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On a Theorem of Intersecting Conics

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Abstract. Given two conics over an infinite field that intersect at the origin, a line through the origin will, in general intersect both conic sections once more each, at points C and D. As the line varies we find that the midpoint of C and D traces out a curve, which is typically a quartic. Intuitively, this locus is the "average" of the two conics from the perspective of an observer at the origin. We give necessary and sufficient conditions for this locus to be a point, line, line minus a point, or a conic itself.

1. Introduction

Consider, in Figure 1, an observer standing on the point A . He wants to find the "average" of the two circles P_1, P_2. He could accomplish his task by facing towards an arbitrary direction, and then measuring his distance to P_1 and P_2 through that direction. The distances may be negative if either circle is behind him. He can then take the average of the two distances and mark it along his chosen direction. As our observer repeats this process, he will eventually trace out the circle ABE_1. Thus, in some sense ABE_1 is the "average" of our two original circles. In this paper we will consider analogous (weighted) averages of two nondegenerate plane conics meeting at a point A. This curve will be termed the "medilocus".

Definition 1 (Nondegenerate conic)**.** A nondegenerate conic is the zero set of a quadratic equation in two variables over an infinite field $\mathbb{F}$ which is not a point and does not contain a line.

In this paper, we shall assume all conics nondegenerate. We thus exclude lines and pairs of lines, for example $xy = 0$. We also remark here that if a conic consists of more than one point, it must be infinite: we will prove this in Proposition 6.

We will also define a tangent line to a point on a conic:

Definition 2 (Tangent line)**.** The tangent line to a conic represented by the equation $P(x, y) = 0$ at the point (x_0, y_0) is the line that contains the point (x_0, y_0) with (linear) equation $L(x, y) = 0$, whose substitution into $P(x, y) = 0$ gives a quadratic equation in one variable with a double root.

We remark here that the tangent line of a conic at the origin is the homogeneous linear part of the equation for the conic. We will prove that the homogeneous linear part is always nonzero in Lemma 4.

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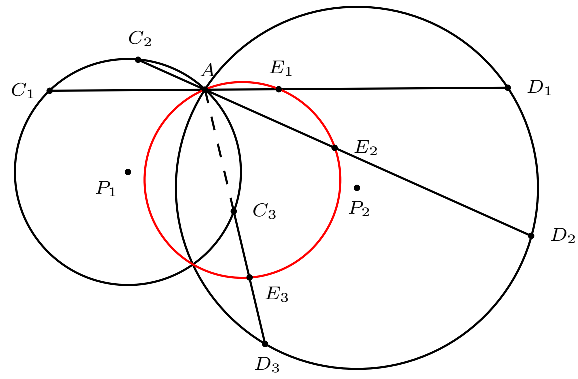

Figure 1. The medilocus of weight 1/2 of the two circles P_1, P_2 is the circle ABE_1. As we rotate a line CD around A into C_1D_1, C_2D_2, C_3D_3, the locus of the midpoint $E_1E_2E_3$ is the medilocus.

Definition 3 (Medilocus)**.** Let P_1, P_2 be two conics meeting at a distinguished intersection point A. The medilocus of weight k is the set of all points of the form $E = kC + (1-k)D$, where C, A, D are points on some line L through A, with $C \in P_1$ and $D \in P_2$. $C = A$ or $D = A$ is possible if and only if L is tangent to P_1 or P_2 respectively at A. The medilocus is denoted $M(P_1, P_2, A, k)$.

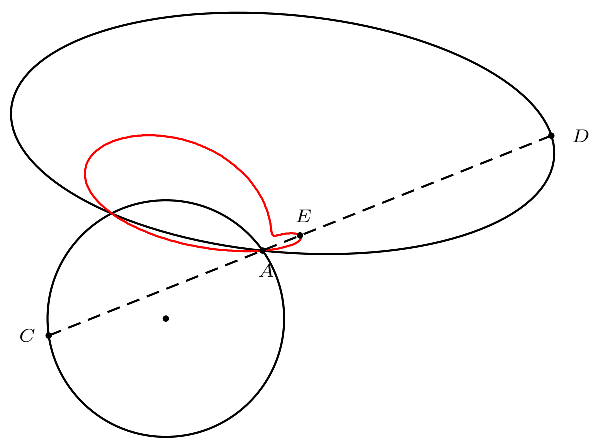

Figure 2. The red curve is the medilocus weight $1/2$ of the circle and the ellipse, with distinguished intersection point at A.

We comment here that given a line L, C and D are uniquely defined, if they exist. Clearly any line intersects a conic at most twice, and the following proposition will demonstrate that C or D cannot be multiply defined even when L is tangent to P_1 or P_2.

Proposition 1. *A line tangent to a conic at the origin cannot intersect the conic section again at another point.*

Proof. Choose coordinates so that the tangent line at the origin is $x = 0$. Restricting the conic to $x = 0$ gives us a quadratic polynomial in y, which has at most two zeroes with multiplicity. But since $x = 0$ is tangent to the conic at the origin, there is a double zero at $y = 0$, and so there cannot be any more. □

If P_1 and P_2 are the same conic, we can see that their medilocus is the conic P_1. The medilocus of weight 0 is always the conic P_1, and the medilocus of weight 1 is always the conic P_2. Figure 1 shows the medilocus of weight $1/2$ of two circles. While the mediloci we have seen so far have been circles, if we make different choices of intersecting conics we usually obtain a more interesting medilocus. Figure 2 is an example. Note in this case that our medilocus is not even a conic. Our research began as an attempt to answer the question, when is the medilocus of two conic sections itself either a conic, or another "well-behaved" curve? We addressed this question by introducing an equivalence relation on conics.

Definition 4 (Medisimilarity)**.** Two conics are medisimilar if the homogeneous quadratic part of the equation of the first conic is a nonzero scalar multiple of the homogeneous quadratic part of the equation of the second conic. Equivalently, we can say that two conics are medisimilar if their equations can be written in such a way that their homogeneous quadratic parts are equal.

The proposition that follows describes medisimilarity in the real plane, so that the reader may gain some geometric intuition of what medisimilarity means. For the sake of brevity, this proposition is stated without proof. All that a proof requires is the understanding of the role that the homogeneous quadratic part of the equation of a conic plays in its geometry. Both [3] and [2] serve as useful references in this regard.

Proposition 2. *Two intersecting conics in $\mathbb{R}^2$ can be medisimilar only if they are both ellipses, both hyperbolas, or both parabolas.*
(1) *Two intersecting ellipses are medisimilar only if they have the same eccentricity, and their respective major axes are parallel.*
(2) *Two hyperbolas are medisimilar if and only if the two asymptotes of the first hyperbola have the same slopes as the two asymptotes of the second hyperbola.*
(3) *Two parabolas are medisimilar if and only if their directrices are parallel.*

Our main theorem can be stated thus:

Theorem 3. *The medilocus weight $k \neq 0, 1$ of two conics P_1, P_2 over an infinite field is a conic, a line, a line with a missing point, or a point if and only if P_1, P_2 are medisimilar.*

2. Preliminaries

Here we establish some conventions and prove certain lemmas that will streamline the proofs in the following section. Firstly, whenever two conics intersect and we wish to describe the medilocus, we shall choose coordinates so that the distinguished intersection point is at the origin. We will label the two conics P_1 and P_2

and express them as follows:

$$P_1 : \ Q_1(x,y) + L_1(x,y) = 0, \tag{1}$$

$$P_2 : \ Q_2(x,y) + L_2(x,y) = 0, \tag{2}$$

where Q_1, Q_2 are quadratic forms, and L_1, L_2 are linear forms.

We are now prepared to demonstrate some brief lemmas about the conics P_1, P_2.

Lemma 4. *Neither L_1 nor L_2 can be identically zero.*

Proof. If L_1 is identically zero, then the equation of P_1 is a homogeneous quadratic form. Thus if $P_1(a,b) = 0$, then either $a = b = 0$ or the line $\{(at, bt), t \in \mathbb{F}\}$ is contained in P_1. Thus P_1 is either a point or contains a line, neither of which is acceptable by our definition of conic. An analogous contradiction occurs when L_2 is identically zero. □

Lemma 5. *L_1 cannot divide Q_1, and L_2 cannot divide Q_2.*

Proof. If L_1 divides Q_1, then L_1 divides P_1 as well. But according to our definition, a conic cannot contain a zero set of a linear equation. A completely analogous proof works for P_2. □

Proposition 6. *If a quadratic equation in two variables over an infinite field has two distinct solutions, it has infinitely many.*

Proof. Let $P(x,y)$ be a quadratic polynomial with two distinct solutions. We may choose coordinates so that one of the solutions is the origin, and the other is (a,b), where a is nonzero. Substitute $y = mx$ into $P(x,y)$. We then get

$$P(x,mx) = xL(m) + x^2Q(m),$$

where L, Q are linear and quadratic functions respectively. Note that P has no constant term since $(0,0)$ is a solution. We know when $m = b/a$, $P(x,mx)$ has two distinct solutions: this implies that $Q(b/a)$ is not zero. We also deduce that $-L(b/a)/Q(b/a) = a$, so $L(b/a)$ is not zero. In particular, we now know that neither Q nor L is identically zero. Q has at most two solutions, m_1, m_2. Thus for every choice of $m \neq m_1, m_2$, $P(x,mx)$ has solutions $x = 0$ and $x = -L(m)/Q(m)$. We have infinitely many choices for m, and so P must have infinitely many solutions. □

Lemma 7. *If we define conics P_1, P_2 intersecting at the origin as in* (1) *and* (2), *then the medilocus is either the zero set of the equation*

$$1 = k\left(-\frac{L_1(x,y)}{Q_1(x,y)}\right) + (1-k)\left(-\frac{L_2(x,y)}{Q_2(x,y)}\right), \tag{3}$$

or the union of the zero set of this equation with a point at the origin. The origin is in the medilocus if and only if there exist $a, b \in \mathbb{F}$ not both zero for which

$$0 = k\left(-\frac{L_1(a,b)}{Q_1(a,b)}\right) + (1-k)\left(-\frac{L_2(a,b)}{Q_2(a,b)}\right), \tag{4}$$

in which case that medilocus point on the origin is given by the weighted average of the intersections of the line $\{(at, bt) | t \in \mathbb{F}\}$ with P_1 and P_2.

Proof. Firstly, we write a parametric equation for a line through the origin and a non-origin point (a, b) as

$$l = \{(at, bt), t \in \mathbb{F}\}. \tag{5}$$

We define points C, D as in Definition 3, such that the line l intersects P_1 at the origin and at C, and intersects P_2 at the origin and at D. We wish to find the coordinates of C, D in terms of t. Substituting (5) into (1), we obtain the quadratic

$$Q_1(at, bt) + L_1(at, bt) = 0.$$

Provided $Q_1(at, bt) \neq 0$, this means either $t = 0$ or

$$t = -\frac{L_1(a, b)}{Q_1(a, b)}.$$

If b/a is the slope of L_1 (if $a = 0$, we consider b/a to be the "slope" of $x = 0$) then l is tangent to P_1 at the origin, and so we set C at the origin. This is consistent with the equation since $L_1(a, b) = 0$. We cannot have $Q_1(a, b) = 0$ for that same a and b, otherwise $Q_1(x, y)$ is a multiple of $L_1(x, y)$, contradicting Lemma 5.

Thus l intersects P_1 at $t = 0$ and $t = -L_1(a, b)/Q_1(a, b)$, and so C is at $t = -L_1(a, b)/Q_1(a, b)$. Similarly, the t-coordinate of D is $t = -L_2(a, b)/Q_2(a, b)$.

Note that for certain choices of a, b the denominators can be zero. If this happens we know that the line through the origin with that slope b/a does not intersect P_1 or P_2, and so C or D is undefined and the medilocus does not intersect l except, perhaps, at the origin. For example, when we are working on the field $\mathbb{R}$ the denominator Q_1 will be zero when P_1 is a hyperbola and b/a is the slope its asymptote or when P_1 is a parabola and b/a is the slope of its axis of symmetry.

Let T be a function of a, b such that T is the t-coordinate of the non-origin point of intersection between the medilocus and the line $\{at, bt\}$. By the definition of the medilocus we know

$$T = k\left(-\frac{L_1(a, b)}{Q_1(a, b)}\right) + (1 - k)\left(-\frac{L_2(a, b)}{Q_2(a, b)}\right). \tag{6}$$

Substitute $T = t, x = at, y = bt$. If $T \neq 0$ we can divide by t, and so the points of the medilocus that are not on the origin satisfy (3). If $T = 0$, then the medilocus contains the origin and (4) holds for a, b.

Conversely, let (a, b) be any solution of (3). We consider the line expressed parametrically as $\{(at, bt) | t \in \mathbb{F}\}$. It intersects P_1 at the origin and when $t = -L_1(a, b)/Q_1(a, b)$, and it intersects P_2 at the origin and when $t = -L_2(a, b)/Q_2(a, b)$. Thus there is a point of the medilocus at

$$t = k\left(-\frac{L_1(a, b)}{Q_1(a, b)}\right) + (1 - k)\left(-\frac{L_2(a, b)}{Q_2(a, b)}\right) = 1,$$

but then $t = 1$ is the point (a, b), and so every solution to (3) is indeed in the medilocus.

If (a,b) is a non-origin solution for (4), then we note that the line $\{(at,bt)|t \in \mathbb{F}\}$ intersects P_1, P_2 at $t = -L_1(a,b)/Q_1(a,b)$, $t = -L_2(a,b)/Q_2(a,b)$ respectively. But by (4), the weighted average of the two values of t is 0, and so the medilocus does indeed contain the origin. □

Some mediloci intersect the distinguished intersection point, and some do not. The medilocus in Figure 1 contains the distinguished intersection point. If we consider two parabolas in $\mathbb{R}^2$, $P_1 = x^2 - 2x - y, P_2 = x^2 - 2x + y$ with distinguished intersection point at the origin, we find that the medilocus weight $1/2$ is the line $x = 2$ which does not contain the origin.

3. A criterion for medisimilarity

Proposition 8. *The medilocus of two intersecting medisimilar conics is a conic section, a line, a line with a missing point, or a point.*

Proof. If $k = 0$ or $k = 1$ we are done, so let us assume k is not equal to 0 or 1. With a proper choice of coordinates, we may translate the distinguished intersection point to the origin. We may thus represent the two conics as in (1), (2). By Lemma 7, we know that the medilocus is the zero set of (3), possibly union the origin. And since P_1 is medisimilar to P_2, we may scale their equations appropriately so that $Q_1 = Q_2$. If we set $Q = Q_1 = Q_2$, $L = -kL_1 - (1-k)L_2$, (3) and (4) in Lemma 7 respectively reduce to

$$1 = \frac{L(x,y)}{Q(x,y)}, \tag{7}$$

$$0 = \frac{L(x,y)}{Q(x,y)}. \tag{8}$$

Five natural cases arise:

(1) If L is identically zero, then there are no solutions to (7). Thus the medilocus is either empty or the point at the origin. But since Q cannot be identically zero, there must exist (a,b) such that $Q(a,b) \neq 0$, in which case (a,b) would be a solution to (8), and hence the medilocus is precisely the point at the origin.

(2) If L is nonzero and Q is irreducible, we consider the line $L(x,y) = 0$ through the origin and write it in the form $\{(at,bt)\}$. But then $L(a,b) = 0$, and so (a,b) satisfies (8). Thus by Lemma 7 the medilocus contains a point at the origin. But then (7) simply reduces to $Q - L = 0$, and so again by Lemma 7 the medilocus is the zero set of $Q - L = 0$, and thus a conic.

(3) If L is nonzero and Q factors into linear terms M, N, neither of which is a multiple of L, we have by Lemma 7 that the medilocus is the zero set of $1 = L/MN$, possibly union a point in the origin. Reasoning identical to that of the previous case tells us that the origin is indeed in the medilocus. Since M, N aren't multiples of L, we add no erroneous solutions (except for the origin, which we know is in the medilocus) by clearing denominators. This shows that the medilocus is the zero set of $MN - L = 0$ and thus is a conic.

(4) If L is nonzero and Q factors into ML with M not a multiple of L, then Lemma 7 gives us that the medilocus is the zero set of $1 = L/ML$, possibly union

the origin. We show that the medilocus does not contain the origin. Assume instead for some a, b that (8) is satisfied. This implies $Q(a, b)$ nonzero $L(a, b) = 0$, which means that $L(x, y)$ has a root which is not a root of $Q(x, y)$. This is impossible since $L|Q$. Thus the medilocus cannot contain the origin. But then the zero set of $1 = L/ML$ is the zero set of $M = 1$, a line not through the origin, minus the zero set of $L = 0$. This is a line missing a point.

(5) If L nonzero, and $Q = L^2$, then by Lemma 7 the medilocus is the zero set of $1 = L/L^2$, possibly union a point at the origin. By reasoning similar to the previous part, we can conclude that the medilocus does not contain a point at the origin. Thus the medilocus is the zero set of $L = 1$, minus the zero set of $L = 0$. Since the lines $L = 1, L = 0$ are disjoint, we can conclude that the medilocus is just the zero set of the line $L = 1$. □

We will now demonstrate examples for these various cases in $\mathbb{R}^2$.

Example 1. Figure 1 gives us an example of two medisimilar conics having a medilocus that is a conic.

Example 2. If we consider two parabolas in $\mathbb{R}^2$, $P_1 = x^2 - 2x - y, P_2 = x^2 - 2x + y$ with distinguished intersection point at the origin, we find that the medilocus weight $1/2$ is the line $x = 2$.

Example 3. If we consider two hyperbolas in $\mathbb{R}^2$, $P_1 = yx - x - y, P_2 = yx + x - y$ with distinguished intersection point at the origin, we find that the medilocus weight $1/2$ is the line $x = 1$ missing the point $(1, 0)$.

Example 4. The parabolas $P_1 = y - x^2, P_2 = y + x^2$ in $\mathbb{R}^2$ have the single point at the origin as their medilocus weight $1/2$.

Before we proceed to prove the other direction of Theorem 3, let us first demonstrate a case where that direction comes literally a point away from failing. This proposition will also be used in the proof for Proposition 11.

Proposition 9. *Consider two conics intersecting the origin,*

$$P_1 : h(x, y)c_1(x, y) + d_1(x, y) = 0,$$
$$P_2 : h(x, y)c_2(x, y) + d_2(x, y) = 0,$$

where h, c_1, c_2, d_1, d_2 are all linear forms of x, y through the origin. Let $h(x, y)$, $c_1(x, y)$ and $c_2(x, y)$ not be scalar multiples of each other, so that in particular P_1, P_2 are not medisimilar. Also assume $kc_1(x, y)d_2(x, y)+(1-k)c_2(x, y)d_1(x, y)$ is a multiple of $h(x, y)$. Then the medilocus weight k of P_1, P_2 is a conic missing exactly one point.

Proof. We invoke Lemma 7, so we know that points of the medilocus away from the origin can be expressed in the form

$$1 = \frac{-kc_1(x, y)d_2(x, y) - (1 - k)c_2(x, y)d_1(x, y)}{h(x, y)c_1(x, y)c_2(x, y)}.$$

If we denote $kc_1(x,y)d_2(x,y) + (1-k)c_2(x,y)d_1(x,y) = h(x,y)g(x,y)$ for another linear equation $g(x,y)$ through the origin, we can write the medilocus except for the origin as the zero set of the quadratic equation

$$c_1(x,y)c_2(x,y) + g(x,y) = 0, \tag{9}$$

subtracting the points on the line $h(x,y) = 0$. Since c_1, c_2 are not scalar multiples, $c_1(x,y), c_2(x,y)$ cannot divide $g(x,y)$ without contradicting Lemma 5, and so $c_1(x,y) = 0, c_2(x,y) = 0$ do not intersect the curve defined by (9) except at the origin.

First, assume that $g(x,y), h(x,y)$ are not scalar multiples of each other. Recall that $g(x,y)$ is not a multiple of $c_1(x,y)$ or $c_2(x,y)$. (4) is written as

$$0 = \frac{h(a,b)g(a,b)}{h(a,b)c_1(a,b)c_2(a,b)}. \tag{10}$$

If we express $g(x,y) = 0$ as $\{(a't, b't)\}$, then $g(a',b') = 0$ and since h is not a scalar multiple of g, $h(a',b') \neq 0$ and thus $a = a', b = b'$ will solve (10). Thus by Lemma 7 the medilocus contains the origin.

Additionally, assume that $h(x,y)$ is not simply a multiple of y. In that case, we can write $c_1(x,y), c_2(x,y), g(x,y)$ respectively as $m_1h(x,y) + \alpha y, m_2h(x,y) + \beta y, m_3h(x,y) + \gamma y$ where $m_1, m_2, m_3, \alpha, \beta, \gamma$ are constants, and α, β, γ are nonzero. But substituting these equations into (9), this implies that at a point on $y = -\gamma/\alpha\beta$, (a nonzero value for y) $h(x,y) = 0$ intersects the curve (9). Thus the medilocus must be a conic subtracting one point.

If $h(x,y)$ is a multiple of y, we instead write $c_1(x,y), c_2(x,y), g(x,y)$ respectively as $m_1h(x,y) + \alpha x, m_2h(x,y) + \beta x, m_3h(x,y) + \gamma x$ and proceed analogously.

Now consider the case where $g = sh$, for some nonzero constant s. In this case certainly $h = 0$ does not intersect (9) other than the origin, and so the solutions to (9) must be precisely the points of the medilocus away from the origin. We claim that the medilocus cannot contain the origin. By Lemma 7 the medilocus contains the origin if and only if for some a, b not both zero

$$0 = -\frac{kd_1(a,b)c_2(a,b) + (1-k)d_2(a,b)c_1(a,b)}{h(a,b)c_1(a,b)c_2(a,b)} = -\frac{sh(a,b)^2}{h(a,b)c_1(a,b)c_2(a,b)}.$$

But then clearly the denominator of the right hand side is zero whenever the numerator is zero, so this equation can never be satisfied. We conclude that the medilocus is the conic represented by (9) missing a point at the origin. □

Example 5. The medilocus weight $1/2$ of $x^2 - yx + y = 0$, $x^2 + yx + y = 0$ is $y = y^2 - x^2$, missing the point at $(0,1)$.

We will now need to invoke the definition of variety in our next lemma.

Definition 5 (Variety)**.** For an algebraic equation $f = 0$ over a field $\mathbb{F}$ we use the notation $V(f)$, the *variety* or *zero set* of f to represent the subset of $\mathbb{F}[x,y]$ for which f is zero.

This definition is given in [1, p.8].

Lemma 10. *Let $l, g, f \in \mathbb{F}[x, y]$ be polynomials with degree $l = 1$, degree $g = 2$ and $V(g)$ infinite.*
(i) *If $V(l) \subset V(f)$, then $l|f$.*
(ii) *If $V(g) \subset V(f)$ and $V(g)$ is not a line, then $g|f$.*

Proof. (i) First, we claim that if l is linear and $V(l) \subset V(f)$, then $l|f$. Corollary 1 of Proposition 2 in Chapter 1 of [1] says that if l is an irreducible polynomial in a closed field $\overline{\mathbb{F}}$, $V(l) \subset V(f)$ and $V(l)$ infinite, then we may conclude that $l|f$ in $\overline{\mathbb{F}}[x, y]$. Thus $l|f$ in $\mathbb{F}[x, y]$ as well.

(ii) We consider then the case where $g = l_1 l_2$ factors as a product of distinct linear factors. We then have $V(g) = V(l_1) \cup V(l_2) \subset V(f)$, so this implies $l_1|f$ and $l_2|f$, therefore $l_1 l_2|f$ and so $g|f$.

If g is irreducible over $\mathbb{F}$, suppose g is reducible over $\overline{\mathbb{F}}$. Then $g = st$ for some linear terms s, t, and we have $V(g) = V(s) \cup V(t)$ over $\overline{\mathbb{F}}^2$. Thus $V(g)$ over $\mathbb{F}^2$ is empty, one point, two points, a point and line, a line, or two lines. The first three cases are not possible because $V(g)$ over $\mathbb{F}^2$ is infinite. The last three cases are not possible by our observation in the first paragraph of this proof, because then a linear equation divides g, contradicting irreducibility in $\mathbb{F}$. We conclude that g is irreducible over $\overline{\mathbb{F}}$. $V(g) \cap V(f)$ infinite implies $g|f$ again by Corollary 1 of Proposition 2 in Chapter 1 of [1] □

Proposition 11. *The medilocus weight k of two conics P_1, P_2 is a conic itself only if $k = 0$, $k = 1$, or P_1, P_2 are medisimilar.*

Proof. If $k = 0, 1$ we are done; we may henceforth assume k is neither 0 nor 1.

We may set the conics P_1, P_2 as in (1),(2). By Lemma 7, we know that we can represent the medilocus by (3) for every point except for the origin. Clearing the denominators, we get the quartic $f(x, y) = 0$ where

$$f(x, y) = Q_1(x, y)Q_2(x, y) + kL_1(x, y)Q_2(x, y) + (1 - k)L_2(x, y)Q_1(x, y). \tag{11}$$

We claim that the zero set of this equation contains the medilocus.

Note that we might have added some erroneous points by clearing denominators this way: for example, if $Q_1(x, y), Q_2(x, y)$ have a linear common factor, then $f(x, y) = 0$ will contain the solutions of that linear factor, even if the medilocus itself does not. It is clear, however that every point of the medilocus is in the zero set of $f(x, y) = 0$. In particular, the origin is in that zero set whether or not it is in the medilocus.

Whether $f(x, y) = 0$ is the medilocus or merely contains the medilocus, by Lemma 10 and Proposition 6, $f(x, y)$ contains a conic only if it has a quadratic factor. So assume that the quartic f factors into quadratics f_1, f_2. Let us define

$$f_1 = q_1 + w_1 + c_1,$$
$$f_2 = q_2 + w_2 + c_2.$$

The c_i are constants, w_i are homogeneous linear terms in x, y, and the q_i are homogeneous quadratic terms in x, y. First, we note that since $f = f_1 f_2$ has no quadratic, linear or constant terms, either $c_1 = 0$ or $c_2 = 0$. We claim that this implies c_1, c_2, w_1w_2 are all zero. Without loss of generality, let us start with the assumption that $c_1 = 0$. But then c_2w_1 is the homogeneous linear part of f, and so either $c_2 = 0$ or $w_1 = 0$. If $c_2 = 0$ then w_1w_2 is the homogeneous quadratic part of f, and so $w_1w_2 = 0$. If $w_1 = 0$, then $c_2q_1 = 0$, and since by this point $f_1 = q_1$, q_1 must be nonzero and we must have $c_2 = 0$.

And so we must have $c_1 = 0$, $c_2 = 0$ and $w_1w_2 = 0$. Without loss of generality we let $w_1 = 0$. We now have

$$f_1 = q_1,$$
$$f_2 = q_2 + w_2.$$

Given that $f = f_1f_2$, the homogeneous quartic and homogeneous cubic parts must match (with reference to (11)). This gives us

$$q_1(x,y)q_2(x,y) = Q_1(x,y)Q_2(x,y), \tag{12}$$
$$q_1(x,y)w_2(x,y) = kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y). \tag{13}$$

From (12), we have three possibilities: either $q_1(x,y)$ divides $Q_1(x,y)$, $q_1(x,y)$ divides $Q_2(x,y)$ or $q_1(x,y)$ factors into two homogeneous linear terms $u_1(x,y)$,$v_1(x,y)$ and we have both $u_1(x,y)|Q_1(x,y)$ and $v_1(x,y)|Q_2(x,y)$. We handle these three cases one by one.

(1) If $q_1(x,y)|Q_1(x,y)$, we consider (13) and conclude that it must be the case that

$$q_1(x,y)|kL_1(x,y)Q_2(x,y).$$

This is possible only if a linear factor of $q_1(x,y)$ is a scalar multiple of $L_1(x,y)$, or $q_1(x,y)|Q_2(x,y)$. If a linear factor of $q_1(x,y)$ is a scalar multiple of $L_1(x,y)$, then $L_1(x,y)$ divides $P_1(x,y)$, violating (5). If $q_1(x,y)|Q_2(x,y)$, then $Q_2(x,y)$ is a scalar multiple of $q_1(x,y)$ and hence of $Q_1(x,y)$, implying that P_1, P_2 are medisimilar.

(2) If $q_1(x,y)|Q_2(x,y)$, we consider (13) and conclude that it must be the case that

$$q_1(x,y) = (1-k)L_2(x,y)Q_1(x,y).$$

By a proof completely analogous to that of part (a), this implies that P_1, P_2 are medisimilar.

(3) If we have $u_1(x,y)|Q_1(x,y)$ and $v_1(x,y)|Q_2(x,y)$, then by (13) it must be the case that

$$u_1(x,y)|kL_1(x,y)Q_2(x,y).$$

This is possible only if $u_1(x,y)$ is a scalar multiple of $L_1(x,y)$, or $u_1(x,y)$ divides $Q_2(x,y)$. If $u_1(x,y)$ is a scalar multiple of $L_1(x,y)$, then $L_1(x,y)$ divides $P_1(x,y)$, violating Lemma 5.

