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The Area of the Diagonal Point Triangle

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Abstract. In this note we derive a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the sides of the quadrilateral, and prove a characterization of trapezoids.

1. Introduction

If the diagonals in a convex quadrilateral intersect at E and the extensions of the opposite sides intersect at F and G, then the triangle EFG is called the *diagonal point triangle* [3, p.79] or sometimes just the diagonal triangle [8], see Figure 1. Here it's assumed the quadrilateral has no pair of opposite parallel sides. If it's a cyclic quadrilateral, then the diagonal point triangle has the property that its orthocenter is also the circumcenter of the quadrilateral [3, p.197].

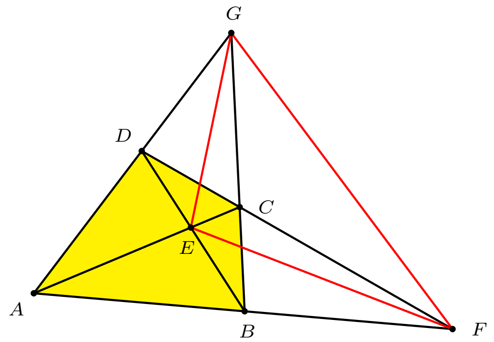

Figure 1. The diagonal point triangle EFG

How about the area of the diagonal point triangle? In a note in an old number of the MONTHLY [2, p.32] reviewing formulas for the area of quadrilaterals, we found a formula and a reference to an even older paper on quadrilaterals by Georges Dostor [4, p.272]. He derived the following formula for the area T of the diagonal point triangle belonging to a cyclic quadrilateral,

$$T = \frac{4a^2b^2c^2d^2K}{(a^2b^2 - c^2d^2)(a^2d^2 - b^2c^2)}$$

where a, b, c, d are the sides of the cyclic quadrilateral and K its area. To derive this formula was also a problem in [7, p.208].[1] However, the formula is wrong, and the mistake Dostor made in his derivation was assuming that two angles are equal when they in fact are not. The purpose of this note is to derive the correct formula.

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[1] But it had a misprint, the 4 was missing.

As far as we have been able to find out, this triangle area has received little interest over the years. But we have found one important result. In [1, pp.13–17] Hugh ApSimon made a delightful derivation of a formula for the area of the diagonal point triangle belonging to a general convex quadrilateral $ABCD$. His formula (which used other notations) is

$$T = \frac{2T_1T_2T_3T_4}{(T_1+T_2)(T_1-T_4)(T_2-T_4)}$$

where T_1, T_2, T_3, T_4 are the areas of the four overlapping triangles ABC, ACD, ABD, BCD respectively. In [6, p.163] Richard Guy rewrote this formula using $T_1 + T_2 = T_3 + T_4$ in the more symmetric form

$$T = \frac{2T_1T_2T_3T_4}{K(T_1T_2 - T_3T_4)} \tag{1}$$

using other notations, where $K = T_1 + T_2$ is the area of the convex quadrilateral.

2. The area of the diagonal point triangle belonging to a cyclic quadrilateral

We will use (1) to derive a formula for the area of the diagonal point triangle belonging to a cyclic quadrilateral in terms of the sides of the quadrilateral.

Theorem 1. *If a, b, c, d are the consecutive sides of a cyclic quadrilateral, then its diagonal point triangle has the area*

$$T = \frac{2abcdK}{|a^2-c^2||b^2-d^2|}$$

if it's not an isosceles trapezoid, where

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

is the area of the quadrilateral and $s = \frac{a+b+c+d}{2}$ is its semiperimeter.

Proof. In a cyclic quadrilateral $ABCD$ opposite angles are supplementary, so $\sin C = \sin A$ and $\sin D = \sin B$. From these we get that $\sin A = \frac{2K}{ad+bc}$ and $\sin B = \frac{2K}{ab+cd}$ by dividing the quadrilateral into two triangles by a diagonal in two different ways. Using (1) and the formula for the area of a triangle, we have

$$\begin{aligned} T &= \frac{2\cdot\frac{1}{2}ad\sin A\cdot\frac{1}{2}bc\sin C\cdot\frac{1}{2}ab\sin B\cdot\frac{1}{2}cd\sin D}{K\left(\frac{1}{2}ad\sin A\cdot\frac{1}{2}bc\sin C-\frac{1}{2}ab\sin B\cdot\frac{1}{2}cd\sin D\right)} \\ &= \frac{\frac{1}{2}abcd\sin^2 A\sin^2 B}{K\left(\sin^2 A-\sin^2 B\right)} = \frac{\frac{1}{2}abcd}{K\left(\frac{1}{\sin^2 B}-\frac{1}{\sin^2 A}\right)} \\ &= \frac{2abcdK}{(ab+cd)^2-(ad+bc)^2} = \frac{2abcdK}{(ab+cd+ad+bc)(ab+cd-ad-bc)} \\ &= \frac{2abcdK}{(a+c)(b+d)(a-c)(b-d)} = \frac{2abcdK}{(a^2-c^2)(b^2-d^2)}. \end{aligned}$$

Since we do not know which of the opposite sides is longer, we put absolute values around the subtractions in the denominator to cover all cases. The area K of the cyclic quadrilateral is given by the well known Brahmagupta formula [5, p.24]. The formula for the area of the diagonal point triangle does not apply when two opposite sides are congruent. Then the other two sides are parallel, so the cyclic quadrilateral is an isosceles trapezoid.[2] $\square$

3. A characterization of trapezoids

In (1) we see that if $T_1T_2 = T_3T_4$, then the area of the diagonal point triangle is infinite. This suggests that two opposite sides of the quadrilateral are parallel, giving a condition for when a quadrilateral is a trapezoid. We prove this in the next theorem.

Theorem 2. *A convex quadrilateral is a trapezoid if and only if the product of the areas of the triangles formed by one diagonal is equal to the product of the areas of the triangles formed by the other diagonal.*

Proof. We use the same notations as in the proof of Theorem 1. The following statements are equivalent.

$$\begin{aligned}
&T_1T_2 = T_3T_4,\\
&\tfrac{1}{2}ad\sin A \cdot \tfrac{1}{2}bc\sin C = \tfrac{1}{2}ab\sin B \cdot \tfrac{1}{2}cd\sin D,\\
&\sin A\sin C = \sin B\sin D,\\
&\tfrac{1}{2}\left(\cos(A-C) - \cos(A+C)\right) = \tfrac{1}{2}\left(\cos(B-D) - \cos(B+D)\right),\\
&\cos(A-C) = \cos(B-D),\\
&A - C = B - D \quad \text{or} \quad A - C = -(B-D),\\
&A + D = B + C = \pi \quad \text{or} \quad A + B = C + D = \pi.
\end{aligned}$$

Here we have used that $\cos(A+C) = \cos(B+D)$, which follows from the sum of angles in a quadrilateral. The equalities in the last line are respectively equivalent to $AB \parallel DC$ and $AD \parallel BC$ in a quadrilateral $ABCD$. $\square$

References

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[6] R. K. Guy, ApSimon's Diagonal Point Triangle Problem, *Amer. Math. Monthly*, 104 (1997) 163–167.

[2] Or in special cases a rectangle or a square.

[7] E. W. Hobson, *A Treatise on Plane and Advanced Trigonometry*, Seventh Edition, Dover Publications, New York, 1957.
[8] E. W. Weisstein, Diagonal Triangle, *MathWorld* – A Wolfram web resource, `http://mathworld.wolfram.com/DiagonalTriangle.html`

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