Forum Geometricorum
Volume 11 (2011) 65–82.

FORUM GEOM
ISSN 1534-1178

More Characterizations of Tangential Quadrilaterals

Martin Josefsson

Abstract. In this paper we will prove several not so well known conditions for a quadrilateral to have an incircle. Four of these are different excircle versions of the characterizations due to Wu and Vaynshtejn.

1. Introduction

In the wonderful paper [13] Nicuşor Minculete gave a survey of some known characterizations of tangential quadrilaterals and also proved a few new ones. This paper can to some extent be considered a continuation to [13].

A tangential quadrilateral is a convex quadrilaterals with an incircle, that is a circle tangent to all four sides. Other names for these quadrilaterals are[1] tangent quadrilateral, circumscribed quadrilateral, circumscribable quadrilateral, circumscribing quadrilateral, inscriptable quadrilateral and circumscriptible quadrilateral. The names inscriptible quadrilateral and inscribable quadrilateral have also been used, but sometimes they refer to a quadrilateral with a circumcircle (a cyclic quadrilateral) and are not good choices because of this ambiguity. To avoid confusion with so many names we suggest that only the names *tangential quadrilateral* (or tangent quadrilateral) and circumscribed quadrilateral be used. This is supported from the number of hits on Google[2] and the fact that both MathWorld and Wikipedia uses the name tangential quadrilateral in their encyclopedias.

Not all quadrilaterals are tangential,[3] hence they must satisfy some condition. The most important and perhaps oldest such condition is the *Pitot theorem*, that a quadrilateral $ABCD$ with consecutive sides a, b, c and d is tangential if and only if the sums of opposite sides are equal: $AB + CD = BC + DA$, that is

$$a + c = b + d. \tag{1}$$

It is named after the French engineer Henri Pitot (1695-1771) who proved that this is a necessary condition in 1725; that it is also a sufficient condition was proved by the Swiss mathematician Jakob Steiner (1796-1863) in 1846 according to F. G.-M. [7, p.319].

Publication Date: March 18, 2011. Communicating Editor: Paul Yiu.

[1]In decreasing order of the number of hits on Google.

[2]Tangential, tangent and circumscribed quadrilateral represent about 80 % of the number of hits on Google, so the other six names are rarely used.

[3]For example, a rectangle has no incircle.

The proof of the direct theorem is an easy application of the *two tangent theorem*, that two tangents to a circle from an external point are of equal length. We know of four different proofs of the converse to this important theorem, all beautiful in their own way. The first is a classic that uses a property of the perpendicular bisectors to the sides of a triangle [2, pp.135-136], the second is a proof by contradiction [10, pp.62-64], the third uses an excircle to a triangle [12, p.69] and the fourth is an exquisite application of the Pythagorean theorem [1, pp.56-57]. The first two of these can also be found in [3, pp.65-67].

Two similar characterizations are the following ones. If $ABCD$ is a convex quadrilateral where opposite sides AB and CD intersect at E, and the sides AD and BC intersect at F (see Figure 1), then $ABCD$ is a tangential quadrilateral if and only if either of

$$BE + BF = DE + DF,$$
$$AE - AF = CE - CF.$$

These are given as problems in [3] and [14], where the first condition is proved using contradiction in [14, p.147]; the second is proved in the same way[4].

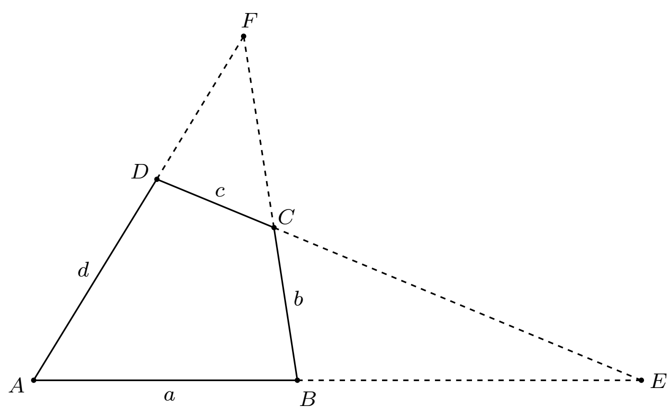

Figure 1. The extensions of the sides

2. Incircles in a quadrilateral and its subtriangles

One way of proving a new characterization is to show that it is equivalent to a previously proved one. This method will be used several times henceforth. In this section we prove three characterizations of tangential quadrilaterals by showing that they are equivalent to (1). The first was proved in another way in [19].

Theorem 1. *A convex quadrilateral is tangential if and only if the incircles in the two triangles formed by a diagonal are tangent to each other.*

[4]In [3, pp.186-187] only the direct theorems (not the converses) are proved.

Proof. In a convex quadrilateral $ABCD$, let the incircles in triangles ABC, CDA, BCD and DAB be tangent to the diagonals AC and BD at the points X, Y, Z and W respectively (see Figure 2). First we prove that

$$ZW = \tfrac{1}{2}\big|a-b+c-d\big| = XY.$$

Using the two tangent theorem, we have $BW = a - w$ and $BZ = b - z$, so

$$ZW = BW - BZ = a - w - b + z.$$

In the same way $DW = d - w$ and $DZ = c - z$, so

$$ZW = DZ - DW = c - z - d + w.$$

Adding these yields

$$2ZW = a - w - b + z + c - z - d + w = a - b + c - d.$$

Hence

$$ZW = \tfrac{1}{2}\big|a-b+c-d\big|$$

where we put an absolute value since Z and W can "change places" in some quadrilaterals; that is, it is possible for W to be closer to B than Z is. Then we would have $ZW = \frac{1}{2}(-a+b-c+d)$.

