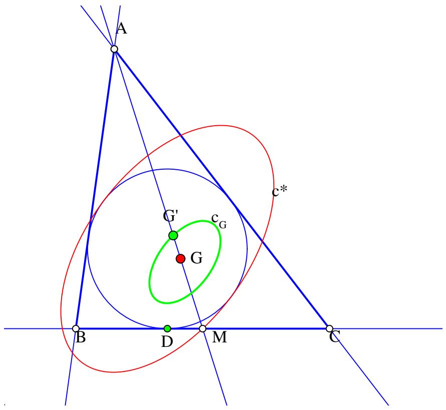

Figure 3. Locus of centroids G'

(7) The previous remark implies that if ratio $k \neq -2$ then every triangle ABC circumscribed in c and having its median AM passing through G has its other medians intersecting AG at the same point $G' \neq G$. Figure 3 illustrates this remark by displaying the locus of the centroid G' of triangles ABC which are circumscribed in c, their median AM passes through G and is divided by it in ratio k but later does not coincide with the centroid (*i.e.*, $k \neq -2$). It is easily seen that the locus of the centroid G' in such a case is part of a conic c_G which is homothetic to c^* with respect to G and in ratio $k' = \frac{2+k}{3}$. Obviously for $k = -2$ this conic collapses to the point G and triangles like ABC circumscribed in c have all their vertices on c^{**}.

(8) Figure 4 shows a case in which the points $A \in c^{**}$ which are outside c fall into four connected arcs of a hyperbola. In this example $k = -2$ and, by the symmetry of the condition, it is easily seen that if a point A is on one of these arcs then the other vertices of ABC are also on respective arcs of the same hyperbola. The fact to notice here is that conics c^* and c^{**} are defined directly from the basic configuration consisting of the circle c and the homothety f. It is though not possible for every point of c^{**} to be vertex of a triangle circumscribed in c and with centroid at G. I summarize the results obtained so far in the following proposition.

Proposition 3. *All triangles ABC sharing the same incircle c and centroid G have their side-middles on a conic c^* and their vertices on a conic $c^{**} = f(c^*)$ homothetic to c^* by the homothety f centered at the centroid with ratio $k = -2$.*

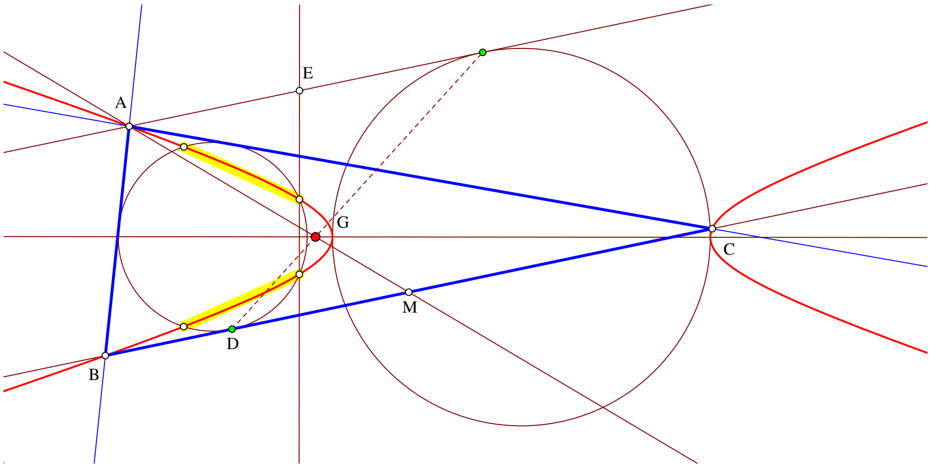

Figure 4. Arcs: $A = f(M) \in c^{**}$ and A outside c

The proof follows from the previous remarks and the fact that in this setting the (anti-)homothety f has center at G and ratio $k = -2$. By Lemma 5 below, the median AM of side BC coincides with the polar p_E of point $E = L \cap p_G$, where L is the parallel to BC from A. The conic c^* generated by point M, as in Lemma 5 , passes now through the middle of BC but also through the middles of the other sides, since the configuration, as already remarked, is independent of the preferred vertex A and depends only on the circle c and the homothety f (see Figure 5).

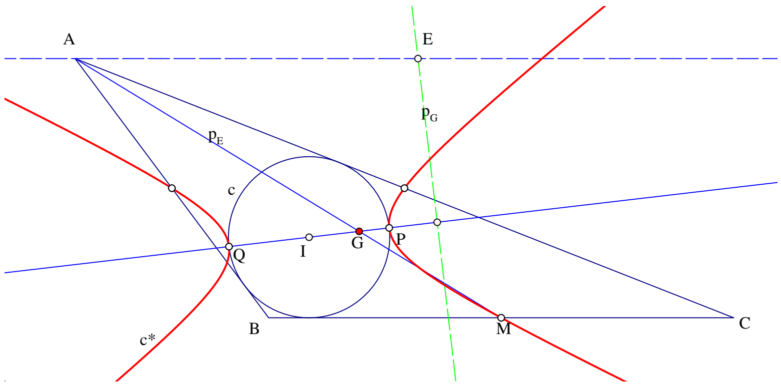

Figure 5. Conic c^*

The conic $c^{**} = f(c^*)$ which is homothetic to c^* will pass through all three vertices of triangle ABC. Since the same argument can be applied to any triangle sharing with ABC the same incircle c and centroid G it follows that all these triangles have their vertices on this conic. Figure 6 displays two triangles sharing the same incircle c and centroid G and having their vertices on conic c^{**} in a case in which this happens to be elliptic. Note that in this case the entire conic consists of vertices of triangles circumscribing c and having their centroid at G. This follows from the fact that in this case c^{**} does not intersect c (see §4) and by applying then the Poncelet's porism [3, p.68].

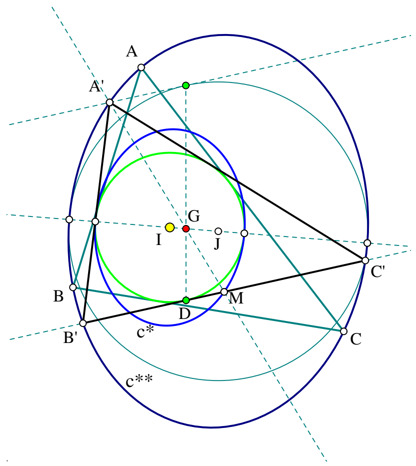

Figure 6. The two conics c^* and c^{**}

Before going further into the study of these conics I devote the next section to a couple of remarks concerning the homography H.

3. The homography H

Proposition 4. *The homography H defined in the previous section is uniquely characterized by its properties:*
(i) *H fixes points $\{P, Q\}$ which are the diameter points of c on the x-axis,*
(ii) *H maps points $\{S, T\}$ to $\{S', T'\}$. Here $\{S, T\}$ are the diameter points of c on a parallel to the y-axis and $\{S', T'\}$ are their orthogonal projections on the diameter of $c' = f(c)$ which is parallel to the y-axis.*

The proof of the proposition (see Figure 7) follows by applying the matrix to the coordinate vectors of these points which are $P(u+r, 0, 1)$, $Q(u-r, 0, 1)$, $S(u, r, 1)$, $T(u, -r, 1)$,
$S'(ku, r, 1)$ and $T'(ku, -r, 1)$, and using the well-known fact that a homography is uniquely determined by prescribing its values at four points in general position [2, Vol.I, p.97].

Remarks. (1) By its proper definition, line XX' for $X \in c$ and $X' = H(X) \in c^* = H(c)$ is tangent to circle c at X.

(2) The form of the matrix H implies that the conic c^* is symmetric with respect to the x-axis and passes through points P and Q having there tangents coinciding respectively with the tangents of circle c. Thus P and Q are vertices of c^* and c coincides with the *auxiliary* circle of c^* if this is a hyperbola. In the case c^* is an ellipse, by the previous remark, follows that it lies entirely outside c hence later is the maximal circle inscribed in the ellipse.

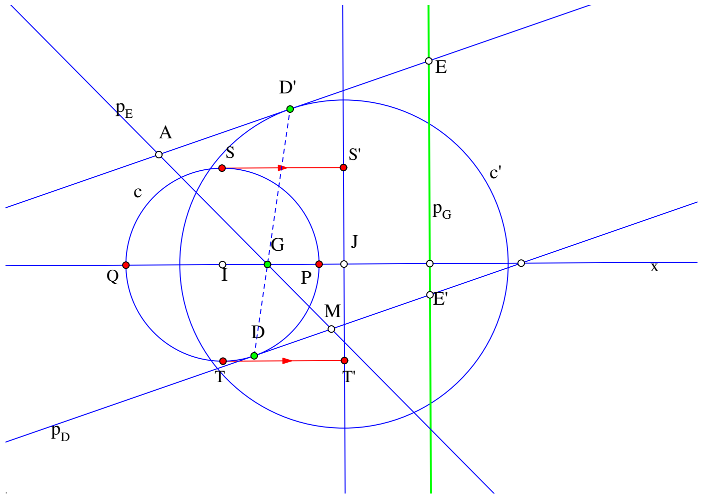

Figure 7. Homography $H : \{P, Q, S, T\} \mapsto \{P, Q, S', T'\}$

(3) The kind of c^* depends on the location of the line L_0 mapped to the line at infinity by H.

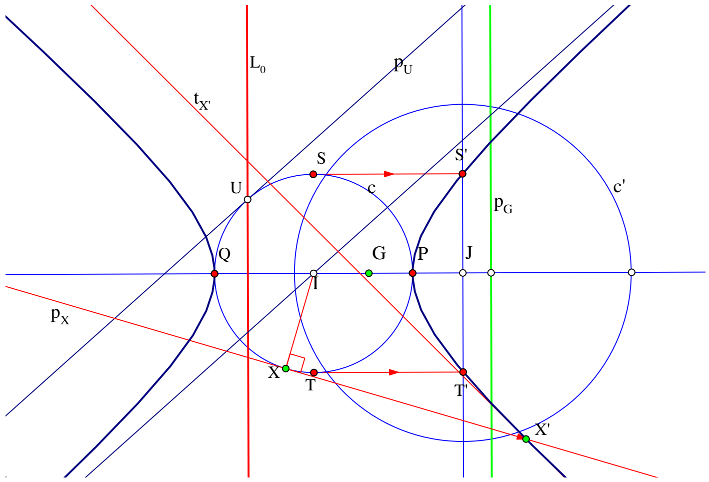

Figure 8. XX' is tangent to c

This line is easily determined by applying H to a point (x, y, z) and requiring that the resulting point (x', y', z') has $z' = 0$, thus leading to the equation of a line parallel to y-axis:

$$u(1-k)x - (r^2 + u^2(1-k))z = 0.$$

The conic c^*, depending on the number n of intersection points of L_0 with circle c, is a hyperbola $(n = 2)$, ellipse $(n = 0)$ or becomes a degenerate parabola consisting of the pair of parallel tangents to c at $\{P, Q\}$.

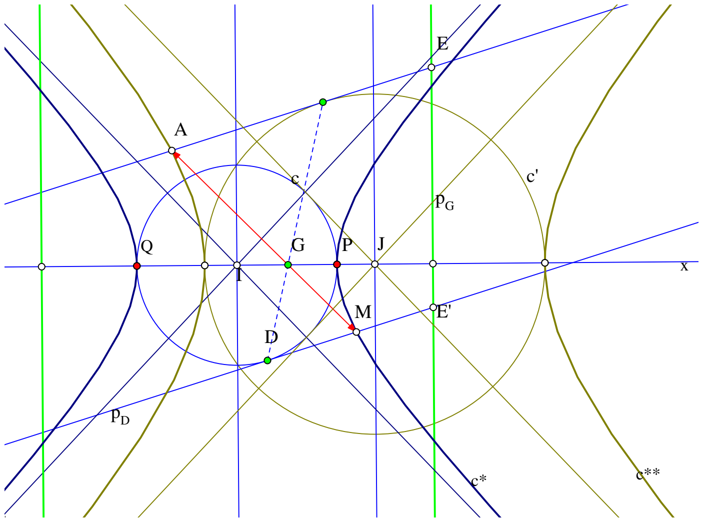

Figure 9. Homothetic conics c^* and c^{**}

(4) The tangent p_X of the circle c at X maps via H to the tangent $t_{X'}$ at the image point $X' = H(X)$ of c^*. In the case c^* is a hyperbola this implies that each one of its asymptotes is parallel to the tangent p_U of c at an intersection point $U \in c \cap L_0$ (see Figure 8).

(5) Figure 9 displays both conics c^* and $c^{**} = f(c^*)$ in a case in which these are hyperbolas and suggests that circle c is tangent to the asymptotes of c^{**}. This is indeed so and is easily seen by first observing that H maps the center I of c to the center J of $c' = f(c)$. In fact, line ST maps by H to $S'T'$ (see Figure 10) and line PQ is invariant by H. Thus $H(I) = J$. Thus all lines through I map under H to lines through J. In particular the antipode U' of U on line IU maps to a point $H(U')$ on the tangent to U' and U maps to the point at infinity on p_U. Thus line UU' maps to $U'J$ which coincides with $p_{U'}$ and is parallel to the asymptote of c^*. Since c^* and c^{**} are homothetic by f line $U'J$ is an asymptote of c^{**} thereby proving the claim.

(6) The arguments of the last remark show that line L_0 is the symmetric with respect to the center I of c (see Figure 10) of line $U'V'$ which is the polar of point J with respect to circle c. They show also that the intersection point K of $U'V'$ with the x-axis is the image via H of the axis point at infinity. An easy calculation using the matrices of the previous section shows that these remarks about the symmetry of $\{UV, p_J\}$ and the location of K is true also in the cases in which J is inside c and there are no real tangents from it to c.

(7) The homography H demonstrates a remarkable behavior on lines parallel to the coordinate axes (see Figure 11). As is seen from its matrix it preserves the point at infinity of the y-axis hence permutes the lines parallel to this axis. In particular, the arguments in (5) show that the parallel to the y-axis from I is simply orthogonally projected onto the parallel through point J. More generally points X moving on a parallel L to the y-axis map via H to points $X' \in L' = H(L)$,

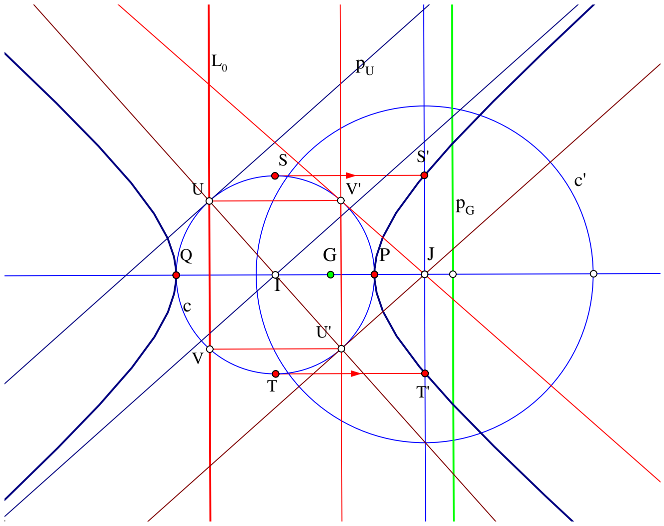

Figure 10. Location of asymptotes

such that the line XX' passes through a point X_L depending only on L and being harmonic conjugate with respect to $\{P, Q\}$ to the intersection X_0 of L with the x-axis.

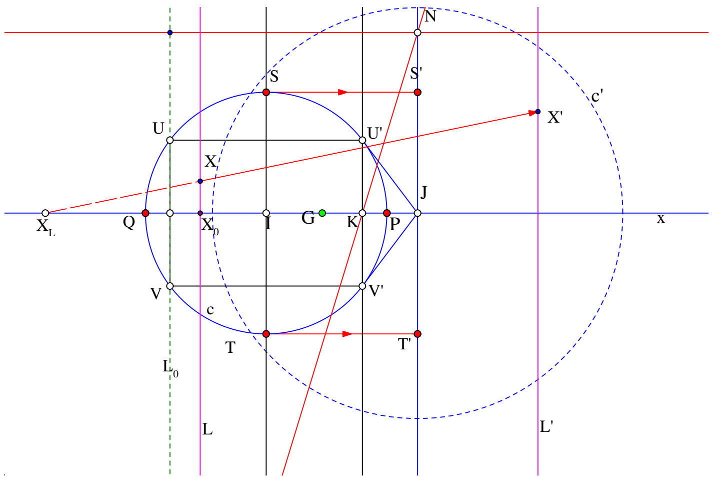

Figure 11. Mapping parallels to the axes

(8) Regarding the parallels to the x-axis and different from it their images via H are lines passing through K and also through their intersection N with the parallel to y-axis through J (see Figure 11). The x-axis itself is invariant under H and the action of this map on it is completely determined by the triple of points $\{P, Q, I\}$ and their images $\{P, Q, J\}$. Next lemma and its corollary give an insight into the difference of a general ratio $k \neq -2$ from the centroid case, in which $k =$

-2, by focusing on the behavior of the tangents to c from $A = f(M)$ and their intersections with the variable tangent p_D of circle c.

Lemma 5. *Consider a hyperbola c^* and its auxiliary circle c. Draw two tangents $\{t, t'\}$ parallel to the asymptotes intersecting at a point J. Then every tangent p to the auxiliary circle intersects lines $\{t, t'\}$ correspondingly at points $\{Q, R\}$ and the conic at two points $\{S, T\}$ of which one, S say, is the* middle *of QR.*

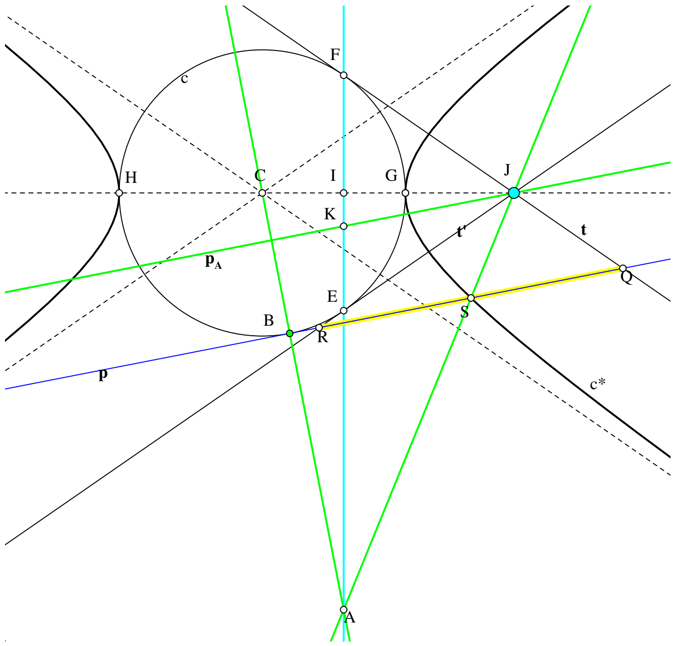

Figure 12. Hyperbola property

The proof starts by defining the polar FE of J with respect to the conic c^*. This is simultaneously the polar of J with respect to circle c since $(G, H, I, J) = -1$ are harmonic with respect to either of the curves (see Figure 12). Take then a tangent of c at B as required and consider the intersection point A of CB with the polar EF. The polar p_A of A with respect to the circle passes through J (by the reciprocity of relation pole-polar) and is parallel to tangent p. Besides $(E, F, K, A) = -1$ build a harmonic division, thus the pencil of lines at $J : J(E, F, K, A)$ defines a harmonic division on every line it meets. Apply this to the tangent p. Since JK is parallel to this tangent, S is the harmonic conjugate with respect to $\{R, Q\}$ of the point at infinity of line p. Hence it is the middle of RQ.

Corollary 6. *Under the assumptions of Lemma 5 , in the case conic c^* is a hyperbola, point $M = c^* \cap p_D$ is the common middle of segments $\{NO, KL, BC\}$ on the tangent p_D of circle c. Of these segments the first NO is intercepted by the asymptotes of c^{**}, the second KL is intercepted by the conic c^{**} and the third BC is intercepted by the tangents to c from A.*

The proof for the first segment NO follows directly from Lemma 5. The proof for the second segment KL results from the well known fact [4, p.267], according to which KL and NO have common middle for every secant of the hyperbola. The proof for the third segment BC follows from the definition of A and its properties as these are described by Lemma 1.

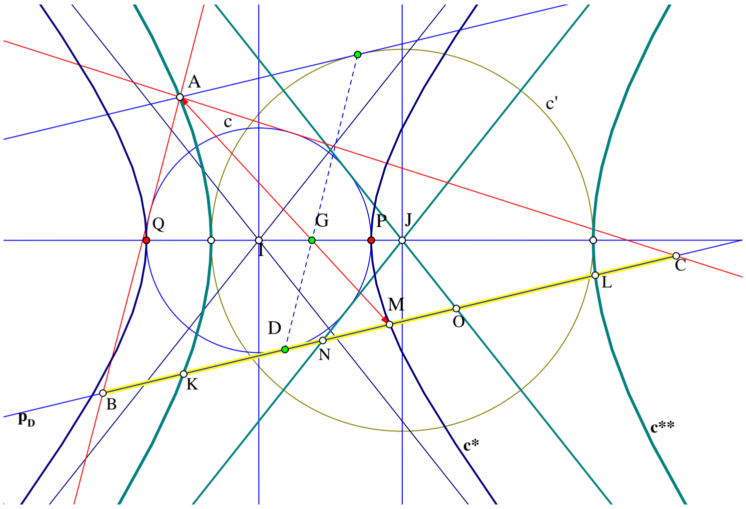

Figure 13. Common middle M

Remark. (9) When the homothety ratio $k = -2$ point B coincides with K (see Figure 13) and point L coincides with C. Inversely if $B \equiv K$ and $C \equiv L$ are points of c^{**} then the corresponding points $f^{-1}(B) \in AC$ and $f^{-1}(C) \in AB$ are middles of the sides, $k = -2$ and G is the centroid of ABC. It even suffices one triangle satisfying this identification of points to make this conclusion.

4. The kind of the conic c^*

In this section I examine the kind of the conic c^* by specializing the remarks made in the previous sections for the case of ratio $k = -2$, shown equivalent to the fact that there is a triangle circumscribed in c having its centroid at G, its side-middles on conic c^* and its vertices on $c^{**} = f(c^*)$. First notice that circle $c' = f(c)$ is the Nagel circle ([13]), its center $J = f(I)$ is the *Nagel* point ([10, p. 8]), its radius is twice the radius r of the incircle and it is tangent to the circumcircle. Line IG is the *Nagel* line of the triangle ([15]). The homography H defined in the second section obtains in this case $(k = -2, u = GI)$ the form.

$$H = \begin{pmatrix} 3u^2 - r^2 & 0 & -3u(u^2 - r^2) \\ 0 & -r^2 & 0 \\ 3u & 0 & -(r^2 + 3u^2) \end{pmatrix}.$$

As noticed in §2 the line L_0 sent to the line at infinity by H is orthogonal to line IG and its x-coordinate is determined by

$$x_0 = \frac{r^2 + 3u^2}{3u},$$

where $|u|$ is the distance of G from the incenter I. The kind of conic c^{**} is determined by the location of x_0 relative to the incircle. In the case $|x_0 - u| < r \iff |u| > \frac{r}{3}$ we obtain hyperbolas. In the case $|u| < \frac{r}{3}$ we obtain ellipses and in the case $|u| = \frac{r}{3}$ we obtain two parallel lines orthogonal to line GI. Since the hyperbolic case was discussed in some extend in the previous sections here I examine the two other cases.

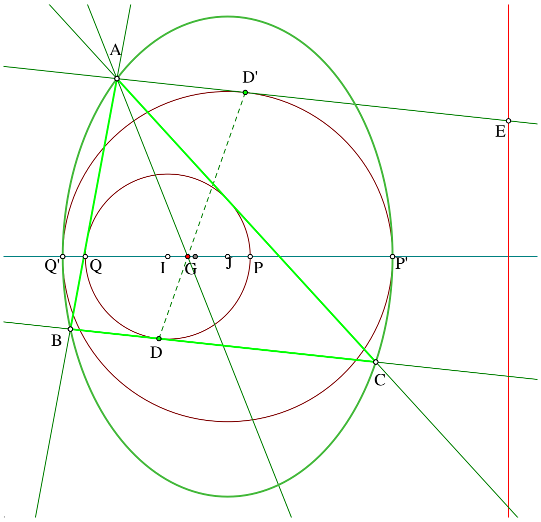

Figure 14. The elliptic case

First, the elliptic case characterized by the condition $u < \frac{r}{3}$ and illustrated by Figure 14. In this case it is easily seen that circle $c' = f(c)$ encloses entirely circle c and since c' is the maximal inscribed in c^{**} circle the points of this conic are all on the outside of circle c. Thus, from all points A of this conic there exist tangents to the circle c defining triangles ABC with incircle c and centroid G. Besides these common elements triangles ABC share also the same Nagel point which is the center J of the ellipse. Note that this ellipse can be easily constructed as a conic passing through five points $\{A, B, C, P', Q'\}$, where $\{P', Q'\}$ are the diametral points of its Nagel circle c' on IG.

Figure 15 illustrates the case of the singular conic, which can be considered as a degenerate parabola.

From the condition $|u| = \frac{r}{3}$ follows that c and $c' = f(c)$ are tangent at one of the diametral points $\{P, Q\}$ of c and the tangent there carries two of the vertices of the triangle.

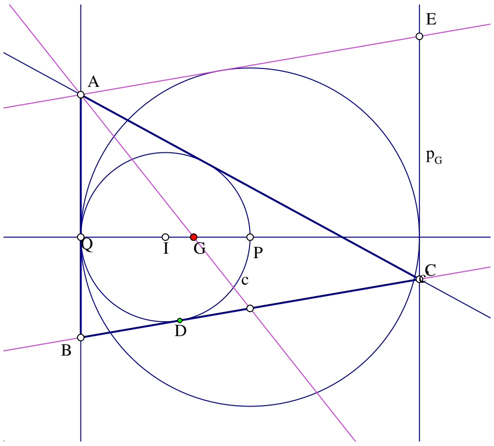

Figure 15. The singular case

Figure 16 displays two particular triangles of such a degenerate case. The isosceles triangle ABC characterized by the ratio of its sides $\frac{CA}{AB} = \frac{3}{2}$ and the right-angled $A'B'C'$ characterized by the ratio of its orthogonal sides $\frac{C'A'}{A'B'} = \frac{4}{3}$, which is similar to the right-angled triangle with sides $\{3, 4, 5\}$.

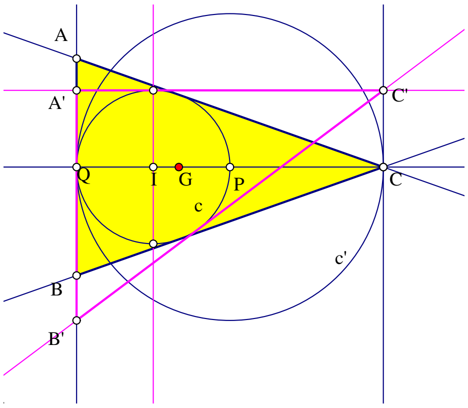

Figure 16. Two special triangles

It can be shown [3, p. 82] that triangles belonging to this degenerate case have the property to posses sides such that the sum of two of them equals three times the length of the third. This class of triangles answers also the problem [9] of finding all triangles such that their Nagel point is a point of the incircle.

5. The locus of the circumcenter

In this section I study the locus c_2 described by the circumcenter of all triangles sharing the same incircle c and centroid G. Since the homothety f, centered at G with ratio $k = -2$, maps the circumcenter O to the center E_u of the *Euler* circle

c_E, the problem reduces to that of finding the locus c_1 of points E_u. The clue here is the tangency of the Euler circle c_E to the incircle c at the *Feuerbach* point F_e of the triangle ([12]). Next proposition indicates that the Euler circle is also tangent to another fixed circle $c_0(S, r_0)$, whose center S (see Figure 17 and Figure 19) is on the Nagel line IG. I call this circle the *secondary* circle of the configuration (or of the triangle). As will be seen below this circle is homothetic to the incircle c with respect to the Nagel point N_a and at a certain ratio κ determined from the given data.

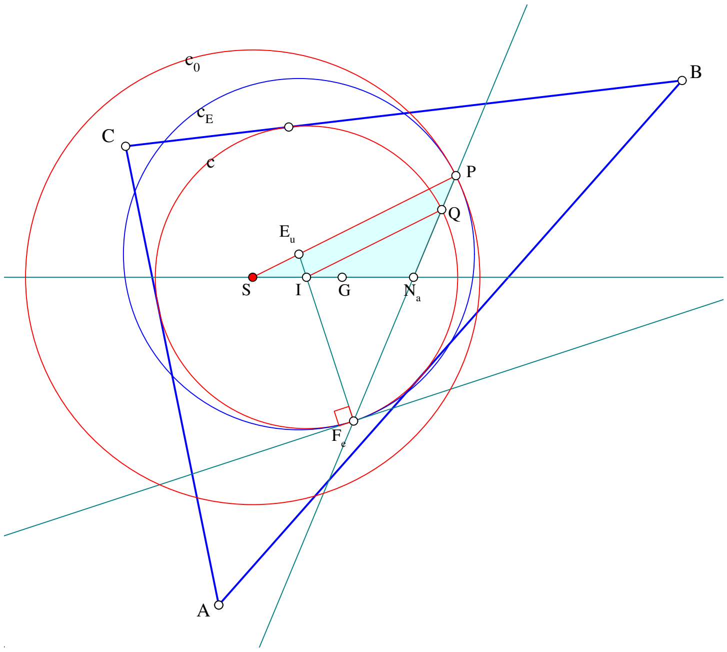

Figure 17. Invariant distance SP

Proposition 7. *Let $\{E_u, F_e, N_a\}$ be correspondingly the center of the Euler circle, the Feuerbach and the Nagel point of the triangle ABC. Let Q be the second intersection point of line F_eN_a and the incircle c. Then the parallel E_uP from E_u to line IQ intersects the Nagel line IG at a point S such that segments SN_a, SP have constant length for all triangles ABC sharing the same incircle c and centroid G. As a result all these triangles have their Euler circles c_E simultaneously tangent to the incircle c and a second fixed circle c_0 centered at S.*

The proof proceeds by showing that the ratio of oriented segments

$$\kappa = \frac{N_aS}{N_aI} = \frac{N_aP}{N_aQ} = \frac{N_aP \cdot N_aF_a}{N_aQ \cdot N_aF_a}$$

is constant. Since the incircle and the Euler circle are homothetic with respect to F_e, point P, being the intersection of line F_eN_a with the parallel to IQ from E_u, is on the Euler circle. Hence the last quotient is the ratio of powers of N_a with respect to the two circles: the variable Euler circle and the fixed incircle c. Denoting by r and R respectively the inradius and the circumradius of triangle ABC ratio κ can be expressed as

$$\kappa = \frac{|N_aE_u|^2 - (R/2)^2}{|N_aI|^2 - r^2}.$$

The computation of this ratio can be carried out using standard methods ([1, p.103], [17, p.87]). A slight simplification results from the fact ([17, p.30]) that homothety f maps the incenter I to the Nagel point N_a, producing the constellation displayed in Figure 18, in which H is the orthocenter and N' is the point on line GO such that $|GN'| : |N'O| = 1 : 3$ and $|IN'| = \frac{|E_uN_a|}{2}$.

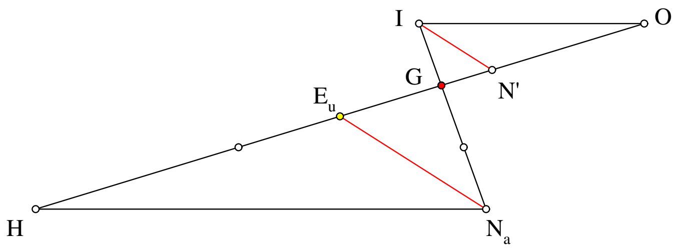

Figure 18. Configuration's symmetry

Thus, the two lengths needed for the determination of κ are $|N_aE_u| = 2|IN'|$ and $|N_aI| = 3|IG|$. In the next four equations $|OI|$ is given by Euler's relation ([17, p.10]), $|IG|, |GO|$ result by a standard calculation ([1, p.111], [14, p.185]), and $|IN'|$ results by applying Stewart's theorem to triangle GIO [5, p.6].

$$\begin{aligned}
|OI|^2 &= R(R-2r),\\
|OG|^2 &= \frac{1}{9}(9R^2 + 2r^2 + 8Rr - 2s^2),\\
|IG|^2 &= \frac{1}{9}(5r^2 + (s^2 - 16Rr)),\\
|IN'|^2 &= \frac{1}{16}(6r^2 - 32Rr + 2s^2 + R^2),
\end{aligned}$$

where s is the semiperimeter of the triangle. Using these relations ratio κ is found to be

$$\kappa = \frac{3r^2 + (s^2 - 16Rr)}{2(4r^2 + (s^2 - 16Rr))},$$

which using the above expression for $u^2 = |GI|^2$ becomes

$$\kappa = \frac{(3u)^2 - 2r^2}{2((3u)^2 - r^2)}.$$

By our assumptions this is a constant quantity, thereby proving the proposition.

The denominator of κ becomes zero precisely when $3|u| = r$. This is the case when point S (see Figure 17) goes to infinity and also the case in which, according to the previous section, the locus of the vertices of ABC is a degenerate parabola of two parallel lines. Excluding this exceptional case of infinite κ, to be handled below, last proposition implies that segments SP, SN_a (see Figure 17) have (constant) corresponding lengths $|SN_a| = |\kappa| \cdot |IN_a|$, $|SP| = |\kappa| \cdot r$. Hence the Euler circle of ABC is tangent simultaneously to the fixed incircle c as well as to the *secondary* circle c_0 with center at S and radius equal to $r_0 = |\kappa| \cdot r$. In the case $\kappa = 0$ the secondary circle collapses to a point coinciding with the Nagel point of the triangle and the Euler circle of ABC passes through that point.

Proposition 8. *The centers E_u of the Euler circles of triangles ABC which share the same incircle $c(I, r)$ and centroid G with $|IG| \neq \frac{r}{3}$ are on a central conic c_1 with one focus at the incenter I of the triangle and the other focus at the center S of the secondary circle. The kind of this conic depends on the value of κ as follows.*
(1) *For $\kappa > 1$ which corresponds to $3|u| < r$ conic c_1 is an ellipse similar to c^* and has great axis equal to $\frac{r_0 - r}{2}$, where r_0 the radius of the secondary circle.*
(2) *The other cases correspond to values of $\kappa < 0$, $0 < \kappa < \frac{1}{2}$ and $\kappa = 0$. In the first two cases the conic c_1 is a hyperbola similar to the conjugate one of the hyperbola c^*.*
(3) *In the case $\kappa = 0$ the Euler circles pass through the Nagel point and the conic c_1 is a rectangular hyperbola similar to c^*.*

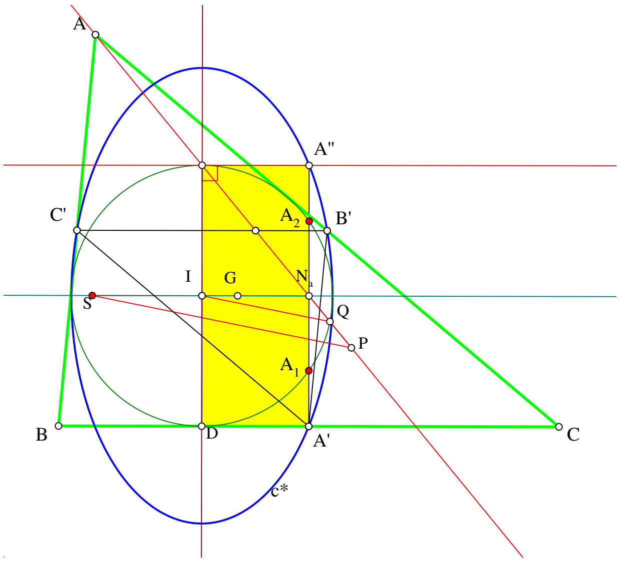

Figure 19. Axes ratio of c^*

The fact that the locus c_1 of points E_u is part of the central conic with foci at $\{I, S\}$ and great axis equal to $\frac{|r_0 \mp r|}{2}$ is a consequence of the simultaneous tangency of the Euler circles with the two fixed circles c and c_0 ([11, vol.I, p.42]). In the case

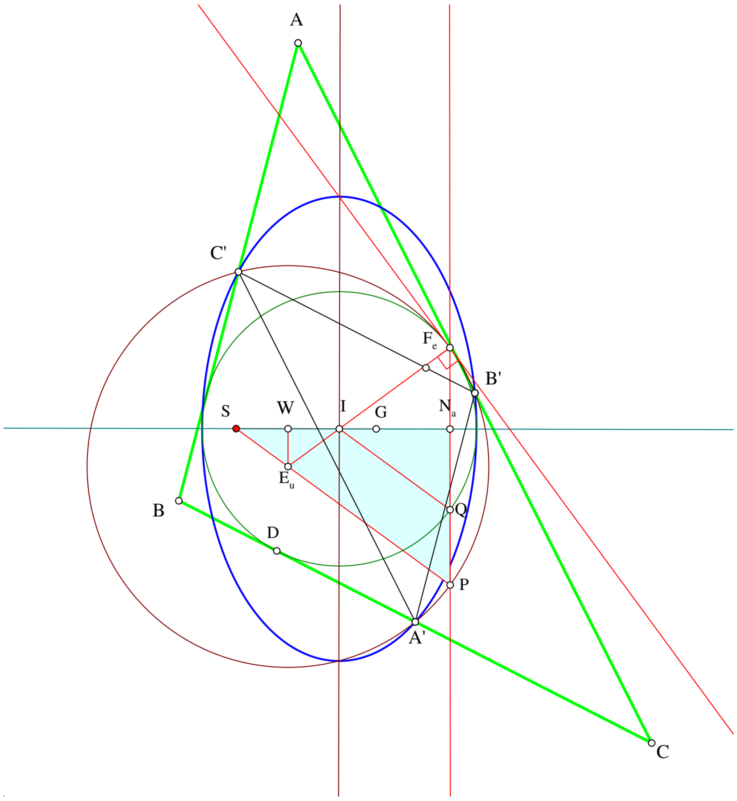

Figure 20. Axes ratio of c_1 (elliptic case)