As for $u_1(x,y)|Q_2(x,y)$, if $u_1(x,y)$ isn't a scalar multiple of $v_1(x,y)$ then we have case (b). If $u_1(x,y)$ is a scalar multiple of $v_1(x,y)$, this means that $u_1(x,y)^2$ divides the right hand side of (13). Note that $u_1(x,y)$ divides $Q_1(x,y)$ and $Q_2(x,y)$. If $u_1(x,y)^2$ does not divide each individual term of the right hand side of (13) we either have P_1, P_2 medisimilar, or we have the case described in

Proposition 9 (where u_1 is h in the notation of that proposition). If $u_1(x,y)^2$ divides each term of the right hand side, we have

$$u_1(x,y)^2 | kL_1(x,y)Q_2(x,y)$$

and

$$u_1(x,y)^2 | (1-k)L_2(x,y)Q_1(x,y).$$

The first equation implies that $u_1(x,y)^2$ is a scalar multiple of $Q_2(x,y)$. If $u_1(x,y)$ is a scalar multiple of $L_2(x,y)$, $u_1(x,y)$ divides $P_2(x,y)$ violating Lemma 5. If $u_1(x,y)$ is not a scalar multiple of $L_2(x,y)$, $Q_1(x,y)$ is a scalar multiple of $u_1(x,y)^2$ and hence $Q_2(x,y)$. Thus P_1, P_2 are medisimilar. □

Proposition 12. *The medilocus of two intersecting conics is a line or a line missing a point only if they are medisimilar.*

Proof. Clearly, $k \neq 0, 1$.

We will need to consider the coefficients of P_1, P_2, so let us again assume that the distinguished intersection points is the origin and that we can express our two conics as

$$P_1 : R_1x^2 + S_1xy + T_1y^2 + V_1x + W_1y = 0, \tag{14}$$

$$P_2 : R_2x^2 + S_2xy + T_2y^2 + V_2x + W_2y = 0, \tag{15}$$

with constants $R_1, S_1, T_1, V_1, W_1, R_2, S_2, T_2, V_2, W_2$.

Through a proper choice of coordinates, we may express the medilocus as a vertical line $x = c$ for some constant c, which may or may not be missing a point. First, we note that c cannot be zero. The line $x = 0$ intersects P_1, P_2 at most twice each, and so the medilocus cannot intersect $x = 0$ infinitely many times.

But if c is nonzero, the medilocus does not intersect $x = 0$ at all. But this means $x = 0$ cannot intersect twice both P_1 and P_2. In algebraic terms, we know that $x = 0$ intersects P_1, P_2 at the origin and at $-W_1/T_1$, $-W_2/T_2$ respectively. Thus at least one of W_1, T_1, W_2, T_2 must be zero. If W_1 is zero, P_1 is tangent to $x = 0$ at the origin, and so $x = 0$ now cannot intersect P_2 twice: in other words, $W_1 = 0$ implies $W_2 = 0$ or $T_2 = 0$. If W_2 is zero, both P_1, P_2 would be tangent to $x = 0$, and so the origin will be in the medilocus. Thus we must have $T_2 = 0$. In other words, $W_1 = 0$ implies $T_2 = 0$, and we similarly have $W_2 = 0$ implying $T_1 = 0$. We thus must have either $T_1 = 0$ or $T_2 = 0$. Without loss of generality we let $T_1 = 0$.

All the points in the medilocus must have x-value $c \neq 0$. By appropriate scaling, we may without loss of generality set $c = 1$. Thus with reference to Lemma 7 and in particular (3), we then find that the coefficients of P_1, P_2 must satisfy

$$1 = k\left(-\frac{V_1 + W_1m}{R_1 + S_1m}\right) + (1-k)\left(-\frac{V_2 + W_2m}{R_2 + S_2m + T_2m^2}\right), \tag{16}$$

for all but at most one m. We may simplify (16) into

$$\begin{aligned} &(R_1 + S_1m)(R_2 + S_2m + T_2m^2) \\ = &-k(R_2 + S_2m + T_2m^2)(V_1 + W_1m) - (1-k)(R_1 + S_1m)(V_2 + W_2m), \end{aligned}$$

which in turn reduces to

$$((R_1 + S_1 m) + k(V_1 + W_1 m))(R_2 + S_2 m + T_2 m^2)$$
$$= -(1-k)(R_1 + S_1 m)(V_2 + W_2 m). \tag{17}$$

Since this equality holds for infinitely many values of m, all the coefficients must be zero. Looking at the m^3 coefficient, we deduce that either $S_1 + kW_1 = 0$, or $T_2 = 0$.

If $T_2 \neq 0$, we must have $S_1 + kW_1 = 0$. Since (17) now implies $(R_1 + S_1 m)(V_2 + W_2 m)$ is a scalar multiple of $R_2 + S_2 m + T_2 m^2$, this means that S_1, W_2 both nonzero. But since we have $(R_1 + S_1 m)(V_2 + W_2 m) = C(R_2 + S_2 m + T_2 m^2)$ for some constant C, we may write $m = y/x$ and then multiply both sides by x^2, to determine that $V_2 x + W_2 y$ divides $R_2 x^2 + S_2 xy + T_2 y^2$, contradicting Lemma 5.