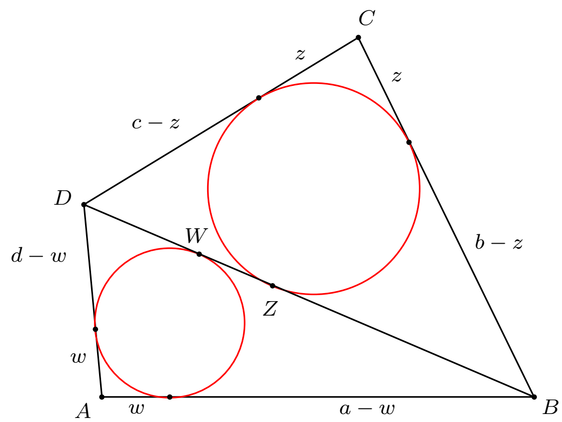

Figure 2. Incircles on both sides of one diagonal

The formula for XY is derived in the same way.

Now two incircles on different sides of a diagonal are tangent to each other if and only if $XY = 0$ or $ZW = 0$. These are equivalent to $a + c = b + d$, which proves the theorem according to the Pitot theorem. $\square$

Another way of formulating this result is that the incircles in the two triangles formed by one diagonal in a convex quadrilateral are tangent to each other if and only if the incircles in the two triangles formed by the other diagonal are tangent to each other. These two tangency points are in general not the same point, see Figure 3, where the notations are different from those in Figure 2.

Theorem 2. *The incircles in the four overlapping triangles formed by the diagonals of a convex quadrilateral are tangent to the sides in eight points, two per side, making one distance between tangency points per side. It is a tangential quadrilateral if and only if the sums of those distances at opposite sides are equal.*

Proof. According to the two tangent theorem, $AZ = AY$, $BS = BT$, $CU = CV$ and $DW = DX$, see Figure 3. Using the Pitot theorem, we get

$$\begin{aligned} &AB + CD = BC + DA \\ \Leftrightarrow\ &AZ + ZS + BS + CV + VW + DW = BT + TU + CU + DX + XY + AY \\ \Leftrightarrow\ &ZS + VW = TU + XY \end{aligned}$$

after cancelling eight terms. This is what we wanted to prove. □

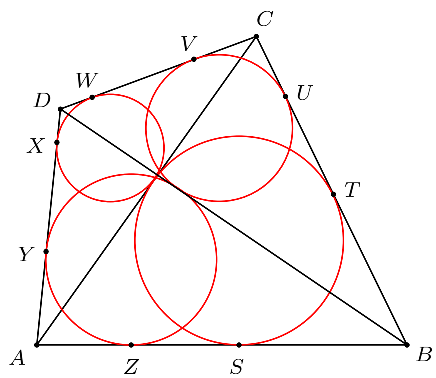

Figure 3. Incircles on both sides of both diagonals

The configuration with the four incircles in the last two theorems has other interesting properties. If the quadrilateral $ABCD$ is cyclic, then the four incenters are the vertices of a rectangle, see [2, p.133] or [3, pp.44-46].

A third example where the Pitot theorem is used to prove another characterization of tangential quadrilaterals is the following one, which is more or less the same as one given as a part of a Russian solution (see [18]) to a problem we will discuss in more detail in Section 4.

Theorem 3. *A convex quadrilateral is subdivided into four nonoverlapping triangles by its diagonals. Consider the four tangency points of the incircles in these triangles on one of the diagonals. It is a tangential quadrilateral if and only if the distance between two tangency points on one side of the second diagonal is equal to the distance between the two tangency points on the other side of that diagonal.*

Proof. Here we cite the Russian proof given in [18]. We use notations as in Figure 4 and shall prove that the quadrilateral has an incircle if and only if $T_1'T_2' = T_3'T_4'$.

By the two tangent theorem we have

$$AT_1 = AT_1'' = AP - PT_1'',$$
$$BT_1 = BT_1' = BP - PT_1',$$

so that

$$AB = AT_1 + BT_1 = AP + BP - PT_1'' - PT_1'.$$

Since $PT_1'' = PT_1'$,

$$AB = AP + BP - 2PT_1'.$$

In the same way

$$CD = CP + DP - 2PT_3'.$$

Adding the last two equalities yields

$$AB + CD = AC + BD - 2T_1'T_3'.$$

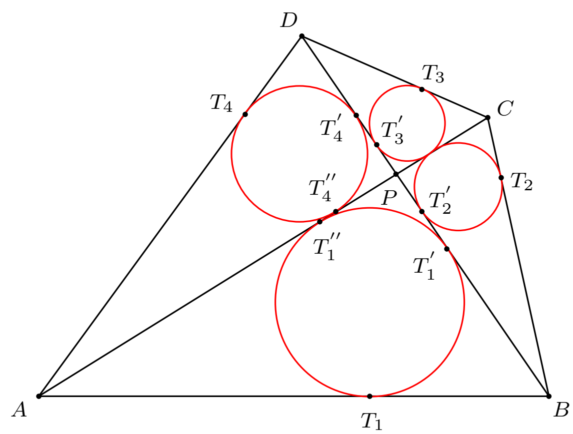

Figure 4. Tangency points of the four incircles

In the same way we get

$$BC + DA = AC + BD - 2T_2'T_4'.$$

Thus

$$AB + CD - BC - DA = -2\left(T_1'T_3' - T_2'T_4'\right).$$

The quadrilateral has an incircle if and only if $AB + CD = BC + DA$. Hence it is a tangential quadrilateral if and only if

$$T_1'T_3' = T_2'T_4' \quad \Leftrightarrow \quad T_1'T_2' + T_2'T_3' = T_2'T_3' + T_3'T_4' \quad \Leftrightarrow \quad T_1'T_2' = T_3'T_4'.$$

Note that both $T_1'T_3' = T_2'T_4'$ and $T_1'T_2' = T_3'T_4'$ are characterizations of tangential quadrilaterals. It was the first of these two that was proved in [18]. □

3. Characterizations with inradii, altitudes and exradii

According to Wu Wei Chao (see [20]), a convex quadrilateral $ABCD$ is tangential if and only if

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4},$$

where r_1, r_2, r_3 and r_4 are the inradii in triangles ABP, BCP, CDP and DAP respectively, and P is the intersection of the diagonals, see Figure 5.

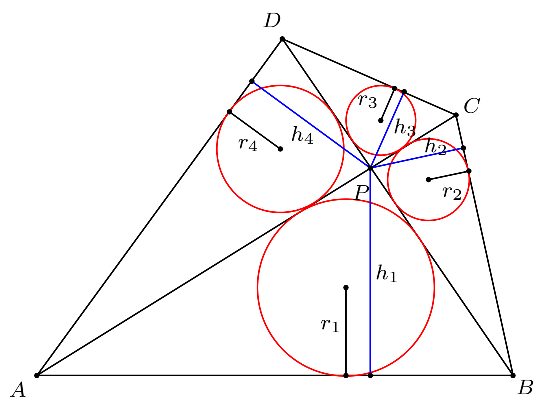

Figure 5. The inradii and altitudes

In [13] Nicuşor Minculete proved in two different ways that another characterization of tangential quadrilaterals is[5]

$$\frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}, \tag{2}$$

where h_1, h_2, h_3 and h_4 are the altitudes in triangles ABP, BCP, CDP and DAP from P to the sides AB, BC, CD and DA respectively, see Figure 5. These two characterizations are closely related to the following one.

Theorem 4. *A convex quadrilateral $ABCD$ is tangential if and only if*

$$\frac{1}{R_1}+\frac{1}{R_3}=\frac{1}{R_2}+\frac{1}{R_4}$$

where R_1, R_2, R_3 and R_4 are the exradii to triangles ABP, BCP, CDP and DAP opposite the vertex P, the intersection of the diagonals AC and BD.

Proof. In a triangle, an exradius R_a is related to the altitudes by the well known relation

$$\frac{1}{R_a}=-\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}. \tag{3}$$

[5] Although he used different notations.

If we denote the altitudes from A and C to the diagonal BD by h_A and h_C respectively and similar for the altitudes to AC, see Figure 6, then we have

$$\begin{aligned}\frac{1}{R_1} &= -\frac{1}{h_1}+\frac{1}{h_A}+\frac{1}{h_B},\\ \frac{1}{R_2} &= -\frac{1}{h_2}+\frac{1}{h_B}+\frac{1}{h_C},\\ \frac{1}{R_3} &= -\frac{1}{h_3}+\frac{1}{h_C}+\frac{1}{h_D},\\ \frac{1}{R_4} &= -\frac{1}{h_4}+\frac{1}{h_D}+\frac{1}{h_A}.\end{aligned}$$

Using these, we get

$$\frac{1}{R_1}+\frac{1}{R_3}-\frac{1}{R_2}-\frac{1}{R_4} = -\left(\frac{1}{h_1}+\frac{1}{h_3}-\frac{1}{h_2}-\frac{1}{h_4}\right).$$

Hence

$$\frac{1}{R_1}+\frac{1}{R_3}=\frac{1}{R_2}+\frac{1}{R_4} \quad\Leftrightarrow\quad \frac{1}{h_1}+\frac{1}{h_3}=\frac{1}{h_2}+\frac{1}{h_4}.$$

Since the equality to the right is a characterization of tangential quadrilaterals according to (2), so is the equality to the left. $\square$

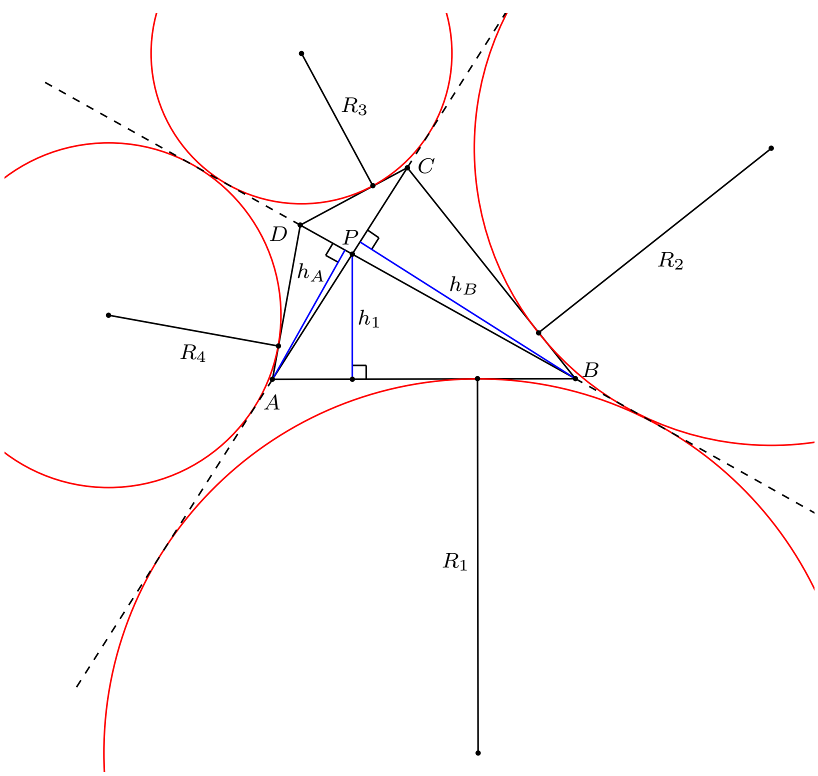

Figure 6. Excircles to four subtriangles

4. Christopher Bradley's conjecture and its generalizations

Consider the following problem:

In a tangential quadrilateral $ABCD$, let P be the intersection of the diagonals AC and BD. Prove that the incenters of triangles ABP, BCP, CDP and DAP form a cyclic quadrilateral. See Figure 7.

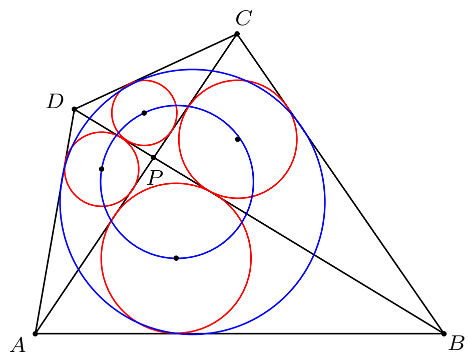

Figure 7. Christopher Bradley's conjecture

This problem appeared at the CTK Exchange[6] on September 17, 2003 [17], where it was debated for a month. On Januari 2, 2004, it migrated to the *Hyacinthos* problem solving group at Yahoo [15], and after a week a full synthetic solution with many extra properties of the configuration was given by Darij Grinberg [8] with the help of many others.

So why was this problem called *Christopher Bradley's conjecture*? In November 2004 a paper about cyclic quadrilaterals by the British mathematician Christopher Bradley was published, where the above problem was stated as a conjecture (see [4, p.430]). Our guess is that the conjecture was also published elsewhere more than a year earlier, which explains how it appeared at the CTK Exchange.

A similar problem, that is almost the converse, was given in 1998 by Toshio Seimiya in the Canadian problem solving journal Crux Mathematicorum [16]:

Suppose $ABCD$ is a convex cyclic quadrilateral and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incenters of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic. Prove that $ABCD$ has an incircle.

The next year a beautiful solution by Peter Y. Woo was published in [16]. He generalized the problem to the following very nice characterization of tangential quadrilaterals:

When a convex quadrilateral is subdivided into four nonoverlapping triangles by its two diagonals, then the incenters of the four triangles are concyclic if and only if the quadrilateral has an incircle.

[6]It was formulated slightly different, where the use of the word inscriptable led to a misunderstanding.

There was however an even earlier publication of Woo's generalization. According to [8], the Russian magazine *Kvant* published in 1996 (see [18]) a solution by I. Vaynshtejn to the problem we have called Christopher Bradley's conjecture and its converse (see the formulation by Woo). [18] is written in Russian, so neither we nor many of the readers of Forum Geometricorum will be able to read that proof. But anyone interested in geometry can with the help of the figures understand the equations there, since they are written in the Latin alphabet.

Earlier we saw that Minculete's characterization with incircles was also true for excircles (Theorem 4). Then we might wonder if Vaynshtejn's characterization is also true for excircles? The answer is yes and it was proved by Nikolaos Dergiades at [6], even though he did not state it as a characterization of tangential quadrilaterals. The proof given here is a small expansion of his.

Theorem 5 (Dergiades)**.** *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if the four excenters to triangles ABP, BCP, CDP and DAP opposite the vertex P are concyclic.*

Proof. In a triangle ABC with sides a, b, c and semiperimeter s, where I and J_1 are the incenter and excenter opposite A respectively, and where r and R_a are the radii in the incircle and excircle respectively, we have $AI = \frac{r}{\sin\frac{A}{2}}$ and $AJ_1 = \frac{R_a}{\sin\frac{A}{2}}$. Using Heron's formula $T^2 = s(s-a)(s-b)(s-c)$ and other well known formulas[7], we have

$$AI \cdot AJ_1 = r \cdot R_a \cdot \frac{1}{\sin^2\frac{A}{2}} = \frac{T}{s} \cdot \frac{T}{s-a} \cdot \frac{bc}{(s-b)(s-c)} = bc. \tag{4}$$

Similar formulas hold for the other excenters.

Returning to the quadrilateral, let I_1, I_2, I_3 and I_4 be the incentes and J_1, J_2, J_3 and J_4 the excenters opposite P in triangles ABP, BCP, CDP and DAP respectively. Using (4) we get (see Figure 8)

$$\begin{aligned} PI_1 \cdot PJ_1 &= PA \cdot PB, \\ PI_2 \cdot PJ_2 &= PB \cdot PC, \\ PI_3 \cdot PJ_3 &= PC \cdot PD, \\ PI_4 \cdot PJ_4 &= PD \cdot PA. \end{aligned}$$

From these we get

$$PI_1 \cdot PI_3 \cdot PJ_1 \cdot PJ_3 = PA \cdot PB \cdot PC \cdot PD = PI_2 \cdot PI_4 \cdot PJ_2 \cdot PJ_4.$$

Thus

$$PI_1 \cdot PI_3 = PI_2 \cdot PI_4 \qquad \Leftrightarrow \qquad PJ_1 \cdot PJ_3 = PJ_2 \cdot PJ_4.$$

In his proof [16], Woo showed that the quadrilateral has an incircle if and only if the equality to the left is true. Hence the quadrilateral has an incircle if and only if the equality to the right is true. Both of these equalities are conditions for the four points I_1, I_2, I_3, I_4 and J_1, J_2, J_3, J_4 to be concyclic according to the converse of the intersecting chords theorem. $\square$

[7] Here and a few times later on we use the half angle theorems. For a derivation, see [9, p.158].

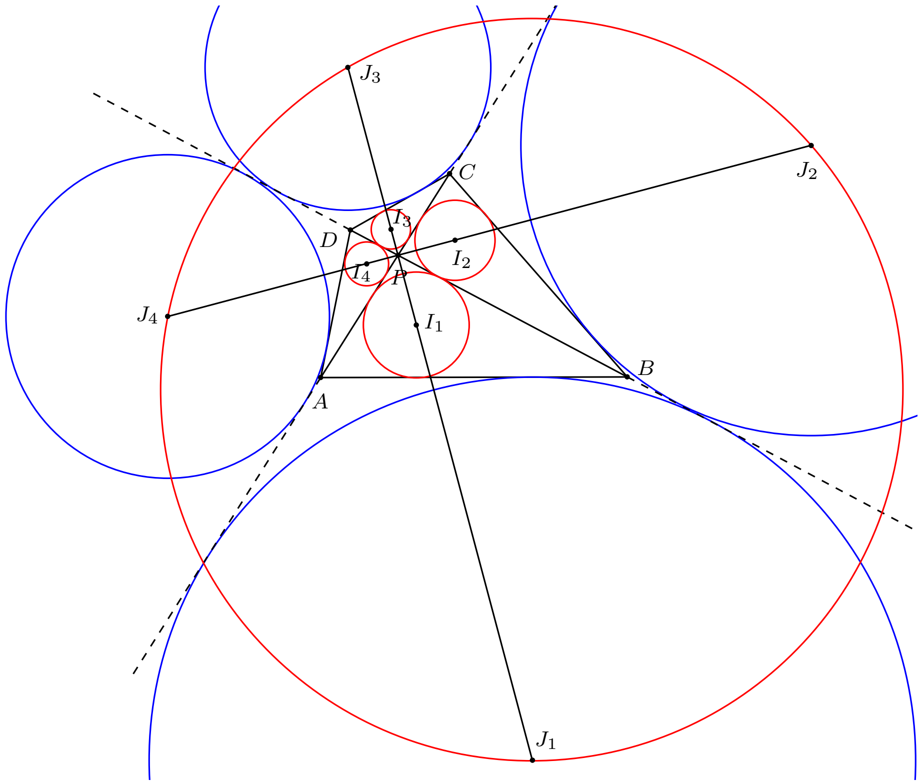

Figure 8. An excircle version of Vaynshtejn's characterization

Figure 8 suggests that $J_1J_3 \perp J_2J_4$ and $I_1I_3 \perp I_2I_4$. These are true in all convex quadrilaterals, and the proof is very simple. The incenters and excenters lies on the angle bisectors to the angles between the diagonals. Hence we have $\angle J_4PJ_1 = \angle I_4PI_1 = \frac{1}{2}\angle DPB = \frac{\pi}{2}$.

Another characterization related to the configuration of Christopher Bradley's conjecture is the following one. This is perhaps not one of the nicest characterizations, but the connection between opposite sides is present here as well as in many others. That the equality in the theorem is true in a tangential quadrilateral was established at [5].

Theorem 6. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if*

$$\frac{(AP+BP-AB)(CP+DP-CD)}{(AP+BP+AB)(CP+DP+CD)} = \frac{(BP+CP-BC)(DP+AP-DA)}{(BP+CP+BC)(DP+AP+DA)}.$$

Proof. In a triangle ABC with sides a, b and c, the distance from vertex A to the incenter I is given by

$$AI = \frac{r}{\sin\frac{A}{2}} = \frac{\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}}{\sqrt{\frac{(s-b)(s-c)}{bc}}} = \sqrt{\frac{bc(s-a)}{s}} = \sqrt{\frac{bc(-a+b+c)}{a+b+c}}. \qquad (5)$$

In a quadrilateral $ABCD$, let the incenters in triangles ABP, BCP, CDP and DAP be I_1, I_2, I_3 and I_4 respectively. Using (5), we get

$$PI_1 = \sqrt{\frac{PA \cdot PB(AP + BP - AB)}{AP + BP + AB}},$$
$$PI_3 = \sqrt{\frac{PC \cdot PD(CP + DP - CD)}{CP + DP + CD}}.$$

Thus in all convex quadrilaterals

$$(PI_1 \cdot PI_3)^2 = \frac{AP \cdot BP \cdot CP \cdot DP(AP + BP - AB)(CP + DP - CD)}{(AP + BP + AB)(CP + DP + CD)}$$

and in the same way we have

$$(PI_2 \cdot PI_4)^2 = \frac{AP \cdot BP \cdot CP \cdot DP(BP + CP - BC)(DP + AP - DA)}{(BP + CP + BC)(DP + AP + DA)}.$$

In [16], Woo proved that $PI_1 \cdot PI_3 = PI_2 \cdot PI_4$ if and only if $ABCD$ has an incircle. Hence it is a tangential quadrilateral if and only if

$$\frac{AP \cdot BP \cdot CP \cdot DP(AP + BP - AB)(CP + DP - CD)}{(AP + BP + AB)(CP + DP + CD)}$$
$$= \frac{AP \cdot BP \cdot CP \cdot DP(BP + CP - BC)(DP + AP - DA)}{(BP + CP + BC)(DP + AP + DA)}$$

from which the theorem follows. □

5. Iosifescu's characterization

In [13] Nicuşor Minculete cites a trigonometric characterization of tangential quadrilaterals due to Marius Iosifescu from the old Romanian journal [11]. We had never seen this nice characterization before and suspect no proof has been given in English, so here we give one. Since we have had no access to the Romanian journal we don't know if this is the same proof as the original one.

Theorem 7 (Iosifescu)**.** *A convex quadrilateral $ABCD$ is tangential if and only if*

$$\tan\frac{x}{2} \cdot \tan\frac{z}{2} = \tan\frac{y}{2} \cdot \tan\frac{w}{2}$$

where $x = \angle ABD$, $y = \angle ADB$, $z = \angle BDC$ and $w = \angle DBC$.

Proof. Using the trigonometric formula

$$\tan^2\frac{u}{2} = \frac{1 - \cos u}{1 + \cos u},$$

we get that the equality in the theorem is equivalent to

$$\frac{1 - \cos x}{1 + \cos x} \cdot \frac{1 - \cos z}{1 + \cos z} = \frac{1 - \cos y}{1 + \cos y} \cdot \frac{1 - \cos w}{1 + \cos w}.$$

This in turn is equivalent to

$$(1 - \cos x)(1 - \cos z)(1 + \cos y)(1 + \cos w)$$
$$= (1 - \cos y)(1 - \cos w)(1 + \cos x)(1 + \cos z). \qquad (6)$$

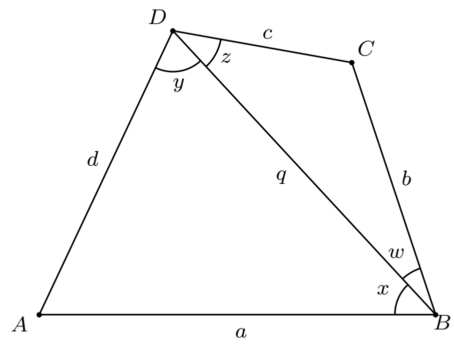

Figure 9. Angles in Iosifescu's characterization

Let $a = AB$, $b = BC$, $c = CD$, $d = DA$ and $q = BD$. From the law of cosines we have (see Figure 9)

$$\cos x = \frac{a^2 + q^2 - d^2}{2aq},$$

so that

$$1 - \cos x = \frac{d^2 - (a-q)^2}{2aq} = \frac{(d+a-q)(d-a+q)}{2aq}$$

and

$$1 + \cos x = \frac{(a+q)^2 - d^2}{2aq} = \frac{(a+q+d)(a+q-d)}{2aq}.$$

In the same way

$$1 - \cos y = \frac{(a+d-q)(a-d+q)}{2dq}, \qquad 1 + \cos y = \frac{(d+q+a)(d+q-a)}{2dq},$$

$$1 - \cos z = \frac{(b+c-q)(b-c+q)}{2cq}, \qquad 1 + \cos z = \frac{(c+q+b)(c+q-b)}{2cq},$$

$$1 - \cos w = \frac{(c+b-q)(c-b+q)}{2bq}, \qquad 1 + \cos w = \frac{(b+q+c)(b+q-c)}{2bq}.$$

Thus (6) is equivalent to

$$\frac{(d+a-q)(d-a+q)^2}{2aq} \cdot \frac{(b+c-q)(b-c+q)^2}{2cq} \cdot \frac{(d+q+a)}{2dq} \cdot \frac{(b+q+c)}{2bq}$$
$$= \frac{(a+d-q)(a-d+q)^2}{2dq} \cdot \frac{(c+b-q)(c-b+q)^2}{2bq} \cdot \frac{(a+q+d)}{2aq} \cdot \frac{(c+q+b)}{2cq}.$$

This is equivalent to

$$P\left((d-a+q)^2(b-c+q)^2 - (a-d+q)^2(c-b+q)^2\right) = 0 \qquad (7)$$

where

$$P = \frac{(d+a-q)(b+c-q)(d+q+a)(b+q+c)}{16abcdq^4}$$

is a positive expression according to the triangle inequality applied in triangles ABD and BCD. Factoring (7), we get

$$P\big((d-a+q)(b-c+q)+(a-d+q)(c-b+q)\big)$$
$$\cdot\big((d-a+q)(b-c+q)-(a-d+q)(c-b+q)\big)=0.$$

Expanding the inner parentheses and cancelling some terms, this is equivalent to

$$4qP(b+d-a-c)\left((d-a)(b-c)+q^2\right)=0. \tag{8}$$

The expression in the second parenthesis can never be equal to zero. Using the triangle inequality, we have $q>a-d$ and $q>b-c$. Thus $q^2 \gtrless (a-d)(b-c)$.

Hence, looking back at the derivation leading to (8), we have proved that

$$\tan\frac{x}{2}\cdot\tan\frac{z}{2}=\tan\frac{y}{2}\cdot\tan\frac{w}{2} \qquad \Leftrightarrow \qquad b+d=a+c$$

and Iosifescu's characterization is proved according to the Pitot theorem. □

6. Characterizations with other excircles

We have already seen two characterizations concerning the four excircles opposite the intersection of the diagonals. In this section we will study some other excircles. We begin by deriving a characterization similar to the one in Theorem 6, not for its own purpose, but because we will need it to prove the next theorem.

Theorem 8. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if*

$$\frac{(AB+AP-BP)(CD+CP-DP)}{(AB-AP+BP)(CD-CP+DP)}=\frac{(BC-BP+CP)(DA-DP+AP)}{(BC+BP-CP)(DA+DP-AP)}.$$

Proof. It is well known that in a triangle ABC with sides a, b and c,

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}}$$

where s is the semiperimeter [9, p.158]. Now, if P is the intersection of the diagonals in a quadrilateral $ABCD$ and x, y, z, w are the angles defined in Theorem 7, we have

$$\tan\frac{x}{2}=\sqrt{\frac{(AB+AP-BP)(BP+AP-AB)}{(AB+AP+BP)(BP-AP+AB)}},$$

$$\tan\frac{z}{2}=\sqrt{\frac{(CD+CP-DP)(DP+CP-CD)}{(CD+CP+DP)(DP-CP+CD)}},$$

$$\tan\frac{y}{2}=\sqrt{\frac{(DA+AP-DP)(DP+AP-DA)}{(DA+AP+DP)(DP-AP+DA)}},$$

$$\tan\frac{w}{2}=\sqrt{\frac{(BC+CP-BP)(BP+CP-BC)}{(BC+CP+BP)(BP-CP+BC)}}.$$

From Theorem 7 we have the equality[8]

$$\tan^2\frac{x}{2}\cdot\tan^2\frac{z}{2}=\tan^2\frac{y}{2}\cdot\tan^2\frac{w}{2}$$

and putting in the expressions above we get

$$\frac{(AB+AP-BP)(BP+AP-AB)(CD+CP-DP)(DP+CP-CD)}{(AB+AP+BP)(BP-AP+AB)(CD+CP+DP)(DP-CP+CD)}$$
$$=\frac{(DA+AP-DP)(DP+AP-DA)(BC+CP-BP)(BP+CP-BC)}{(DA+AP+DP)(DP-AP+DA)(BC+CP+BP)(BP-CP+BC)}.$$

Now using Theorem 6, the conclusion follows.[9] □

Lemma 9. *If J_1 is the excenter opposite A in a triangle ABC with sides a, b and c, then*

$$\frac{(BJ_1)^2}{ac}=\frac{s-c}{s-a}$$

where s is the semiperimeter.

