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Perspective Isoconjugate Triangle Pairs, Hofstadter Pairs, and Crosssums on the Nine-Point Circle

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Abstract. The r-Hofstadter triangle and the $(1-r)$-Hofstadter triangle are proved perspective, and homogeneous trilinear coordinates are found for the perspector. More generally, given a triangle DEF inscribed in a reference triangle ABC, triangles $A'B'C'$ and $A''B''C''$ derived in a certain manner from DEF are perspective to each other and to ABC. Trilinears for the three perspectors, denoted by P^*, P_1, P_2 are found (Theorem 1) and used to prove that these three points are collinear. Special cases include (Theorems 4 and 5) this: if X and X' are an antipodal pair on the circumcircle, then the perspector $P^* = X \oplus X'$, where $\oplus$ denotes crosssum, is on the nine-point circle. Taking X to be successively the vertices of a triangle DEF inscribed in the circumcircle thus yields a triangle $D'E'F'$ inscribed in the nine-point circle. For example, if DEF is the circumtangential triangle, then $D'E'F'$ is an equilateral triangle.

1. Introduction and main theorem

We begin with a very general theorem about three triangles, one being the reference triangle ABC with sidelengths a, b, c, and the other two, denoted by $A'B'C'$ and $A''B''C''$, which we shall now proceed to define. Suppose DEF is a triangle inscribed in ABC; that is, the vertices are given by homogeneous trilinear coordinates (henceforth simply *trilinears*) as follows:

$$D = 0 : y_1 : z_1, \quad E = x_2 : 0 : z_2, \quad F = x_3 : y_3 : 0, \tag{1}$$

where $y_1z_1x_2z_2x_3y_3 \neq 0$ (this being a quick way to say that none of the points is A, B, C). For any point $P = p : q : r$ for which $pqr \neq 0$, define

$$D' = 0 : \frac{1}{qy_1} : \frac{1}{rz_1}, \quad E' = \frac{1}{px_2} : 0 : \frac{1}{rz_2}, \quad F' = \frac{1}{px_3} : \frac{1}{qy_3} : 0.$$

Define $A'B'C'$ and introduce symbols for trilinears of the vertices A', B', C' :

$$A' = CF \cap BE' = u_1 : v_1 : w_1,$$
$$B' = AD \cap CF' = u_2 : v_2 : w_2,$$
$$C' = BE \cap AD' = u_3 : v_3 : w_3,$$

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and similarly,

$$A'' = CF' \cap BE = \frac{1}{pu_1} : \frac{1}{qv_1} : \frac{1}{rw_1},$$
$$B'' = AD' \cap CF = \frac{1}{pu_2} : \frac{1}{qv_2} : \frac{1}{rw_2},$$
$$C'' = BE' \cap AD = \frac{1}{pu_3} : \frac{1}{qv_3} : \frac{1}{rw_3}.$$

Thus, triangles $A'B'C'$ and $A''B''C''$ are a P-isoconjugate pair, in the sense that every point on each is the P-isoconjugate of a point on the other (except for points on a sideline of ABC). (The P-isoconjugate of a point $X = x : y : z$ is the point $1/(px) : 1/(qy) : 1/(rz)$; this is the isogonal conjugate of X if P is the incenter, and the isotomic conjugate of X if $P = X_{31} = a^2 : b^2 : c^2$. Here and in the sequel, the indexing of triangles centers in the form X_i follows that of [4].)

Theorem 1. *The triangles ABC, $A'B'C'$, $A''B''C''$ are pairwise perspective, and the three perspectors are collinear.*

Proof. The lines CF given by $-y_3\alpha + x_3\beta = 0$ and BE' given by $px_2\alpha - rz_2\gamma = 0$, and cyclic permutations, give

$$A' = CF \cap BE' = u_1 : v_1 : w_1 = rx_3z_2 : ry_3z_2 : px_2x_3,$$
$$B' = AD \cap CF' = u_2 : v_2 : w_2 = qy_1y_3 : px_3y_1 : px_3z_1,$$
$$C' = BE \cap AD' = u_3 : v_3 : w_3 = qy_1x_2 : rz_2z_1 : qy_1z_2;$$
$$A'' = CF' \cap BE = qx_2y_3 : px_2x_3 : qy_3z_2,$$
$$B'' = AD' \cap CF = rz_1x_3 : ry_3z_1 : qy_3y_1,$$
$$C'' = BE' \cap AD = rz_1z_2 : px_2y_1 : pz_1x_2.$$

