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On a Certain Cubic Geometric Inequality

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Abstract. We provide a proof of a geometric inequality relating the cube of the distances of an arbitrary point from the vertices of a triangle to the cube of the inradius of the triangle. Comparable versions of the inequality involving second and fourth powers may be obtained by modifying our arguments.

1. Introduction

Given triangle ABC and a point P in its plane, let R_1, R_2, and R_3 denote the respective distances AP, BP, and CP and r denote the inradius of triangle ABC. In this note, we prove the following result.

Theorem 1. *If ABC is a triangle and P is a point, then*

$$R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3 \geqslant 72r^3. \tag{1}$$

This answers a conjecture raised by Wu, Zhang, and Chu at the end of [5] in the affirmative. Furthermore, there is equality in (1) if and only if triangle ABC is equilateral with P its center. By homogeneity, one may take $r = 1$ in (1), which we will assume. Modifying our proof, one can generalize inequality (1) somewhat and establish versions of it for exponents 2 and 4. For other related inequalities of the Erdős-Mordell type involving powers of the R_i, see, for example, [1–5].

Note that in Theorem 1, one may assume that the point P lies on or within the triangle ABC since for any P lying outside triangle ABC, one can find a point Q lying on or within the triangle, all of whose distances to the vertices are strictly less than or equal the respective distances for P. To see this, consider the seven regions of the plane formed by extending the sides of triangle ABC indefinitely in both directions. Suppose first that P, say, lies in the region inside of $\angle BAC$ but outside of triangle ABC. Let K be the foot of P on line BC. If K lies between B and C, then take $Q = K$; otherwise, take Q to be the closer of B or C to the point K. On the other hand, suppose that the point P lies in a region bounded by one of the vertical angles opposite an interior angle of triangle ABC, say in the vertical angle to $\angle BAC$. Let J be the foot of P on the line ℓ passing through A and parallel to line BC; note that all of the distances from J to the vertices of triangle ABC are strictly less than the respective distances for P. Now proceed as in the prior case using the point J.

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To reduce the number of cases, we will assume that the point P lies on or within the triangle ABC.

2. Preliminary results

Lemma 2. *Suppose B and C are fixed points on the x-axis. Consider all possible triangles ABC having inradius 1. If $A = (x, y)$, where $y > 0$, then y achieves its minimum value only when triangle ABC is isosceles.*

Proof. Let P denote the point of tangency of the incircle of triangle ABC with side BC. Without loss of generality, we may let $B = (0, 0)$, $P = (a, 0)$ and $C = (a + b, 0)$, where a and b are positive numbers such that $ab > 1$. It is then a routine exercise to show that

$$A = \left(\frac{b(a^2 - 1)}{ab - 1}, \frac{2ab}{ab - 1}\right),$$

upon first noting that cot $IBP = a$ and cot $ICP = b$, where I denotes the incenter. Thus, the y-coordinate of A is $2 + \frac{2}{ab-1}$, and this has minimum value when ab is maximized, which occurs only when $a = b$ since $BC = a+b$ is of fixed length. $\square$

For a point P in the plane of triangle ABC, we define the *power of the point P* (with respect to triangle ABC) as

$$f^P_{\triangle ABC} := R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3.$$

Lemma 3. *Fix base BC. Let triangle $A'BC$ have inradius 1, and P' be a point on or within this triangle. Then*

$$f^{P'}_{\triangle A'BC} \geq f^P_{\triangle ABC}$$

for an isosceles triangle ABC with inradius 1 and some point P on the altitude from A to BC.

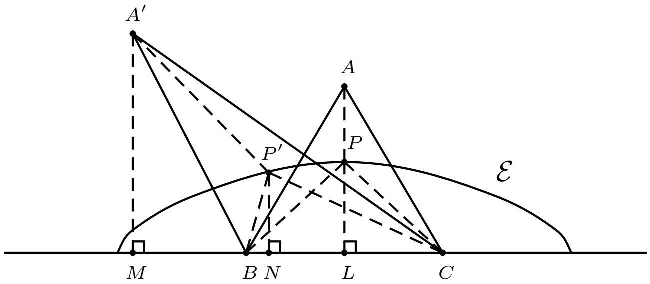

Figure 1. Triangles ABC and $A'BC$.