$\kappa > 1$ it is readily verified that the Euler circles contain circle c and are contained in c_0 and the sum of the distances of E_u from $\{I, S\}$ is $r_0 - r$. In the case $\kappa < 0$ the Nagel point is between S and I and the Euler circle contains c and is outside c_0. Thus the difference of distances of E_u from $\{I, S\}$ is equal to $r_0 + r$. In the case $0 < \kappa < \frac{1}{2}$ the Euler circle contains both circles c and c_0 and the difference of distances of E_u from $\{I, S\}$ is $r - r_0$. Finally in the case $\kappa = 0$ the difference of the distances from $\{I, S\}$ is equal to r. This and the condition $0 = \kappa = (3u)^2 - 2r^2$ imply easily that c_1 is a rectangular hyperbola. To prove the similarity to c^* in the case $\kappa > 1$ it suffices to compare the ratios of the axes of the two conics. By the remarks of §3, in this case ($|IG| = |u| < \frac{r}{3}$), the incircle c is the maximal circle contained in c^* and the ratio of its axes is (see Figure 19) $q = \frac{|A_1A_2|}{|A'A''|}$. Here $A'A''$ is the chord parallel to the great axis and equal to the diameter of the incircle and A_1A_2 is the intersection of this chord with the incircle. It is readily seen that this ratio is equal to $\sqrt{1 - \frac{IN_a^2}{ID^2}} = \sqrt{1 - \frac{(3u)^2}{r^2}}$. The corresponding ratio for c_1 can be realized by considering the special position of ABC for which the corresponding point E_u becomes a vertex of c_1 *i.e.*, the position for which the projection W of E_u on line IG is the middle of the segment IS (see Figure 20). By this position of ABC the ratio of axes of conic c_1 is realized as $\frac{E_uW}{E_uI}$. By the resulting similarity at I this is equal to $\frac{F_eN_a}{F_eI} = \frac{\sqrt{r^2-IN_a^2}}{r} = \sqrt{1-\frac{(3u)^2}{r^2}}$, thereby proving the claim in the case $3|u| < r$. In the case $3|u| > r$, *i.e.*, when c^* is a hyperbola, it was seen in Remark (5) of §3 that the asymptotes of the hyperbola are the tangents to c from the Nagel point N_a (see Figure 21). The proof results in this case by taking the position of the variable triangle ABC in such a way that the center of the Euler circle goes to infinity. In this case points $\{P, Q\}$ defining the parallels $\{IP, E_uQ\}$ tend respectively to points $\{P_0, Q_0\}$, which are the projections of $\{I, S\}$ on the asymptote and the parallels tend respectively to $\{IP_0, SQ_0\}$, which are parallel to an asymptote of the conic c_1. Analogously is verified also that the other asymptote of c_1 is orthogonal to the corresponding other asymptote of c^*. This proves the claim on the conjugacy of c_1 to c.

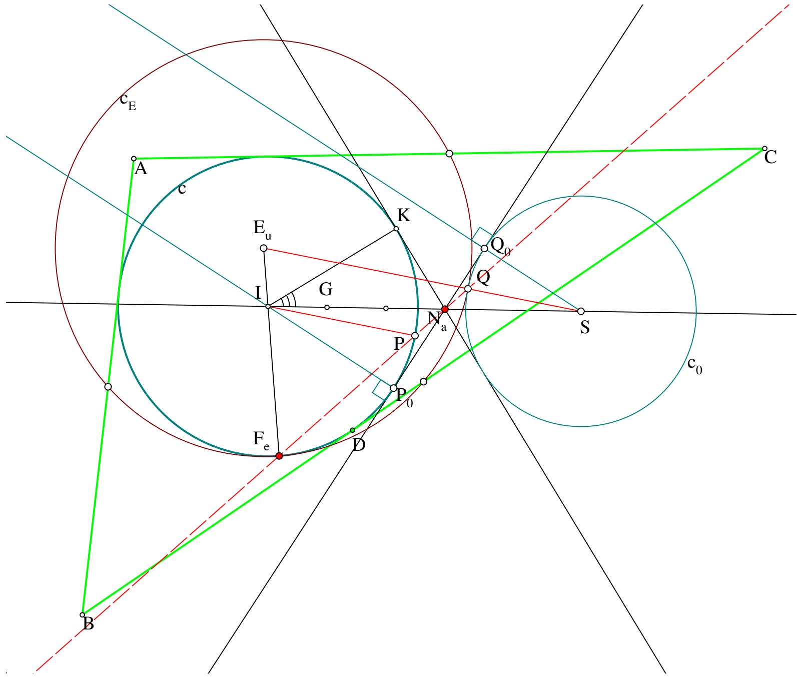

Figure 21. Axes ratio of c_1 (hyperbolic case)

Remarks. (1) The fact that the interval $(\frac{1}{2}, 1)$ represents a gap for the values of κ is due to the formula representing κ as a Moebius transformation of $(3u)^2$, whose the asymptote parallel to the x-axis is at $y = \frac{1}{2}$.

(2) An easy exploration of the same formula shows that $\{c, c_0\}$ are always disjoint, except in the case $u = -\frac{2r}{3}$, in which $\kappa = -\frac{1}{3}$ and the two circles become externally tangent at the Spieker point S_p of the triangle ABC, which is the middle of IN_a.

I turn now to the singular case corresponding to $|GI| = |u| = \frac{r}{3}$ for which κ becommes infinite. Following lemma should be known, I include though its proof for the sake of completeness of the exposition.

Lemma 9. *Let circle c, be tangent to line L_0 at its point C and line L_1 be parallel to L_0 and not intersecting c. From a point H on L_1 draw the tangents to c which intersect L_0 at points $\{K, L\}$. Then $CL \cdot CK$ is constant.*

Figure 22, illustrating the lemma, represents an interesting configuration and the problem at hand gives the opportunity to list some of its properties.
(1) Circle c', passing through the center A of the given circle c and points $\{K, L\}$, has its center P on the line AH.
(2) The other intersection points $\{Q, R\}$ of the tangents $\{HL, HK\}$ with circle c' define line QR, which is symmetric to L_0 with respect to line AH.
(3) Quadrangle $LRKQ$, which is inscribed in c', is an isosceles trapezium.
(4) Points $(A, M, O, H) = -1$ define a harmonic division. Here M is the diametral of A and $O = L_0 \cap QR$.
(5) Point M, defined above, is an excenter of triangle HLK.
(6) Points $(A, N, C, F) = -1$ define also a harmonic division, where N is the other intersection point with c' of AC and $F = AC \cap L_1$.

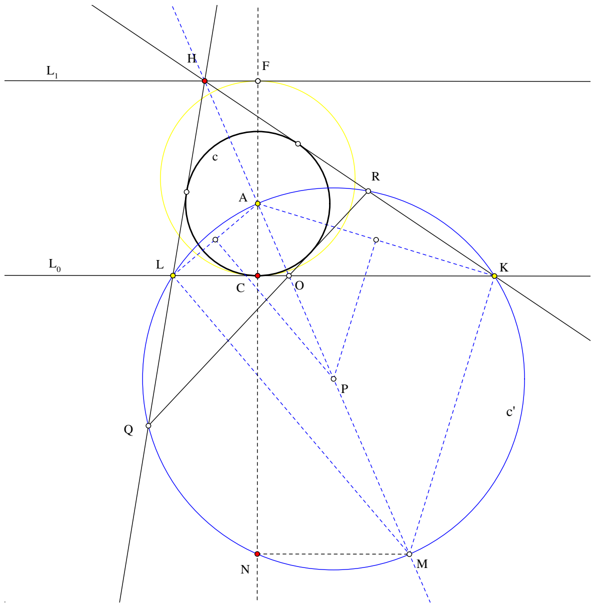

Figure 22. $CL \cdot CK$ is constant

(1) is seen by drawing first the medial lines of segments $\{AL, AK\}$ which meet at P and define there the circumcenter of ALK. Their parallels from $\{L, K\}$ respectively meet at the diametral M of A on the circumcircle c' of ALK. Since they are orthogonal to the bisectors of triangle HLK they are external bisectors of its angles and define an excenter of triangle HLK. (2), (3), (5) are immediate consequences. (4) follows from the standard way ([5, p.145]) to construct the polar of a point H with respect to a conic c'. (6) follows from (4) and the parallelism of lines $\{L_0, L_1, CO, NM\}$. The initial claim is a consequence of (6). This claim is also equivalent to the orthogonality of circle c' to the circle with diameter FC.