Then the line $B'B''$ is given by $d_1\alpha + d_2\beta + d_3\gamma = 0$, where

$$d_1 = px_3y_3(qy_1^2 - rz_1^2), \quad d_2 = prx_3^2y_1^2 - q^2y_1^2y_3^2, \quad d_3 = ry_1z_1(qy_3^2 - px_3^2),$$

and the line $C'C''$ by $d_4\alpha + d_5\beta + d_6\gamma = 0$, where

$$d_4 = px_2z_2(rz_1^2 - qy_1^2), \quad d_5 = qy_1z_1(rz_2^2 - px_2^2), \quad d_6 = pqx_2^2y_1^2 - r^2z_1^2z_2^2.$$

The perspector of $A'B'C'$ and $A''B''C''$ is $B'B'' \cap C'C''$, with trilinears

$$d_2d_6 - d_3d_5 : d_3d_4 - d_1d_6 : d_1d_5 - d_2d_4.$$

These coordinates share three common factors, which cancel, leaving the perspector

$$P^* = qx_2y_1y_3 + rx_3z_1z_2 : ry_3z_1z_2 + px_2x_3y_1 : px_2x_3z_1 + qy_1y_3z_2. \quad (2)$$

Next, we show that the lines AA', BB', CC' concur. These lines are given, respectively, by

$$-px_2x_3\beta + ry_3z_2\gamma = 0, \quad pz_1x_3\alpha - qy_3y_1\gamma = 0, \quad -rz_1z_2\alpha + qx_2y_1\beta = 0,$$

from which it follows that the perspector of ABC and $A'B'C'$ is the point

$$P_1 = AA' \cap BB' \cap CC' = qx_2y_1y_3 : ry_3z_1z_2 : px_2x_3z_1. \quad (3)$$

The same method shows that the lines AA'', BB'', CC'' concur in the P-isoconjugate of P_1 :

$$P_2 = AA'' \cap BB'' \cap CC'' = rx_3z_1z_2 : py_1x_2x_3 : qy_1y_3z_2. \tag{4}$$

Obviously,

$$\begin{vmatrix} qx_2y_1y_3 + rx_3z_1z_2 & ry_3z_1z_2 + px_2x_3y_1 & px_2x_3z_1 + qy_1y_3z_2 \\ qx_2y_1y_3 & ry_3z_1z_2 & px_2x_3z_1 \\ rx_3z_1z_2 & py_1x_2x_3 & qy_1y_3z_2 \end{vmatrix} = 0,$$

so that the three perspectors are collinear. □

Example 1. Let $P = 1 : 1 : 1$ (the incenter), and let DEF be the cevian triangle of the centroid, so that $D = 0 : ca : ab$, etc. Then

$$P_1 = \frac{c}{b} : \frac{a}{c} : \frac{b}{a} \quad \text{and} \quad P_2 = \frac{b}{c} : \frac{c}{a} : \frac{a}{b},$$

these being the 1st and 2nd Brocard points, and

$$P^* = a(b^2 + c^2) : b\left(c^2 + a^2\right) : c(a^2 + b^2),$$

the midpoint X_{39} of segment P_1P_2.

2. Hofstadter triangles

Suppose r is a nonzero real number. Following ([3], p 176, 241), regard vertex B as a pivot, and rotate segment BC toward vertex A through angle rB. Let L_{BC} denote the line containing the rotated segment. Similarly, obtain line L_{CB} by rotating segment BC about C through angle rC. Let $A' = L_{BC} \cap L_{CB}$, and obtain similarly points B' and C'. The r-Hofstadter triangle is $A'B'C'$, and the $(1-r)$-Hofstadter triangle $A''B''C''$ is formed in the same way using angles $(1-r)A$, $(1-r)B$, $(1-r)C$.

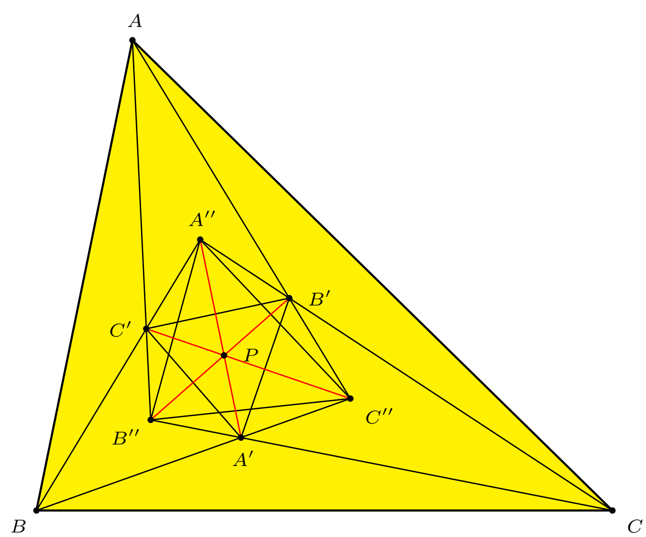

Figure 1. Hofstadter triangles $A'B'C'$ and $A''B''C''$

Trilinears for A' and A'' are easily found, and appear here in rows 2 and 3 of an equation for line $A'A''$:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \sin rB \sin rC & \sin rB \sin(C-rC) & \sin rC \sin(B-rB) \\ \sin(B-rB)\sin(C-rC) & \sin rC \sin(B-rB) & \sin rB \sin(C-rC) \end{vmatrix} = 0.$$

Lines $B'B''$ and $C'C''$ are similarly obtained, or obtained from $A'A''$ by cyclic permutations of symbols. It is then found by computer that the perspectivity determinant is 0 and that the perspector of the r- and $(1-r)$-Hofstadter triangles is the point

$$\begin{aligned} P(r) = \; & \sin(A-rA)\sin rB \sin rC + \sin rA \sin(B-rB)\sin(C-rC) \\ & : \sin(B-rB)\sin rC \sin rA + \sin rB \sin(C-rC)\sin(A-rA) \\ & : \sin(C-rC)\sin rA \sin rB + \sin rC \sin(A-rA)\sin(B-rB). \end{aligned}$$

The domain of P excludes 0 and 1. When r is any other integer, it can be checked that $P(r)$, written as $u : v : w$, satisfies

$$u \sin A + v \sin B + w \sin C = 0,$$

which is to say that $P(r)$ lies on the line L^∞ at infinity. For example, $P(2)$, alias $P(-1)$, is the point X_{30} in which the Euler line meets L^∞. Also, $P\left(\frac{1}{2}\right) = X_1$, the incenter, and $P\left(-\frac{1}{2}\right) = P\left(\frac{3}{2}\right) = X_{1770}$. Regarding $r = 1$ and $r = 0$, we obtain, *as limits*, the Hofstadter one-point and Hofstadter zero-point:

$$\begin{aligned} P(1) = \; & X_{359} = \frac{a}{A} : \frac{b}{B} : \frac{c}{C}, \\ P(0) = \; & X_{360} = \frac{A}{a} : \frac{B}{b} : \frac{C}{c}, \end{aligned}$$

remarkable because of the “exposed” vertex angles A, B, C. Another example is $P\left(\frac{1}{3}\right) = X_{356}$, the centroid of the Morley triangle.

This scattering of results can be supplemented by a more systematic view of selected points $P(r)$. In Figure 2, the specific triangle $(a, b, c) = (6, 9, 13)$ is used to show the points $P\left(r + \frac{1}{2}\right)$ for $r = 0, 1, 2, 3, \ldots, 5000$.

If the swing angles $rA, rB, rC, (1-r)B, (1-r)B, (1-r)C$ are generalized to $rA+\theta$, $rB+\theta$, $rC+\theta$, $(1-r)B+\theta$, $(1-r)B+\theta$, $(1-r)C+\theta$, then the perspector is given by

$$\begin{aligned} P(r,\theta) = \; & \sin(A-rA+\theta)\sin(rB-\theta)\sin(rC-\theta) \\ & \quad + \sin(rA-\theta)\sin(B-rB+\theta)\sin(C-rC+\theta) \\ & : \sin(B-rB+\theta)\sin(rC-\theta)\sin(rA-\theta) \\ & \quad + \sin(rB-\theta)\sin(C-rC+\theta)\sin(A-rA+\theta) \\ & : \sin(C-rC+\theta)\sin(rA-\theta)\sin(rB-\theta) \\ & \quad = + \sin(rC-\theta)\sin(A-rA+\theta)\sin(B-rB+\theta). \end{aligned}$$