Proof. Let $R'_1 = A'P'$, $R'_2 = BP'$, and $R'_3 = CP'$. Without loss of generality, we may assume $R'_1 = \min\{R'_1, R'_2, R'_3\}$, for we may relabel the figure otherwise. Note that P' lies on the ellipse $\mathcal{E}$ with foci at points B and C having major axis length $d = R'_2 + R'_3$. Let L denote the midpoint of BC and suppose the perpendicular to BC at L intersects ellipse $\mathcal{E}$ at the point P. Let A be the point (on the same side of BC as P) such that triangle ABC is isosceles and has inradius 1. See Figure 1.

Let $R_1 = AP$, $R_2 = BP$, and $R_3 = CP$; note that $2R_2 = R_2 + R_3 = d$ since P and P' lie on the same ellipse. Let M and N denote, respectively, the feet of points A' and P' on line BC; note that $P'N \leqslant PL$ since P is the highest point on the ellipse $\mathcal{E}$. We now consider two cases concerning the position of P:
(i) P lies between points A and L (as illustrated above), or
(ii) P lies past point A on line AL.

In case (i), we have, by Lemma 2,

$$R_1 = AL - PL \leqslant A'M - PL \leqslant A'M - P'N \leqslant R_1',$$

as the difference in the vertical heights of points A' and P' (with respect to BC) is no more than the distance between them.

Since $R_2 + R_3 = R_2' + R_3'$, with $R_1' = \min\{R_1', R_2', R_3'\}$ and $R_2 = R_3$, it is a routine exercise to verify

$$R_1'^3 + R_2^3 + R_3^3 + 6R_1'R_2R_3 \leqslant R_1'^3 + R_2'^3 + R_3'^3 + 6R_1'R_2'R_3'.$$

Since $R_1 \leqslant R_1'$, this implies

$$R_1^3 + R_2^3 + R_3^3 + 6R_1R_2R_3 \leqslant R_1'^3 + R_2'^3 + R_3'^3 + 6R_1'R_2'R_3',$$

which completes the proof in case (i).

In case (ii), we may replace P with vertex A since clearly $f^A_{\triangle ABC} \leqslant f^P_{\triangle ABC}$. Furthermore, note that $f^A_{\triangle ABC} \leqslant f^{P'}_{\triangle A'BC}$, by the argument in the prior case, since now $R_2 + R_3 = AB + AC \leqslant R_2' + R_3'$ with $R_1 = 0$. $\square$

Lemma 4. *Suppose ABC is an isosceles triangle having inradius 1 and base BC. Let P be any point on the altitude from A to BC. Then for all possible P, we have*

$$f^P_{\triangle ABC} \geqslant 72.$$

Equality holds if and only if triangle ABC is equilateral with P the center.

Proof. Let $\angle ABC = 2\alpha$ and AD be the altitude from vertex A to side BC. Then $BD = \cot\alpha$ and thus

$$AD = BD \cdot \tan 2\alpha = \tan 2\alpha \cot\alpha = \frac{2}{1 - \tan^2\alpha}.$$

Let $x = \tan^2\alpha$ and $y = R_1 = AP$. Note that

$$\begin{aligned} R_2 = R_3 = \ & BP = \sqrt{DP^2 + BD^2} = \sqrt{(AD - AP)^2 + BD^2} \\ &= \sqrt{\left(\frac{2}{1-x} - y\right)^2 + \frac{1}{x}}. \end{aligned}$$

See Figure 2.

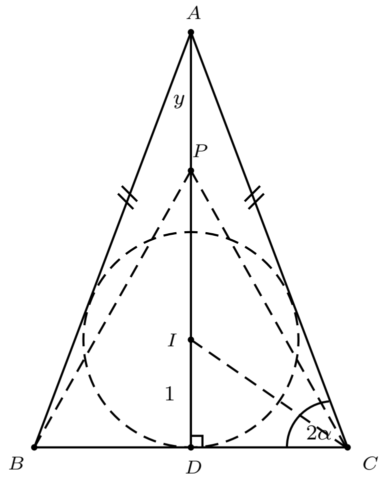

Figure 2. Isosceles triangle ABC.

To show $R_1^3+R_2^3+R_3^3+6R_1R_2R_3 \geqslant 72$, we need only establish the inequality

$$2\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]^{3/2}+6y\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]+y^3 \geqslant 72, \qquad (2)$$

for all $0<x<1$ and $y \geqslant 0$.

To prove (2), we consider the function

$$f(x,y)=2\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]^{3/2}+6y\left[\left(\frac{2}{1-x}-y\right)^2+\frac{1}{x}\right]+y^3$$

and study its extreme points. Direct calculations give

$$f_x(x,y)=3\cdot\frac{8x^2-(1-x)^3-4x^2(1-x)y}{(1-x)^3x^2}\left(\sqrt{\frac{1}{x}+\frac{(2-y+xy)^2}{(1-x)^2}}+2y\right),$$

$$f_y(x,y)=\frac{6(1+x)^2}{(1-x)^2x}-\frac{48}{1-x}y+21y^2-\frac{6(2-y+xy)}{1-x}\sqrt{\frac{1}{x}+\frac{(2-y+xy)^2}{(1-x)^2}}.$$

Since $0<x<1$ and $y \geqslant 0$, we see that the equation $f_x(x,y)=0$ implies $y=\frac{8x^2-(1-x)^3}{4x^2(1-x)}$. Substituting this expression for y into the equation $f_y(x,y)=0$, we obtain, after several straightforward algebraic operations, the equality

$$\begin{aligned}&(1+x)(7x^5+31x^4-70x^3+106x^2-49x+7)\\&=2(1-x)^4\sqrt{(1+x)(1-5x+11x^2+x^3)},\end{aligned}$$

which, since $0<x<1$, implies either $x=x_0=\frac{1}{3}$ or $g(x)=0$, where

$$\begin{aligned}g(x)=\ &15x^{10}+162x^9+279x^8-1256x^7+3150x^6-2644x^5\\&+694x^4+1752x^3-1533x^2+450x-45.\end{aligned}$$

By Maple or any mathematical programming (or, by hand, upon considering eight derivatives), it can be shown that $g'(x) > 0$ for all $0 < x < 1$; see graph above. Thus $g(x)$ is an increasing function with $g(0) < 0$ and $g(1) > 0$. Hence the equation $g(x) = 0$ has a unique root $x = x_1$ on the interval $(0, 1)$. By any numerical method, one may estimate this root to be $x_1 = 0.2558509455\cdots$.

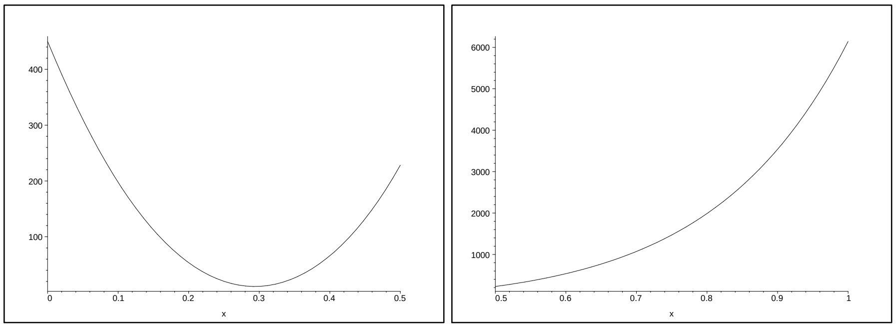

Figure 3. Graph of $g'(x)$ on the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Note that $\lim_{x\to 0^+} f(x, y) = \lim_{x\to 1^-} f(x, y) = \lim_{y\to\infty} f(x, y) = \infty$. Observe further that the function

$$f(x, 0) = 2\left(\frac{4}{(1-x)^2} + \frac{1}{x}\right)^{3/2}$$

has a unique extreme point at $x = -2 + \sqrt{5}$ on the interval $0 < x < 1$, and in this case we have $f(-2 + \sqrt{5}, 0) = \frac{4+2\sqrt{5}}{(\sqrt{5}-2)^{\frac{3}{2}}} \approx 73.8647611$. On the other hand, the function $f(x, y)$ has two internal extreme points, namely $(x_0, y_0) = (\frac{1}{3}, 2)$ and $(x_1, y_1) = (0.2558509455, 0.5727531365)$, with $f(x_0, y_0) = 72$ and $f(x_1, y_1) \approx 77.5182420$. This shows $f(x, y) \geqslant 72$ for all $0 < x < 1$ and $y \geqslant 0$, as required. There is equality only when $(x, y) = (\frac{1}{3}, 2)$, which corresponds to the case when triangle ABC is equilateral with P its center. □

We invite the reader to seek a less technical proof of inequality (2). Meanwhile, Theorem 1 follows quickly from Lemmas 3 and 4.

3. Proof of Theorem 1

Suppose P is a point in the plane of the arbitrary triangle ABC having inradius 1, where BC is fixed. We may assume P lies on or within triangle ABC, by the observation in the introduction. From Lemma 3, we see that the power of P with respect to triangle ABC is at least as large as the power of some point Q with respect to the isosceles triangle having inradius 1 and the same base, where Q lies on the altitude to it, which, by Lemma 4, is at least 72. Since BC was arbitrary, the theorem is proven. We also have that there is equality only in the case when triangle ABC is equilateral and P is the center. □

4. Some corollaries

We now state a few simple consequences of Theorem 1. First, combining with the arithmetic-geometric mean inequality, we obtain the following result.

Corollary 5. *If ABC is a triangle and P is a point, then*

$$R_1^3 + R_2^3 + R_3^3 \geqslant 24r^3. \tag{3}$$

Taking P to be the circumcenter of triangle ABC, we see that Theorem 1 reduces to Euler's inequality $R \geq 2r$, where R denotes the circumradius. Taking P to be the incenter, we obtain the following trigonometric inequality.

Corollary 6. *If α, β, and γ denote the half-angles of a triangle ABC, then*

$$\csc^3\alpha + \csc^3\beta + \csc^3\gamma + 6\csc\alpha\csc\beta\csc\gamma \geqslant 72. \tag{4}$$

Inequality (4) may also be realized by observing

$$\csc^3\alpha+\csc^3\beta+\csc^3\gamma+6\csc\alpha\csc\beta\csc\gamma \geqslant 9\csc\alpha\csc\beta\csc\gamma = 9\left(\frac{4R}{r}\right) \geqslant 72,$$

using Euler's inequality and the fact $\frac{r}{4R} = \sin\alpha\sin\beta\sin\gamma$.

Finally, taking P to be the centroid of triangle ABC, we obtain the following inequality.

Corollary 7. *If triangle ABC has median lengths m_a, m_b, and m_c, then*

$$m_a^3 + m_b^3 + m_c^3 + 6m_am_bm_c \geqslant 243r^3. \tag{5}$$

Using the proof above of Theorem 1, it is possible to find lower bounds for the general sum $R_1^\alpha + R_2^\alpha + R_3^\alpha + k(R_1R_2R_3)^{\frac{\alpha}{3}}$, where α and k are positive constants, in several particular instances. For example, one may generalize Theorem 1 as follows, *mutatis mutandis*.

Theorem 8. *If ABC is a triangle and P is a point, then*

$$R_1^3 + R_2^3 + R_3^3 + kR_1R_2R_3 \geqslant 8(k+3)r^3 \tag{6}$$

for $k = 0, 1, \ldots, 6$.

Furthermore, it appears that inequality (6) would hold for all *real numbers* k in the interval $[0, 6]$, though we do not have a complete proof. However, for any *given* number in this interval, one could apply the steps in the proof of Lemma 4 to test whether or not (6) holds. Even though it does not appear that Lemma 3 can be generalized to handle the case when $\alpha = 3$ and $k > 6$, numerical evidence suggests that inequality (6) still might hold over a larger range, such as $0 \leqslant k \leqslant 15$.

The proof above can be further modified to show the following results for exponents 2 and 4.

Theorem 9. *If ABC is a triangle and P is a point, then*

$$R_1^2 + R_2^2 + R_3^2 + (R_1R_2R_3)^{\frac{2}{3}} \geqslant 16r^2$$

and

$$R_1^2 + R_2^2 + R_3^2 + 2(R_1R_2R_3)^{\frac{2}{3}} \geqslant 20r^2.$$

Theorem 10. *If ABC is a triangle and P is a point, then*

$$R_1^4 + R_2^4 + R_3^4 + k(R_1R_2R_3)^{\frac{4}{3}} \geqslant 16(k+3)r^4$$

for $k = 0, 1, \dots, 9$.

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