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Cyclic Quadrilaterals Associated With Squares

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Abstract. We discuss a family of problems asking to find the geometrical locus of points P in the plane of a square $ABCD$ having the property that the i-th triangle center in Kimberling's list with respect to triangles ABP, BCP, CDP, DAP are concyclic.

1. A family of problems

A fruitful research line in the Euclidean geometry of the plane refers to a given quadrilateral, to which another one is associated, and the question is whether the new quadrilateral has a specific property, like being a trapezium, or parallelogram, rectangle, rhombus, cyclic quadrilateral, and so on. Besides the target property, the results of this kind differ by the procedure used to generate the new points. A common selection procedure involves two choices: on the one hand, one produces four triangles out of the given quadrilateral, on the other hand, one identifies a point in each resulting triangle. We restrain our discussion to the handiest ways to fulfill each task.

With regards to the choices of the first kind, one option is to consider the four triangles defined by the vertices of the given quadrilateral taken three at a time. A known result in this category is illustrated in [4]. Another frequently used construction starts from a triangulation determined by a point in the given quadrilateral.

When it comes to select a point in the four already generated triangles, the most convenient approach is to pick one out of the triangle centers listed in [3]. The alternative is to invoke a more complicated construction.

Changing slightly the point of view, one has another promising research line, whose basic theme is to find the geometric locus of points P for which the quadrilateral obtained by choosing a point in each of the triangles determined by P and an edge of the given quadrilateral has a specific property. The chances to obtain interesting results in this manner are improved if one starts from a configuration richer in potentially useful properties.

To conclude the discussion, we end this section by stating the following research problem.

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Problem A. *For a point P in the plane of a square $ABCD$, denote by E, F, G, and H the triangle center $X(i)$ in Kimberling's Encyclopedia of triangle centers* [3] *of triangles ABP, BCP, CDP, DAP, respectively. Find the geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral.*

This problem has as many instances as entries in Kimberling's list. It is plausible that they are of various degrees of difficulty to solve. Indeed, in the next section we present solutions for four of the possible specializations of Problem A, and we shall have convincing samples of different techniques needed in the proofs. The goal is to persuade the reader that Problem A is worth studying. Some results are valid for more general quadrangle $ABCD$. From their proofs we learn to what extent the requirement "$ABCD$ square" is a sensible one.

2. Four results

The instance of Problem A corresponding to $X(2)$ (centroid) is easily disposed of. The answer follows from the next result, probably already known. Having no suitable reference, we present its simple proof.

Theorem 1. *Let P be a point in the plane of a quadrilateral $ABCD$, and let E, F, G, and H be the centroid of triangles ABP, BCP, CDP, DAP, respectively. Then $EFGH$ is a parallelogram. In particular, $EFGH$ is cyclic if and only if AC and BD are perpendicular.*

Proof. The quadrilateral $EFGH$ is the image of the Varignon parallelogram under the homothety $\mathsf{h}(P, \frac{2}{3})$, hence is a parallelogram. A parallelogram is cyclic exactly when it is rectangle. □

Notice that the condition "perpendicular diagonals" is necessary and sufficient for $EFGH$ to be a cyclic quadrilateral. This remark shows that for arbitrary quadrilaterals the sought-for geometrical locus may be empty. Therefore, Problem A is interesting under a minimum of conditions.

Proving the next theorem is more complicated and requires a different approach. As the proof is based on the intersecting chords theorem, we recall its statement: In the plane, let points I, J, K, L with $I \neq J$ and $K \neq L$ be such that lines IJ and KL intersect at Ω. Then, I, J, K, L are concyclic if and only if $\overrightarrow{\Omega I} \cdot \overrightarrow{\Omega J} = \overrightarrow{\Omega K} \cdot \overrightarrow{\Omega L}$.

Theorem 2. *For a point P in the plane of a rectangle $ABCD$, denote by E, F, G, H the circumcenters of triangles ABP, BCP, CDP, DAP, respectively. The geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral consists of the circumcircle of the rectangle and the real hyperbola passing through A, B, C, D and whose asymptotes are the bisectors of rectangle's symmetry axes. In particular, if $ABCD$ is a square, the geometrical locus consists of the diagonals and the circumcircle of the square.*

Proof. We first look for P lying on a side of the rectangle. For the sake of definiteness, suppose that P belongs to the line AB and $P \neq A$, B. Then E is at infinity, F and H sit on the perpendicular bisector of side BC, and G is on the perpendicular bisector of side CD. Suppose P belongs to the desired geometrical locus.

The condition E, F, G, H concyclic is tantamount to F, G, H collinear, which in turn amounts to G being the rectangle's center O. It follows that $PO = OA$, which is true only when P is one of the vertices A, B. If $P = A$, say, then E is not determined, it can be anywhere on the perpendicular bisector of the segment AB. Also, H is an arbitrary point on the perpendicular bisector of the segment DA, while both F and G coincide with the center of the rectangle. Therefore, $EFGH$ degenerates to a rectangular triangle. Thus, the only points common to the geometric locus and to sides are the vertices of the rectangle.

We now examine the points from the geometrical locus which do not lie on the sides of the rectangle. We choose a coordinate system with origin at the center and axes parallel to the sides of the given rectangle. Then the vertices have the coordinates $A(-a, -b)$, $B(a, -b)$, $C(a, b)$, $D(-a, b)$ for some a, $b > 0$. Let $P(u, v)$ be a point, not on the sides of the rectangle. Routine computations yield the circumcenters:

$$E\left(0, \frac{u^2+v^2-a^2-b^2}{2(v+b)}\right), \quad F\left(\frac{u^2+v^2-a^2-b^2}{2(u-a)}, 0\right),$$
$$G\left(0, \frac{u^2+v^2-a^2-b^2}{2(v-b)}\right), \quad H\left(\frac{u^2+v^2-a^2-b^2}{2(u+a)}, 0\right).$$

If P is on the circle $(ABCD)$, *i.e.*, $u^2 + v^2 - a^2 - b^2 = 0$, then E, F, G, H coincide with $O(0, 0)$. If P is not on the circle $(ABCD)$, then $E \neq G$, $F \neq H$, and EG, FH intersect at O. Thus, E, F, G, H are concyclic if and only if $\overrightarrow{OE} \cdot \overrightarrow{OG} = \overrightarrow{OF} \cdot \overrightarrow{OH}$. