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When is a Tangential Quadrilateral a Kite?

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Abstract. We prove 13 necessary and sufficient conditions for a tangential quadrilateral to be a kite.

1. Introduction

A *tangential quadrilateral* is a quadrilateral that has an incircle. A convex quadrilateral with the sides a, b, c, d is tangential if and only if

$$a + c = b + d \tag{1}$$

according to the Pitot theorem [1, pp.65–67]. A *kite* is a quadrilateral that has two pairs of congruent adjacent sides. Thus all kites has an incircle since its sides satisfy (1). The question we will answer here concerns the converse, that is, what additional property a tangential quadrilateral must have to be a kite? We shall prove 13 such conditions. To prove two of them we will use a formula for the area of a tangential quadrilateral that is not so well known, so we prove it here first. It is given as a problem in [4, p.29].

Theorem 1. *A tangential quadrilateral with sides a, b, c, d and diagonals p, q has the area*

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac - bd)^2}.$$

Proof. A convex quadrilateral with sides a, b, c, d and diagonals p, q has the area

$$K = \tfrac{1}{4}\sqrt{4p^2q^2 - (a^2 - b^2 + c^2 - d^2)^2} \tag{2}$$

according to [6] and [14]. Squaring the Pitot theorem (1) yields

$$a^2 + c^2 + 2ac = b^2 + d^2 + 2bd. \tag{3}$$

Using this in (2), we get

$$K = \tfrac{1}{4}\sqrt{4(pq)^2 - (2bd - 2ac)^2}$$

and the formula follows. □

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2. Conditions for when a tangential quadrilateral is a kite

In a tangential quadrilateral, a *tangency chord* is a line segment connecting the points on two opposite sides where the incircle is tangent to those sides, and the *tangent lengths* are the distances from the four vertices to the points of tangency (see [7] and Figure 1). A *bimedian* in a quadrilateral is a line segment connecting the midpoints of two opposite sides.

In the following theorem we will prove eight conditions for when a tangential quadrilateral is a kite.

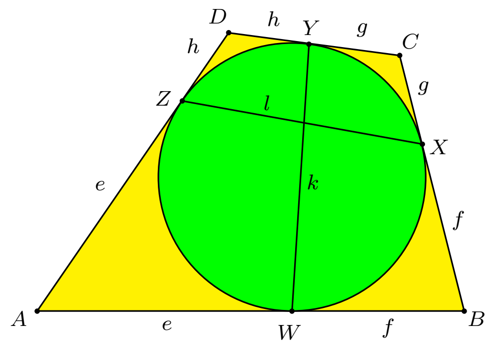

Figure 1. The tangency chords k, l and tangent lengths e, f, g, h

Theorem 2. *In a tangential quadrilateral the following statements are equivalent:*
(i) *The quadrilateral is a kite.*
(ii) *The area is half the product of the diagonals.*
(iii) *The diagonals are perpendicular.*
(iv) *The tangency chords are congruent.*
(v) *One pair of opposite tangent lengths are congruent.*
(vi) *The bimedians are congruent.*
(vii) *The products of the altitudes to opposite sides of the quadrilateral in the nonoverlapping triangles formed by the diagonals are equal.*
(viii) *The product of opposite sides are equal.*
(ix) *The incenter lies on the longest diagonal.*

Proof. Let the tangential quadrilateral $ABCD$ have sides a, b, c, d. We shall prove that each of the statements (i) through (vii) is equivalent to (viii); then all eight of them are equivalent. Finally, we prove that (i) and (ix) are equivalent.

(i) If in a kite $a = d$ and $b = c$, then $ac = bd$. Conversely, in [7, Corollary 3] we have already proved that a tangential quadrilateral with $ac = bd$ is a kite.

(ii) Using Theorem 1, we get

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac - bd)^2} = \tfrac{1}{2}pq \quad \Leftrightarrow \quad ac = bd.$$

(iii) We use the well known formula $K = \frac{1}{2}pq\sin\theta$ for the area of a convex quadrilateral,[1] where θ is the angle between the diagonals p, q. From

$$K = \tfrac{1}{2}\sqrt{(pq)^2 - (ac-bd)^2} = \tfrac{1}{2}pq\sin\theta$$

we get

$$\theta = \frac{\pi}{2} \quad \Leftrightarrow \quad ac = bd.$$

(iv) In a tangential quadrilateral, the tangency chords k, l satisfy

$$\left(\frac{k}{l}\right)^2 = \frac{bd}{ac}$$

according to Corollary 2 in [7]. Hence

$$k = l \quad \Leftrightarrow \quad ac = bd.$$

(v) Let the tangent lengths be e, f, g, h, where $a = e + f$, $b = f + g$, $c = g + h$ and $d = h + e$ (see Figure 1). Then we have

$$\begin{aligned} & ac = bd \\ \Leftrightarrow\quad & (e+f)(g+h) = (f+g)(h+e) \\ \Leftrightarrow\quad & ef - eh - fg + gh = 0 \\ \Leftrightarrow\quad & (e-g)(f-h) = 0 \end{aligned}$$

which is true when (at least) one pair of opposite tangent lengths are congruent.

(vi) In the proof of Theorem 7 in [9] we noted that the length of the bimedians m, n in a convex quadrilateral are

$$\begin{aligned} m &= \tfrac{1}{2}\sqrt{2(b^2+d^2) - 4v^2}, \\ n &= \tfrac{1}{2}\sqrt{2(a^2+c^2) - 4v^2} \end{aligned}$$

where v is the distance between the midpoints of the diagonals. Using these, we have

$$m = n \quad \Leftrightarrow \quad a^2 + c^2 = b^2 + d^2 \quad \Leftrightarrow \quad ac = bd$$

where the last equivalence is due to (3).

(vii) The diagonal intersection P divides the diagonals in parts w, x and y, z. Let the altitudes in triangles ABP, BCP, CDP, DAP to the sides a, b, c, d be h_1, h_2, h_3, h_4 respectively (see Figure 2). By expressing twice the area of these triangles in two different ways we get

$$\begin{aligned} ah_1 &= wy\sin\theta, \\ bh_2 &= xy\sin\theta, \\ ch_3 &= xz\sin\theta, \\ dh_4 &= wz\sin\theta, \end{aligned}$$

[1] For a proof, see [5] or [13, pp.212–213].

where θ is the angle between the diagonals and we used that $\sin(\pi - \theta) = \sin\theta$. These equations yields

$$ach_1h_3 = wxyz\sin^2\theta = bdh_2h_4.$$

Hence

$$h_1h_3 = h_2h_4 \quad \Leftrightarrow \quad ac = bd.$$

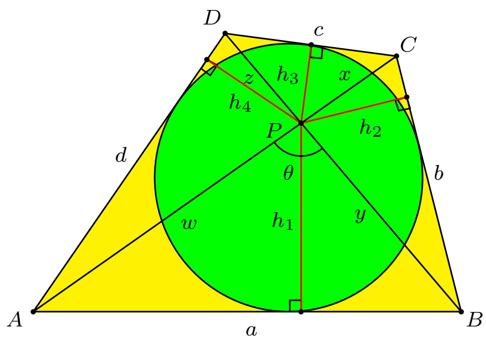

Figure 2. The subtriangle altitudes h_1, h_2, h_3, h_4

(ix) We prove that (i) $\Leftrightarrow$ (ix). A kite has an incircle and the incenter lies on the intersection of the angle bisectors. The longest diagonal is an angle bisector to two of the vertex angles since it divides the kite into two congruent triangles (SSS), hence the incenter lies on the longest diagonal.[2] Conversely, if the incenter lies on the longest diagonal in a tangential quadrilateral (see Figure 3) it directly follows that the quadrilateral is a kite since the longest diagonal divides the quadrilateral into two congruent triangles (ASA), so two pairs of adjacent sides are congruent. □

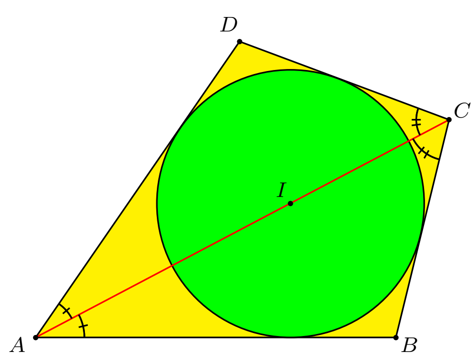

Figure 3. This tangential quadrilateral is a kite

[2] A more detailed proof not assuming that a kite has an incircle is given in [10, pp.92–93].

For those interested in further explorations we note that in a convex quadrilateral where $ac = bd$ there is an interesting angle relation concerning the angles formed by the sides and the diagonals, see [2] and [3]. Atzema calls these *balanced quadrilaterals*.

Theorem 2 (vii) has the following corollary.

Corollary 3. *The sums of the altitudes to opposite sides of a tangential quadrilateral in the nonoverlapping triangles formed by the diagonals are equal if and only if the quadrilateral is a kite.*

Proof. If h_1, h_2, h_3, h_4 are the altitudes from the diagonal intersection P to the sides AB, BC, CD, DA in triangles ABP, BCP, CDP, DAP respectively, then according to [12] and Theorem 1 in [11] (with other notations)

$$\frac{1}{h_1} + \frac{1}{h_3} = \frac{1}{h_2} + \frac{1}{h_4}. \tag{4}$$

From this we get

$$\frac{h_1 + h_3}{h_1 h_3} = \frac{h_2 + h_4}{h_2 h_4}.$$

Hence

$$h_1 h_3 = h_2 h_4 \quad \Leftrightarrow \quad h_1 + h_3 = h_2 + h_4$$

and the proof is compleate. □

Remark. In [8] we attributed (4) to Minculete since he proved this condition in [11]. After the publication of [8], Vladimir Dubrovsky pointed out that condition (4) in fact appeared earlier in the solution of Problem M1495 in the Russian magazine Kvant in 1995, see [12]. There it was given and proved by Vasilyev and Senderov together with their solution to Problem M1495. This problem, which was posed and solved by Vaynshtejn, was about proving a condition with inverse inradii, see (7) later in this paper. In [8, p.70] we incorrectly attributed this inradii condition to Wu due to his problem in [15].

3. Conditions with subtriangle inradii and exradii

In the proof of the next condition for when a tangential quadrilateral is a kite we will need the following formula for the inradius of a triangle.

Lemma 4. *The incircle in a triangle ABC with sides a, b, c has the radius*

$$r = \frac{a + b - c}{2} \tan \frac{C}{2}.$$

Proof. We use notations as in Figure 4, where there is one pair of equal tangent lengths x, y and z at each vertex due to the two tangent theorem. For the sides of the triangle we have $a = y + z$, $b = z + x$ and $c = x + y$; hence $a + b - c = 2z$. CI is an angle bisector, so

$$\tan \frac{C}{2} = \frac{r}{z}$$

and the formula follows. □

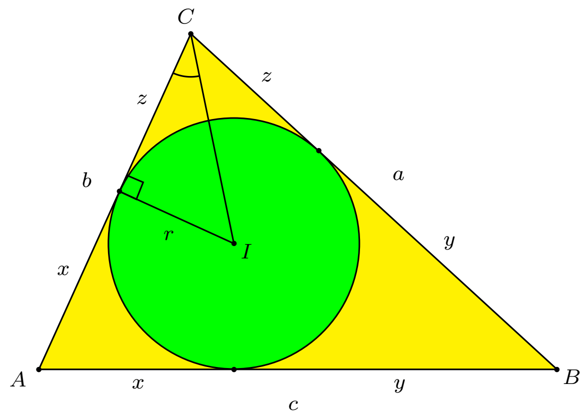

Figure 4. An incircle in a triangle

Theorem 5. *Let the diagonals in a tangential quadrilateral $ABCD$ intersect at P and let the inradii in triangles ABP, BCP, CDP, DAP be r_1, r_2, r_3, r_4 respectively. Then the quadrilateral is a kite if and only if*