Thus it is necessary that $T_2 = 0$. However, as we reevaluate (16), we note that there appear to be no solutions of the medilocus at $m = -R_1/S_1$ and $m = -R_2/S_2$. Since the medilocus is the line $x = c$ missing at most one point, it is necessary that $S_1 = 0$, $S_2 = 0$, or $-R_1/S_1 = -R_2/S_2$. The last case immediately implies that P_1, P_2 are medisimilar, so we consider the first two cases.

If we start by assuming $S_1 = 0$, note now that R_1 must be nonzero, otherwise P_1 has no quadratic terms. Recall that $T_1 = T_2 = 0$. Then by comparing coefficients in (17) we have:

$$S_2 W_1 = 0, \tag{18}$$
$$R_1 S_2 = -k(R_2 W_1 + S_2 V_1) - (1-k) R_1 W_2, \tag{19}$$
$$R_1 R_2 = -kR_2 V_1 - (1-k) R_1 V_2. \tag{20}$$

Based on (18), we are dealing with two subcases: $W_1 = 0$ or $S_2 = 0$.

If $S_2 \neq 0$, $W_1 = 0$. If $R_2 = 0$ as well, then (20) implies that $V_2 = 0$. But then P_2 reduces to $W_2 y + S_2 xy$, contradicting either Lemma 4 or Lemma 5.

Thus we must have R_2 nonzero, and since we assumed $W_1 = 0$ (19) and (20) imply

$$\frac{R_1 W_2}{S_2} = -\frac{R_1 + kV_1}{1-k} = \frac{R_1 V_2}{R_2}.$$

But this implies $W_2/S_2 = V_2/R_2$, again contradicting Lemma 5.

Thus we deduce $S_2 = 0$, and we now have $T_1 = T_2 = S_1 = S_2 = 0$, so P_1, P_2 must now be medisimilar. If we start by assuming $S_2 = 0$, we analogously deduce that $S_1 = 0$ as well, and that P_1, P_2 are medisimilar. □

Proposition 13. *The medilocus of two intersecting conics is a point only if they are medisimilar.*

Proof. We can discount the possibility that the weight k equals 0 or 1. We shall set the distinguished intersection point at the center, and so we may once again define our conics P_1, P_2 as in (1),(2). We claim that if the medilocus is a single point, that point must be in the center. Consider any line of the form $y = mx$ through the center. It intersects the first conic section, P_1 at the origin and at $x = -L_1(1, m)/Q_1(1, m)$, and it intersects the second conic, P_2 at the origin and at

$x = -L_2(1,m)/Q_2(1,m)$. Thus if $Q_1(1,m), Q_2(1,m)$ are both nonzero, then the medilocus has a point on the line $y = mx$ (note that $L_1(1,m) = 0, Q_1(1,m) \neq 0$ or $L_2(1,m) = 0, Q_2(1,m) \neq 0$ respectively imply that $y = mx$ is tangent to P_1 or P_2 at the origin).

But clearly $Q_1(1,m), Q_2(1,m)$ have at most two roots each, and so except for at most four values of m, the medilocus intersects $y = mx$. Thus if the medilocus consists of exactly one point, that point must be the origin.

By Lemma 7, (3) must have no solutions. This means that

$$\frac{kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y)}{Q_1(x,y)Q_2(x,y)}, \tag{21}$$

is either zero or undefined for all choices of x, y. If $Q_1(x,y)$ is zero at (a,b), it must be zero on the line $M_1(x,y)$ through $(0,0)$ and (a,b). Thus $Q_1(x,y) = M_1(x,y)N_1(x,y)$ for some linear form N_1. Similarly, if $Q_2(x,y)$ has a non-origin solution, it must factor into linear forms $Q_2(x,y) = M_2(x,y)N_2(x,y)$ as well. Since $Q_1(x,y)Q_2(x,y)$ cannot be identically zero the expression (21) can be undefined on at most four lines through the origin. But then the expression $kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y)$ is a homogeneous cubic in x and y . Note that if $(a,b) \neq (0,0)$ is a solution to a homogeneous cubic the entire line through $(0,0)$ and (a,b) is as well, and by Lemma 10 the equation for that line must divide the homogeneous cubic. Thus the numerator of (21) is either identically zero, or zero on at most three lines through the origin. We know $\mathbb{F}$ is an infinite field, and so there are infinitely many lines through the origin. Since the expression in (21) must be zero or undefined everywhere, we conclude that

$$kL_1(x,y)Q_2(x,y) + (1-k)L_2(x,y)Q_1(x,y) = 0. \tag{22}$$

By Lemma 5, we know that $L_1(x,y)$ cannot divide $Q_1(x,y)$, and that $L_2(x,y)$ cannot divide $Q_2(x,y)$. Thus it must be true that $L_1(x,y)$ and $L_2(x,y)$ are scalar multiples of each other. We note here that neither $L_1(x,y)$ nor $L_2(x,y)$ can be identically zero by Lemma 4. But then this implies that $Q_1(x,y)$ and $Q_2(x,y)$ are scalar multiples of each other. Thus P_1, P_2 are medisimilar. □

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