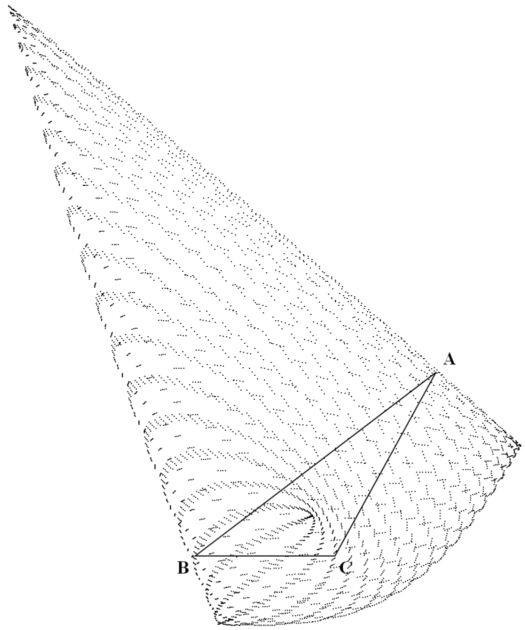

Figure 2

Trilinears for the other two perspectors are given by

$$\begin{aligned} P_1(r,\theta) = &\ \sin(rA-\theta)\csc(A-rA+\theta) \\ &: \sin(rB-\theta)\csc(B-rB+\theta) : \sin(rC-\theta)\csc(A-rC+\theta); \\ P_2(r,\theta) = &\ \sin(A-rA+\theta)\csc(rA-\theta) \\ &: \sin(B-rB+\theta)\csc(rB-\theta) : \sin(A-rC+\theta)\csc(rC-\theta). \end{aligned}$$

If $0 < \theta < 2\pi$, then $P(0,\theta)$ is defined, and taking the limit as $\theta \to 0$ enables a definition of $P(0,0)$. Then, remarkably, the locus of $P(0,\theta)$ for $0 \le \theta < 2\pi$ is the Euler line. Six of its points are indicated here:

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$(1/2)\arccos(5/2)$
$P(0,\theta)$	X_2	X_5	X_4	X_{30}	X_{20}	X_{549}

In general,

$$P\left(0, \frac{1}{2}\arccos t\right) = t\cos A+\cos B\cos C : t\cos B+\cos C\cos A : t\cos C+\cos A\cos B.$$

Among intriguing examples are several for which the angle θ, as a function of a, b, c (or A, B, C), is not constant:

if $\theta =$	then $P(0,\theta) =$
$\omega = \arctan\frac{4\sigma}{a^2+b^2+c^2}$(the Brocard angle)	X_{384}
$\frac{1}{2}\arccos\left(3-\frac{\|OI\|}{2R^2}\right)$	X_{21}
$\frac{1}{2}\arccos\frac{\|OI\|}{2R^2}$	X_{441}
$\frac{1}{2}\arccos(-1-2\cos^2 A\cos^2 B\cos^2 C)$	X_{22}
$\frac{1}{2}\arccos\left(-\frac{1}{3}-\frac{4}{3}\cos^2 A\cos^2 B\cos^2 C\right)$	X_{23}

where

$$\begin{aligned}\sigma &= \text{ area of } ABC,\\ |OI| &= \text{ distance between the circumcenter and the incenter},\\ R &= \text{ circumradius of } ABC.\end{aligned}$$

3. Cevian triangles

The cevian triangle of a point $X = x : y : z$ is defined by (1) on putting $(x_i, y_i, z_i) = (x, y, z)$ for $i = 1, 2, 3$. Suppose that X is a triangle center, so that

$$X = g(a,b,c) : g(b,c,a) : g(c,a,b)$$

for a suitable function $g\,(a,b,c)$. Abbreviating this as $X = g_a : g_b : g_c$, the cevian triangle of X is then given by

$$D = 0 : g_b : g_c, \quad E = g_a : 0 : g_c, \quad F = g_a : g_b : 0,$$

and the perspector of the derived triangles $A'B'C'$ and $A''B''C''$ in Theorem 1 is given by

$$P^* = g_a\left(qg_b^2 + rg_c^2\right) : g_b\left(pg_a^2 + rg_c^2\right) : g_c\left(pg_a^2 + qg_b^2\right),$$

which is a triangle center, namely the crosspoint (defined in the Glossary of [4]) of U and the P-isoconjugate of U. In this case, the other two perspectors are

$$P_1 = qg_ag_b^2 : rg_bg_c^2 : pg_cg_a^2 \text{ and } P_2 = rg_ag_c^2 : pg_bg_a^2 : qg_cg_b^2.$$