$$r_1 + r_3 = r_2 + r_4.$$

Proof. We use the same notations as in Figure 2. The four incircles and their radii are marked in Figure 5. Since $\tan \frac{\pi - \theta}{2} = \cot \frac{\theta}{2}$, where θ is the angle between the diagonals, Lemma 4 yields

$$\begin{aligned} & r_1 + r_3 = r_2 + r_4 \\ \Leftrightarrow \quad & \frac{w+y-a}{2} \tan \frac{\theta}{2} + \frac{x+z-c}{2} \tan \frac{\theta}{2} = \frac{x+y-b}{2} \cot \frac{\theta}{2} + \frac{w+z-d}{2} \cot \frac{\theta}{2} \\ \Leftrightarrow \quad & (w+x+y+z-a-c) \tan \frac{\theta}{2} = (w+x+y+z-b-d) \cot \frac{\theta}{2}. \end{aligned} \tag{5}$$

Using the Pitot theorem $a + c = b + d$, (5) is eqivalent to

$$(w+x+y+z-a-c)\left(\tan \frac{\theta}{2} - \cot \frac{\theta}{2}\right) = 0. \tag{6}$$

According to the triangle inequality applied in triangles ABP and CDP, we have $w + y > a$ and $x + z > c$. Hence $w + x + y + z > a + c$ and (6) is equivalent to

$$\tan \frac{\theta}{2} - \cot \frac{\theta}{2} = 0 \quad \Leftrightarrow \quad \tan^2 \frac{\theta}{2} = 1 \quad \Leftrightarrow \quad \frac{\theta}{2} = \frac{\pi}{4} \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}$$

where we used that $\theta > 0$, so the negative solution is invalid. According to Theorem 2 (iii), a tangential quadrilateral has perpendicular diagonals if and only if it is a kite. □

Corollary 6. *If r_1, r_2, r_3, r_4 are the same inradii as in Theorem 5, then the tangential quadrilateral is a kite if and only if*

$$r_1 r_3 = r_2 r_4.$$

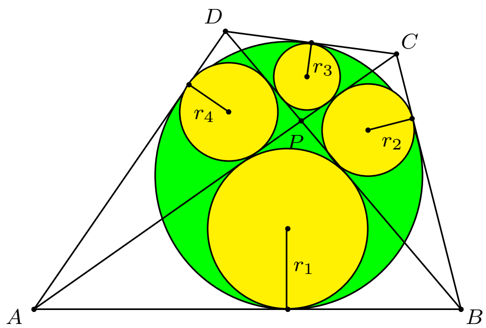

Figure 5. The incircles in the subtriangles

Proof. In a tangential quadrilateral we have according to [12] and [15]

$$\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}. \tag{7}$$

We rewrite this as

$$\frac{r_1+r_3}{r_1r_3}=\frac{r_2+r_4}{r_2r_4}.$$

Hence

$$r_1+r_3=r_2+r_4 \qquad \Leftrightarrow \qquad r_1r_3=r_2r_4$$

which proves this corollary. □

Now we shall study similar conditions concerning the exradii to the same subtriangles.

Lemma 7. *The excircle to side $AB=c$ in a triangle ABC with sides a,b,c has the radius*

$$R_c=\frac{a+b+c}{2}\tan\frac{C}{2}.$$

Proof. We use notations as in Figure 6, where $u+v=c$. Also, according to the two tangent theorem, $b+u=a+v$. Hence $b+u=a+c-u$, so

$$u=\frac{a-b+c}{2}$$

and therefore

$$b+u=\frac{a+b+c}{2}.$$

For the exradius we have

$$\tan\frac{C}{2}=\frac{R_c}{b+u}$$

since CI is an angle bisector, and the formula follows. □

Theorem 8. *Let the diagonals in a tangential quadrilateral $ABCD$ intersect at P and let the exradii in triangles ABP, BCP, CDP, DAP opposite the vertex P be R_1, R_2, R_3, R_4 respectively. Then the quadrilateral is a kite if and only if*

$$R_1+R_3=R_2+R_4.$$

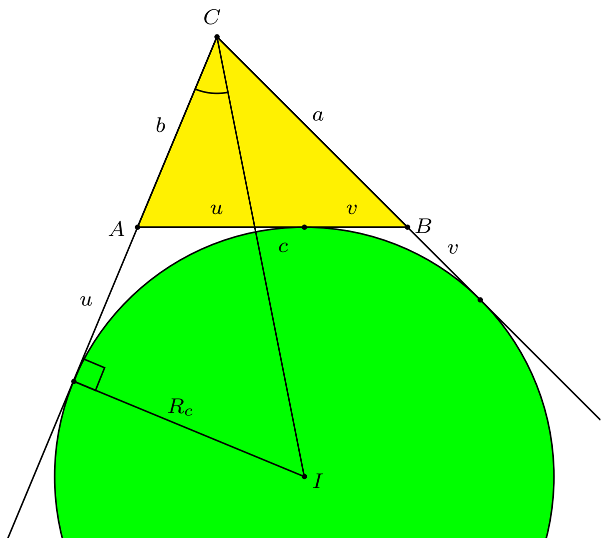

Figure 6. An excircle to a triangle

Proof. The four excircles and their radii are marked in Figure 7. Since $\tan\frac{\pi-\theta}{2} = \cot\frac{\theta}{2}$, where θ is the angle between the diagonals, Lemma 7 yields

$$\begin{aligned} & R_1 + R_3 = R_2 + R_4 \\ \Leftrightarrow \quad & \frac{w+y+a}{2}\tan\frac{\theta}{2} + \frac{x+z+c}{2}\tan\frac{\theta}{2} = \frac{x+y+b}{2}\cot\frac{\theta}{2} + \frac{w+z+d}{2}\cot\frac{\theta}{2} \\ \Leftrightarrow \quad & (w+x+y+z+a+c)\tan\frac{\theta}{2} = (w+x+y+z+b+d)\cot\frac{\theta}{2}. \qquad (8) \end{aligned}$$

Using the Pitot theorem $a+c=b+d$, (8) is eqivalent to

$$(w+x+y+z+a+c)\left(\tan\frac{\theta}{2} - \cot\frac{\theta}{2}\right) = 0. \qquad (9)$$

The first parenthesis is positive. Hence (9) is equivalent to that the second parenthesis is zero and the end of the proof is the same as in Theorem 5. □

Corollary 9. *If R_1, R_2, R_3, R_4 are the same exradii as in Theorem 8, then the tangential quadrilateral is a kite if and only if*

$$R_1 R_3 = R_2 R_4.$$

Proof. In a tangential quadrilateral we have according to Theorem 4 in [8]

$$\frac{1}{R_1} + \frac{1}{R_3} = \frac{1}{R_2} + \frac{1}{R_4}.$$

We rewrite this as

$$\frac{R_1 + R_3}{R_1 R_3} = \frac{R_2 + R_4}{R_2 R_4}.$$

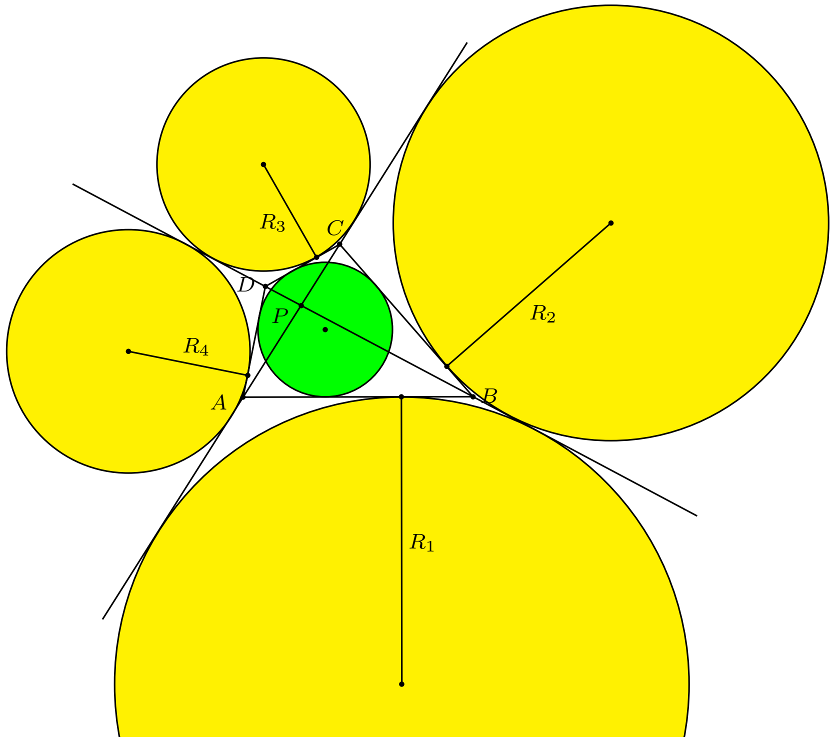

Figure 7. The excircles to the subtriangles

Hence

$$R_1 + R_3 = R_2 + R_4 \quad \Leftrightarrow \quad R_1R_3 = R_2R_4$$

completing the proof. □

References

[1] T. Andreescu and B. Enescu, *Mathematical Olympiad Treasures*, Birkhäuser, Boston, 2004.
[2] E. J. Atzema, A theorem by Giusto Bellavitis on a class of quadrilaterals, *Forum Geom.*, 6 (2006) 181–185.
[3] N. Dergiades, A synthetic proof and generalization of Bellavitis' theorem, *Forum Geom.*, 6 (2006) 225–227.
[4] C. V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell and Sons, London, 1930. New edition by Dover Publications, Mineola, 2003.
[5] J. Harries, Area of a Quadrilateral, *The Mathematical Gazette*, 86 (2002) 310–311.
[6] V. F. Ivanoff, C. F. Pinzka and J. Lipman, Problem E1376: Bretschneider's Formula, *Amer. Math. Monthly*, 67 (1960) 291.
[7] M. Josefsson, Calculations concerning the tangent lengths and tangency chords of a tangential quadrilateral, *Forum Geom.*, 10 (2010) 119–130.
[8] M. Josefsson, More characterizations of tangential quadrilaterals, *Forum Geom.*, 11 (2011) 65–82.
[9] M. Josefsson, The area of a bicentric quadrilateral, *Forum Geom.*, 11 (2011) 155–164.
[10] S. Libeskind, *Euclidean and Transformational Geometry. A Deductive Inquiry*, Jones and Bartlett Publishers, Sudbury, 2008.

[11] N. Minculete, Characterizations of a tangential quadrilateral, *Forum Geom.*, 9 (2009) 113–118.
[12] I. Vaynshtejn, N. Vasilyev and V. Senderov, Problem M1495, *Kvant* (in Russian) no. 6, 1995, 27–28, available at http://kvant.mirror1.mccme.ru/1995/06/resheniya_zadachnika_kvanta_ma.htm
[13] J. A. Vince, *Geometry for Computer Graphics. Formulae, Examples and Proofs*, Springer, London, 2005.
[14] E. W. Weisstein, Bretschneider's formula, *MathWorld*, http://mathworld.wolfram.com/BretschneidersFormula.html
[15] W. C. Wu and P. Simeonov, Problem 10698, *Amer. Math. Monthly*, 105 (1998) 995; solution, ibid., 107 (2000) 657–658.

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