Proof. If R_a is the radius in the excircle opposite A, we have (see Figure 10)

$$\begin{aligned}\sin\frac{\pi-B}{2}&=\frac{R_a}{BJ_1},\\ BJ_1\cos\frac{B}{2}&=\frac{T}{s-a},\\ (BJ_1)^2\cdot\frac{s(s-b)}{ac}&=\frac{s(s-a)(s-b)(s-c)}{(s-a)^2},\end{aligned}$$

and the equation follows. Here T is the area of triangle ABC and we used Heron's formula. □

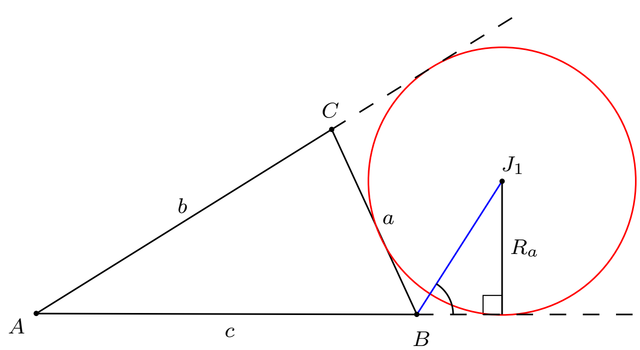

Figure 10. Distance from an excenter to an adjacant vertex

[8]This is a characterization of tangential quadrilaterals, but that's not important for the proof of this theorem.

[9]Here it is important that Theorem 6 is a characterization of tangential quadrilaterals.

Theorem 10. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if the four excenters to triangles ABP, BCP, CDP and DAP opposite the vertices B and D are concyclic.*

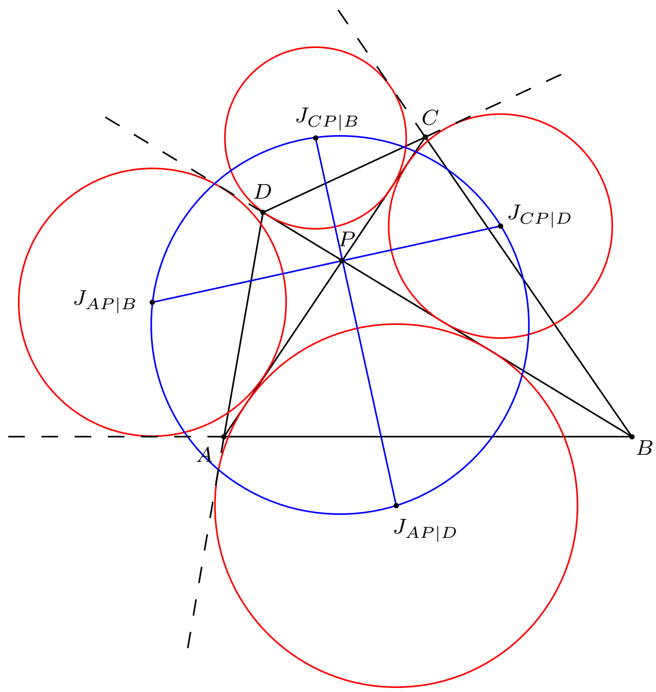

Figure 11. Excircles to subtriangles opposite the vertices B and D

Proof. We use the notation $J_{AP|B}$ for the excenter in the excircle tangent to side AP opposite B in triangle ABP. Using the Lemma in triangles ABP, BCP, CDP and DAP yields (see Figure 11)

$$\frac{(PJ_{AP|B})^2}{AP \cdot BP} = \frac{AB + AP - BP}{AB - AP + BP},$$
$$\frac{(PJ_{CP|D})^2}{CP \cdot DP} = \frac{CD + CP - DP}{CD - CP + DP},$$
$$\frac{(PJ_{CP|B})^2}{CP \cdot BP} = \frac{BC + CP - BP}{BC - CP + BP},$$
$$\frac{(PJ_{AP|D})^2}{AP \cdot DP} = \frac{DA + AP - DP}{DA - AP + DP}.$$

From Theorem 8 we get that $ABCD$ is a tangential quadrilateral if and only if

$$\frac{(PJ_{AP|B})^2}{AP \cdot BP} \cdot \frac{(PJ_{CP|D})^2}{CP \cdot DP} = \frac{(PJ_{CP|B})^2}{CP \cdot BP} \cdot \frac{(PJ_{AP|D})^2}{AP \cdot DP},$$

which is equivalent to

$$PJ_{AP|B} \cdot PJ_{CP|D} = PJ_{CP|B} \cdot PJ_{AP|D}. \tag{9}$$

Now $J_{AP|B}J_{CP|D}$ and $J_{CP|B}J_{AP|D}$ are straight lines through P since they are angle bisectors to the angles between the diagonals in $ABCD$. According to the intersecting chords theorem and its converse, (9) is a condition for the excenters to be concyclic. □

There is of course a similar characterization where the excircles are opposite the vertices A and C.

We conclude with a theorem that resembles Theorem 4, but with the excircles in Theorem 10.