4. Pedal triangles

Suppose $X = x : y : z$ is a point for which $xyz \neq 0$. The pedal triangle DEF of X is given by

$$D = 0 : y{+}xc_1 : z{+}xb_1, \quad E = x{+}yc_1 : 0 : z{+}ya_1, \quad F = x{+}zb_1 : y{+}xa_1 : 0,$$

where

$$\begin{aligned}(a_1, b_1, c_1) &= (\cos A, \cos B, \cos C)\\ &= \left(\frac{b^2+c^2-a^2}{2bc}, \frac{c^2+a^2-b^2}{2ca}, \frac{a^2+b^2-c^2}{2ab}\right).\end{aligned}$$

The three perspectors as in Theorem 1 are given, as in (2)-(4) by

$$P^* = u + u' : v + v' : w + w', \tag{5}$$
$$P_1 = u : v : w, \tag{6}$$
$$P_2 = u' : v' : w', \tag{7}$$

where

$$\begin{aligned} u &= q(x + yc_1)(y + xc_1)(y + za_1), \\ v &= r(y + za_1)(z + ya_1)(z + xb_1), \\ w &= p(z + xb_1)(x + zb_1)(x + yc_1); \\ u' &= r(x + zb_1)(z + xb_1)(z + ya_1), \\ v' &= p(y + xc_1)(x + yc_1)(x + zb_1), \\ w' &= q(z + ya_1)(y + za_1)(y + xc_1). \end{aligned}$$

The perspector P^* is notable in two cases which we shall now consider: when X is on the line at infinity, L^∞, and when X is on the circumcircle, Γ.

Theorem 2. *Suppose DEF is the pedal triangle of a point X on L^∞. Then the perspectors P^*, P_1, P_2 are invariant of X, and P^* lies on L^∞.*

Proof. The three perspectors as in Theorem 1 are given as in (5)-(7) by

$$P^* = a\left(b^2r - c^2q\right) : b\left(c^2p - a^2r\right) : c\left(a^2q - b^2p\right), \tag{8}$$
$$P_1 = qac^2 : rba^2 : pcb^2,$$
$$P_2 = rab^2 : pbc^2 : qca^2.$$

Clearly, the trilinears in (8) satisfy the equation $a\alpha + b\beta + c\gamma = 0$ for L^∞. □

Example 2. For $P = 1 : 1 : 1 = X_1$, we have $P^* = a(b^2 - c^2) : b(c^2 - a^2) : c(a^2 - b^2)$, indexed in ETC as X_{512}. This and other examples are included in the following table.

P	661	1	6	32	663	649	667	19	25	184	48	2
P^*	511	512	513	514	517	518	519	520	521	522	523	788

We turn now to the case that X is on Γ, so that pedal triangle of X is degenerate, in the sense that the three vertices D, E, F are collinear ([1], [3]**)**. The line DEF is known as the pedal line of X. We restrict the choice of P to the point X_{31} :

$$P = a^2 : b^2 : c^2,$$

so that the P-isoconjugate of a point is the isotomic conjugate of the point.

Theorem 3. *Suppose X is a point on the circumcircle of ABC, and $P = a^2 : b^2 : c^2$. Then the perspector P^*, given by*

$$\begin{aligned} P^* = \; & bcx\left(y^2 - z^2\right)\left(ax(bz - cy) + yz(b^2 - c^2)\right) \\ & : cay\left(z^2 - x^2\right)\left(by(cx - az) + zx(c^2 - a^2)\right) \\ & : abz\left(x^2 - y^2\right)\left(cz(ay - bx) + xy(a^2 - b^2)\right), \end{aligned} \tag{9}$$

lies on the nine-point circle.