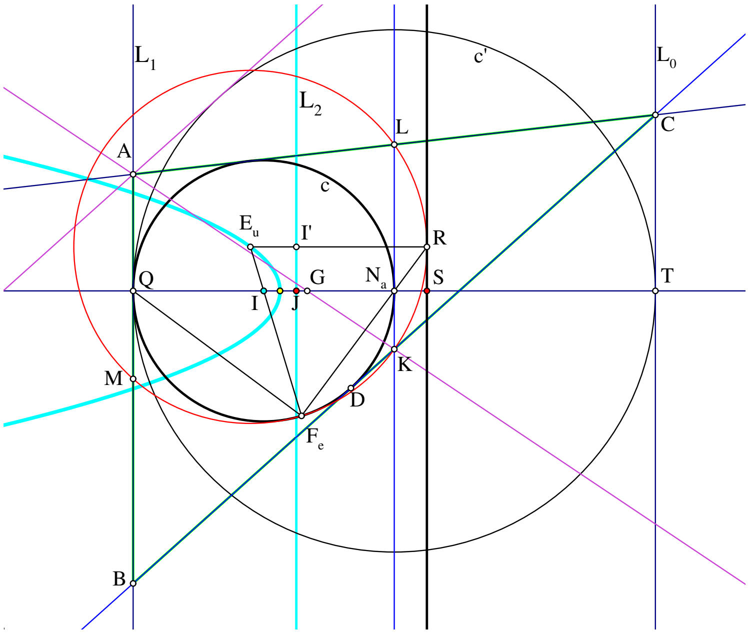

Figure 23. Locus of E_u for $|IG| = \frac{r}{3}$

Proposition 10. *Let $c(I, r)$ be a circle with center at point I and radius r. Let also G be situated at distance $|IG| = \frac{r}{3}$ from I. Then the following statements are valid:*
(1) *The homothety f centered at G and with ratio $k = -2$ maps circle c to circle $c'(N_a, 2r)$ of radius $2r$ and tangent to c at its intersection point Q with line IG such that $GQ : IQ = 4 : 3$.*
(2) *Let T be the diametral point of Q on c' and C be an arbitrary point on the line L_0, which is orthogonal to QT at T. Let ABC be the triangle formed by drawing the tangents to c from C and intersecting them with the parallel L_1 to L_0 from Q. Then G is the centroid of ABC.*
(3) *The center E_u of the Euler circle of ABC, as C varies on line L_0, describes a parabola with focus at I and directrix L_2 orthogonal to GI at a point J such that $IJ : IG = 3 : 4$.*

Statement (1) is obvious. Statement (2) follows easily from the definitions, since the middles $\{L, K\}$ of $\{CA, CB\}$ respectively define line KL which is tangent to c, orthogonal to GI and passes through the center N_a of c' (see Figure 23). Denoting by G' the intersection of AK with IG and comparing the similar triangles AQG' and KN_aG' one identifies easily G' with G. To prove (3) use first Feuerbach's theorem ([12]), according to which the Euler circle is tangent to the incircle c at a point F_e. Let then E_uR be the radius of the Euler circle parallel to GI. Line RF_e passes through N_a, which is the diametral of Q on circle c and which coincides with the Nagel point of the triangle ABC. This follows from Thales theorem for the similar isosceles triangles IF_eN_a and

E_uF_eR. Point R projects on a fixed point S on GI. This follows by first observing that RSF_eQ is a cyclic quadrangle, points $\{F_e, S\}$ viewing RQ under a right angle. It follows that $N_aS \cdot N_aQ = N_aR \cdot N_aF_u$ which is equal to $N_aK \cdot N_aL = \frac{1}{4}QA \cdot QB$ later, according to previous lemma, being constant and independent of the position of C on line L_0. Using the previous facts we see that $E_uI = E_uF_e - IF_e = E_uF_e - r = E_uR - r$. Let point J on GI be such that $SJ = r$ and line L_2 be orthogonal to GI at J. Then the projection I' of E_u on L_2 satisfies $E_uI = E_uI'$, implying that E_u is on the parabola with focus at I and directrix L_2. The claim on the ratio $IJ : IG = 3 : 4$ follows trivially.

6. Miscellanea

In this section I discuss several aspects of the structures involved in the problem at hand. I start with the determination of the perspector of the conic c^{**} in barycentric coordinates. The clue here is the incidence of the conic at the diametral points $\{P, Q\}$ of the Nagel circle c' on the Nagel line GI (see Figure 24). These points can be expressed as linear combinations of $\{G, I\}$:

$$P = p' \cdot G + p'' \cdot I, \qquad Q = q' \cdot G + q'' \cdot I.$$

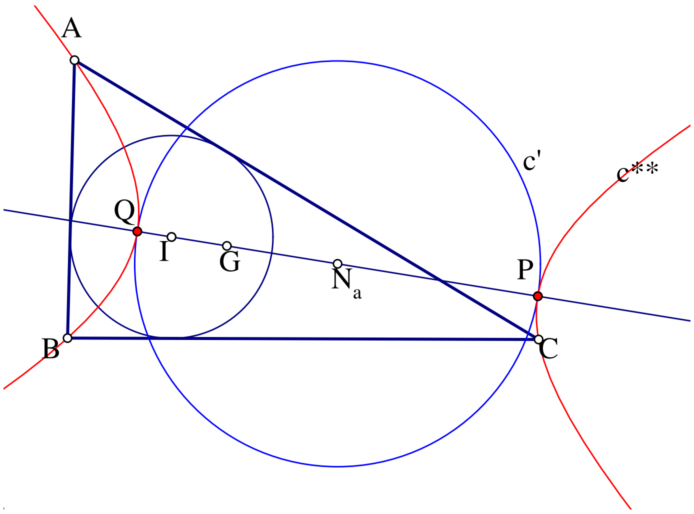

Figure 24. Triangle conic c^{**}

Here the equality is meant as a relation between the barycentric coordinates of the points and on the right are meant the normalized barycentric coordinates $G = \frac{1}{3}(1,1,1)$, $I = \frac{1}{2s}(a,b,c)$, where $\{a,b,c\}$ denote the side-lengths of triangle ABC and s denotes its half-perimeter. From the assumptions follows that

$$\frac{p''}{p'} = \frac{GP}{PI} = \frac{2(r-u)}{-2r+3u}, \qquad \frac{q''}{q'} = \frac{GQ}{QI} = -\frac{2(r+u)}{2r+3u}.$$

The equation of the conic is determined by assuming its general form

$$\alpha yz + \beta zx + \gamma xy = 0,$$

and computing $\{\alpha, \beta, \gamma\}$ using the incidence condition at P and Q. This leads to the system of equations

$$\begin{aligned} \alpha p_y p_z + \beta p_z p_x + \gamma p_x p_y &= 0, \\ \alpha q_y q_z + \beta q_z q_x + \gamma q_x q_y &= 0, \end{aligned}$$

implying

$$(\alpha, \beta, \gamma) = \lambda((q_x q_y p_z p_x - q_z q_x p_x p_y), (q_y q_z p_x p_y - q_x q_y p_y p_z), (q_z q_x p_y p_z - q_y q_z p_z p_x)),$$

where the barycentric coordinates of P and Q are given by

$$\begin{aligned} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} &= p'G + p''I = \frac{-2r+3u}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{r-u}{s}\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \\ \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} &= q'G + q''I = \frac{2r+3u}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{r+u}{s}\begin{pmatrix} a \\ b \\ c \end{pmatrix}. \end{aligned}$$

Making the necessary calculations and elliminating common factors we obtain

$$(\alpha, \beta, \gamma) = ((c-b)((3u(s-a))^2 - r^2(2s-3a)^2), \cdots, \cdots),$$

the dots meaning the corresponding formulas for β, γ resulting by cyclic permutation of the letters $\{a, b, c\}$.