Theorem 11. *A convex quadrilateral $ABCD$ with diagonals intersecting at P is tangential if and only if*

$$\frac{1}{R_a} + \frac{1}{R_c} = \frac{1}{R_b} + \frac{1}{R_d},$$

where R_a, R_b, R_c and R_d are the radii in the excircles to triangles ABP, BCP, CDP and DAP respectively opposite the vertices B and D.

Proof. We use notations on the altitudes as in Figure 12, which are the same as in the proof of Theorem 4. From (3) we have

$$\begin{aligned}
\frac{1}{R_a} &= -\frac{1}{h_B} + \frac{1}{h_A} + \frac{1}{h_1}, \\
\frac{1}{R_b} &= -\frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_2}, \\
\frac{1}{R_c} &= -\frac{1}{h_D} + \frac{1}{h_C} + \frac{1}{h_3}, \\
\frac{1}{R_d} &= -\frac{1}{h_D} + \frac{1}{h_A} + \frac{1}{h_4}.
\end{aligned}$$

These yield

$$\frac{1}{R_a} + \frac{1}{R_c} - \frac{1}{R_b} - \frac{1}{R_d} = \frac{1}{h_1} + \frac{1}{h_3} - \frac{1}{h_2} - \frac{1}{h_4}.$$

Hence

$$\frac{1}{R_a} + \frac{1}{R_c} = \frac{1}{R_b} + \frac{1}{R_d} \quad \Leftrightarrow \quad \frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}.$$

Since the equality to the right is a characterization of tangential quadrilaterals according to (2), so is the equality to the left. □

Even here there is a similar characterization where the excircles are opposite the vertices A and C.

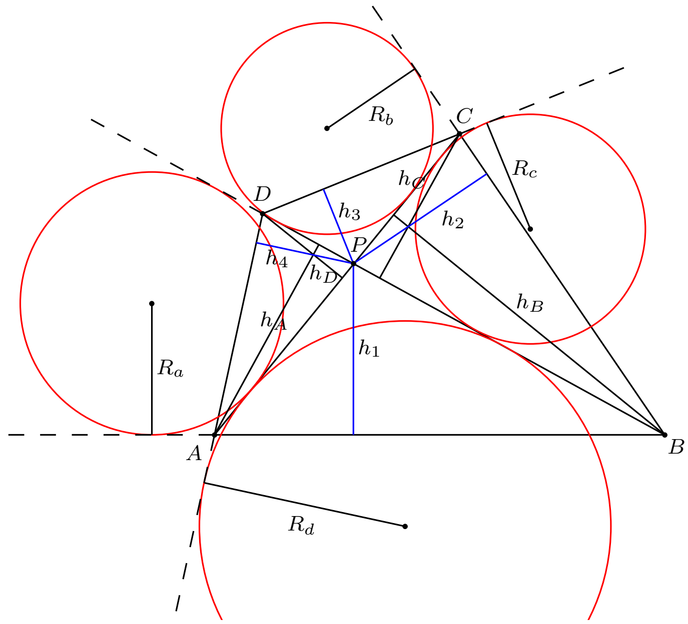

Figure 12. The exradii and altitudes

References

[1] I. Agricola and T. Friedrich, *Elementary geometry*, American Mathematical Society, 2008.
[2] N. Altshiller-Court, *College Geometry*, Dover reprint, 2007.
[3] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, Birkhäuser, Boston, 2004.
[4] C. J. Bradley, Cyclic quadrilaterals, *Math. Gazette*, 88 (2004) 417–431.
[5] crazyman (username), tangential quadrilateral, *Art of Problem Solving*, 2010, `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=334860`
[6] N. Dergiades, Hyacinthos message 8966, January 6, 2004.
[7] F. G.-M, *Exercices de Géométrie*, Cinquième édition (in French), Éditions Jaques Gabay, 1912.
[8] D. Grinberg, Hyacinthos message 9022, January 10, 2004.
[9] E. W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Seventh Edition, Dover Publications, 1957.
[10] R. Honsberger, *In Pólya's Footsteps*, Math. Assoc. Amer., 1997.
[11] M. Iosifescu, Problem 1421, *The Mathematical Gazette* (in Romanian), no. 11, 1954.
[12] K. S. Kedlaya, *Geometry Unbound*, 2006, available at `http://math.mit.edu/~kedlaya/geometryunbound/`
[13] N. Minculete, Characterizations of a tangential quadrilateral, *Forum Geom.*, 9 (2009) 113–118.
[14] V. Prasolov, *Problems in Plane and Solid Geometry*, translated and edited by D. Leites, 2006, available at `http://students.imsa.edu/~tliu/Math/planegeo.pdf`
[15] Rafi (username), Hyacinthos message 8910, January 2, 2004.
[16] T. Seimiya and P. Y. Woo, Problem 2338, *Crux Math.*, 24 (1998) 234; solution, ibid., 25 (1999) 243–245.
[17] sfwc (username), A conjecture of Christopher Bradley, *CTK Exchange*, 2003, `http://www.cut-the-knot.org/cgi-bin/dcforum/forumctk.cgi?az=read_count&om=383&forum=DCForumID6`
[18] I. Vaynshtejn, Problem M1524, *Kvant* (in Russian) no. 3, 1996 pp. 25-26, available at `http://kvant.mccme.ru/1996/03/resheniya_zadachnika_kvanta_ma.htm`

[19] C. Worrall, A journey with circumscribable quadrilaterals, *Mathematics Teacher*, 98 (2004) 192–199.

[20] W. C. Wu and P. Simeonov, Problem 10698, *Amer. Math. Monthly*, 105 (1998) 995; solution, ibid., 107 (2000) 657–658.

Martin Josefsson: Västergatan 25d, 285 37 Markaryd, Sweden
E-mail address: martin.markaryd@hotmail.com