Proof. Since X satisfies $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, we can and do substitute $z=-\frac{cxy}{ay+bx}$ in (8), obtaining

$$P^{*}=\alpha:\beta:\gamma, \tag{10}$$

where

$$\begin{aligned}\alpha=&\ ycb\left(ay+bx-cx\right)\left(ay+bx+cx\right)\left(2abx+a^{2}y+b^{2}y-c^{2}y\right),\\ \beta=&\ xca\left(2aby+a^{2}x+b^{2}x-c^{2}x\right)\left(ay+bx-cy\right)\left(ay+bx+cy\right),\\ \gamma=&\ ab\left(ay+bx\right)\left(b^{3}x-a^{3}y-a^{2}bx+ab^{2}y+ac^{2}y-bc^{2}x\right)\left(x+y\right)\left(y-x\right).\end{aligned}$$

An equation for the nine-point circle [5] is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta-(1/2)(a\alpha+b\beta+c\gamma)(a_{1}\alpha+b_{1}\beta+c_{1}\gamma)=0, \tag{11}$$

and using a computer, we find that P^{*} indeed satisfies (11). Using $ay+by=-\frac{cxy}{z}$, one can verify that the trilinears in (10) yield those in (9). □

A description of the perspector P^{*} in Theorem 3 is given in Theorem 4, which refers to the antipode X' of X, defined as the reflection of X in the circumcenter, O; i.e., X' is the point on Γ that is on the opposite side of the diameter that contains X. Theorem 4 also refers to the crosssum of two points, defined (Glossary of [4]) for points $U=u:v:w$ and $U'=u':v':w'$ by

$$U\oplus U'=vw'+wv':wu'+uw':uv'+vu'.$$

Theorem 4. *Suppose X is a point on the circumcircle of ABC, and let X' denote the antipode of X. Then $P^{*}=X\oplus X'$.*

Proof. [1] Since X is an arbitrary point on Γ, there exists θ such that

$$X=\csc\theta:\csc(C-\theta):-\csc(B+\theta),$$

where θ, understood here a function of a,b,c, is defined ([6], [3, p. 39]) by

$$0\le 2\theta=\measuredangle AOX<\pi,$$

so that the antipode of X is

$$X'=\sec\theta:-\sec(C-\theta):-\sec(B+\theta).$$

The crosssum of the two antipodes is the point $X\oplus X'=\alpha:\beta:\gamma$ given by

$$\begin{aligned}\alpha=&\ -\csc(C-\theta)\sec(B+\theta)+\csc(B+\theta)\sec(C-\theta)\\ \beta=&\ -\csc(B+\theta)\sec\theta-\csc\theta\sec(B+\theta)\\ \gamma=&\ -\csc\theta\sec(C-\theta)+\csc(C-\theta)\sec\theta.\end{aligned}$$

It is easy to check by computer that $\alpha:\beta:\gamma$ satisfies (11). □

[1]This proof includes a second proof that P^{*} lies on the nine-point circle.

Example 3. In Theorems 3 and 4, let $X = X_{1113}$, this being a point of intersection of the Euler line and the circumcircle. The antipode of X is X_{1114}, and we have

$$X_{1113} \oplus X_{1114} = X_{125},$$

the center of the Jerabek hyperbola, on the nine-point circle.

Example 4. The antipode of the Tarry point, X_{98}, is the Steiner point, X_{99}, and

$$X_{98} \oplus X_{99} = X_{2679}.$$

Example 5. The antipode of the point, $X_{101} = \frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b}$ is X_{103}, and

$$X_{101} \oplus X_{103} = X_{1566}.$$

Example 6. The Euler line meets the line at infinity in the point X_{30}, of which the isogonal conjugate on the circumcircle is the point

$$X_{74} = \frac{1}{\cos A - 2\cos B \cos C} : \frac{1}{\cos B - 2\cos C \cos A} : \frac{1}{\cos C - 2\cos A \cos B}.$$

The antipode of X_{74} is the center of the Kiepert hyperbola, given by

$$X_{110} = \csc(B - C) : \csc(C - A) : \csc(A - B),$$

and we have

$$X_{74} \oplus X_{110} = X_{3258}.$$

Our final theorem gives a second description of the perspector $X \oplus X'$ in (9). The description depends on the point $X(\text{medial})$, this being functional notation, read "X of medial (triangle)", in the same way that $f(x)$ is read "f of x"; the variable triangle to which the function X is applied is the cevian triangle of the centroid, whose vertices are the midpoint of the sides of the reference triangle ABC. Clearly, if X lies on the circumcircle of ABC, then $X(\text{medial})$ lies on the nine-point circle of ABC.

Theorem 5. *Let X' be the antipode of X. Then $X \oplus X'$ is a point of intersection of the nine-point circle and the line of the following two points: the isogonal conjugate of X and $X(\textit{medial})$.*

Proof. Trilinears for $X(\text{medial})$ are given ([3, p. 86]) by

$$(by + cz)/a : (cz + ax)/b : (ax + by)/c.$$

Writing $u : v : w$ for trilinears for $X \oplus X'$ and $yz : zx : xy$ for the isogonal conjugate of X, and putting $z = -cxy/(ay + bx)$ because $X \in \Gamma$, we find

$$\begin{vmatrix} u & v & w \\ yz & zx & xy \\ (by + cz)/a & (cz + ax)/b & (ax + by)/c \end{vmatrix} = 0,$$

so that the three points are collinear. □

As a source of further examples for Theorems 4 and 5, suppose D, E, F are points on the circumcircle. Let D', E', F' be the respective antipodes of D, E, F, so that the triangle $D'E'F'$ is the reflection in the circumcenter of triangle DEF. Let

$$D'' = D \oplus D', \quad E'' = E \oplus E', \quad F'' = F \oplus F',$$

so that $D''E''F''$ is inscribed in the nine-point circle.

Example 7. If DEF is the circumcevian triangle of the incenter, then $D''E''F''$ is the medial triangle.

Example 8. If DEF is the circumcevian triangle of the circumcenter, then $D''E''F''$ is the orthic triangle.

Example 9. If DEF is the circumtangential triangle, then $D''E''F''$ is homothetic to each of the three Morley equilateral triangles, as well as the circumtangential triangle (perspector X_2, homothetic ratio $-1/2$) the circumnormal triangle (perspector X_4, homothetic ratio $1/2$), and the Stammler triangle (perspector X_{381}). If DEF is the circumnormal triangle, then $D''E''F''$ is the same as for the circumtangential. (For descriptions of the various triangles, see [5].) The triangle $D''E''F''$ is the second of two equilateral triangles described in the article on the Steiner deltoid at [5]; its vertices are given as follows:

$$\begin{aligned}
D'' = {} & \cos(B-C) - \cos\left(\frac{B}{3} - \frac{C}{3}\right) \\
& : \cos(C-A) - \cos\left(B - \frac{2C}{3}\right) : \cos(A-B) - \cos\left(B - \frac{2C}{3}\right), \\
E'' = {} & \cos(B-C) - \cos\left(C - \frac{2A}{3}\right) \\
& : \cos(C-A) - \cos\left(\frac{C}{3} - \frac{A}{3}\right) : \cos(A-B) - \cos\left(C - \frac{2A}{3}\right), \\
F'' = {} & \cos(B-C) - \cos\left(A - \frac{2B}{3}\right) \\
& : \cos(C-A) - \cos\left(A - \frac{2B}{3}\right) : \cos(A-B) - \cos\left(\frac{A}{3} - \frac{B}{3}\right).
\end{aligned}$$

5. Summary and concluding remarks

If the point X in Section 4 is a triangle center, as defined at [5], then the perspector P^* is a triangle center. If instead of the cevian triangle of X, we use in Section 4 a central triangle of type 1 (as defined in [3], pp. 53-54**)**, then P^* is clearly the same point as obtained from the cevian triangle of X.

Regarding pedal triangles, in Section 4, there, too, if X is a triangle center, then so is P^*, in (8). The same perspector is obtained by various central triangles of type 2. In all of these cases, the other two perspectors, P_1 and P_2, as in (6) and (7) are a bicentric pair [5].

In Examples 3-6, the antipodal pairs are triangle centers. The 90° rotation of such a pair is a bicentric pair, as in the following example.

Example 10. The Euler line meets the circumcircle in the points X_{1113} and X_{1114}. Let X^*_{1113} and X^*_{1114} be their 90° rotations about the circumcenter. Then $X^*_{1113} \oplus X^*_{1114}$ (the perspector of two triangles $A'B'C'$ and $A''B''C''$ as in Theorem 1) lies on the nine-point circle, in accord with Theorems 4 and 5. Indeed $X^*_{1113} \oplus X^*_{1114} = X_{113}$, which is the nine-point-circle-antipode of $X_{125} = X_{1113} \oplus X_{1114}$. Likewise, $X^*_{1379} \oplus X^*_{1380} = X_{114}$ and $X^*_{1381} \oplus X^*_{1382} = X_{119}$.

Example 10 illustrates the following theorem, which the interested reader may wish to prove: *Suppose X and Y are circumcircle-antipodes, with 90° rotations $X^*$and Y^*. Then $X^* \oplus Y^*$is the nine-point-circle-antipode of the center of the rectangular circumhyperbola formed by the isogonal conjugates of the points on the lineXY.*

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