The next proposition depicts another aspect of the configuration, related to a certain pencil of circles passing through the (fixed) Spieker point of the triangle.

Proposition 11. *Let $c(I, r)$ be a circle at I with radius r and G a point. Let also ABC be a triangle having c as incircle and G as centroid. Let further $\{c_E(E_u, r_E), c_0(S, r_0)\}$ be respectively the Euler circle and the secondary circle of ABC and $\{E_u, F_e, N_a, S_p\}$ be respectively the center of the Euler circle, the Feuerbach point, the Nagel point and the Spieker point of ABC. Then the following statements are valid.*
(1) *The pencil $\mathcal{I}$ of circles generated by c and c_0 has limit points $\{S_p, T\}$, where T is the inverse of S_p with respect to c.*
(2) *If P is the other intersection point of the Euler circle c_E with the line F_eN_a, then circle c_t tangent to the radii $\{IF_e, E_uP\}$ respectively at points $\{F_e, P\}$ belongs to the pencil of circles $\mathcal{J}$ which is orthogonal to $\mathcal{I}$ and passes through $\{S_p, T\}$.*
(3) *The polar p_T of T with respect to circles $\{c, c_0\}$ as well as the conic c^* is the same line UV which passes through S_p.*
(4) *In the case $\kappa > 1$ the conic c^* is an ellipse tangent to c_0 at its intersection points with line $p_T = UV$.*

The orthogonality of circle c_t to the three circles $\{c, c_0, c_E\}$ follows from its definition. This implies also the main part of the second claim. To complete the proof of the two first claims it suffices to identify the limit points of the pencil of circles generated by c and c_0. For this use can be made of the fact that these two points are simultaneously harmonic conjugate with respect to the diameter points of the circles c and c_0 on line IG. Denoting provisorily the intersection points of c_t with

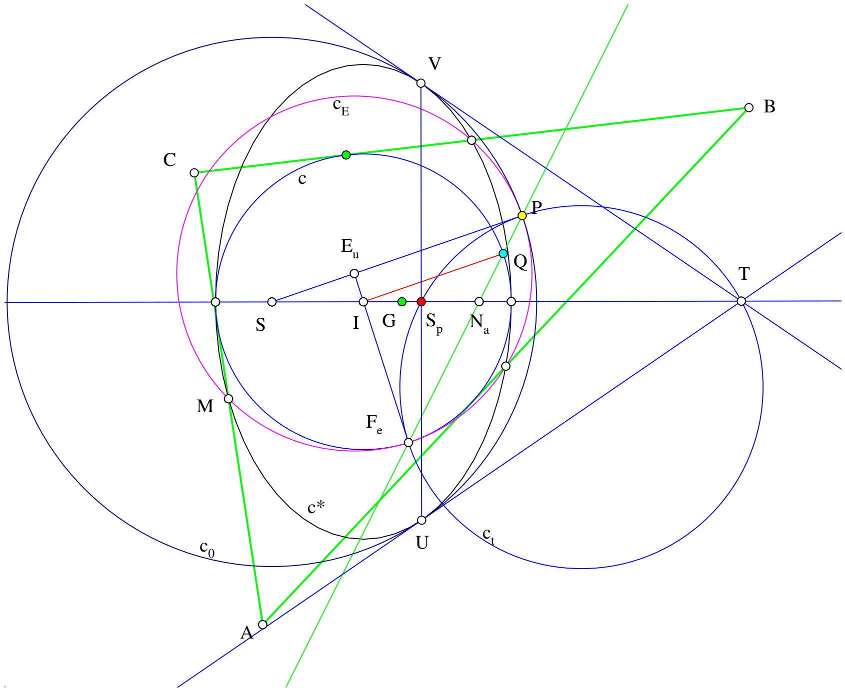

Figure 25. Circle pencil associated to $c(I, r)$ and G

line IG by $\{S', T'\}$ and their distance by d last property translates to the system of equations

$$\begin{aligned} r^2 &= |IS'|(|IS'| + d), \\ r_0^2 &= |SS'|(|SS'| + d). \end{aligned}$$

Eliminating d from these equations, setting $x = IS'$, $SI = SN_a - IN_a = (\kappa - 1)IN_a = (1 - \kappa)(3u)$, and $r_0 = \kappa \cdot r$, we obtain after some calculation, equation

$$6ux^2 + (9u^2 + 4r^2)x + 6ur^2 = 0.$$

One of the roots of this equation is $x = -\frac{3u}{2}$, identifying S' with the Spieker point S_p of the triangle. This completes the proof of the first two claims of the proposition. The third claim is an immediate consequence of the first two and the fact that c^* is bitangent to c_0 at its diametral points with line IG. The fourth claim results from a trivial verification of $\{U, V\} \subset c^*$ using the coordinates of the points.

Remark. Since points $\{G, I\}$ remain fixed for all triangles considered, the same happens for every other point X of the Nagel line IG which can be written as a linear combination $X = \lambda G + \mu I$ of the normalized barycentric coordinates of $\{G, I\}$ and with constants $\{\lambda, \mu\}$ which are independent of the particular triangle. Also fixing such a point X and expressing G as a linear combination of $\{I, X\}$ would imply the constancy of G. This, in particular, considering triangles with the same incircle and Nagel point or the same incircle and Spieker point would convey the discussion back to the present one and force all these triangles to have their vertices on the conic c^{**}.

References

[1] T. Andreescu and D. Andria, *Complex Numbers From A to Z*, Birkhaeuser, Boston, 2006.
[2] M. Berger, *Geometry*, 2 volumes, Springer, Berlin, 1987.
[3] O. Bottema, *Topics in Elementary Geometry*, Springer Verlag, Heidelberg, 2007.
[4] J. Carnoy, *G éométrie Analytique*, Gauthier-Villars, 2876.
[5] H. S. M. Coxeter and S. L. Greitzer, S. *Geometry Revisited*, MAA, 1967.
[6] R. Crane, Another Poristic System of Triangles, *Amer. Math. Monthly*, 33 (1926) 212–214.
[7] W. Gallatly, *The modern geometry of the triangle*, Francis Hodgson, London, 1913.
[8] B. Gibert, Pseudo-Pivotal Cubics and Poristic Triangles, 2010, available at http://pagesperso-orange.fr/bernard.gibert/files/Resources/psKs.pdf
[9] J. T. Groenmann, Problem 1423, *Crux Math.*, 25 (1989) 110.
[10] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, MAA, 1995.
[11] J. Koehler, *Exercices de Geoemetrie Analytique*, 2 volumes, Gauthier-Villars, Paris, 1886.
[12] J. A. Scott, An areal view of Feuerbach's theorem, *Math. Gazette*, 86 (2002) 81–82.
[13] S. Sigur, www.paideiaschool.org/.../Steve_Sigur/resources/pictures.pdf
[14] G. C. Smith, Statics and the moduli space of triangles, *Forum Geom.*, 5 (2005) 181–190.
[15] J. Vonk, On the Nagel line and a prolific polar triangle, *Forum Geom.*, 8 (2008) 183–196.
[16] J. H. Weaver, A system of triangles related to a poristic system, *Amer. Math. Monthly*, 31 (1928) 337–340.
[17] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

Paris Pamfilos: Department of Mathematics, University of Crete, Crete, Greece
E-mail address: pamfilos@math.uoc.gr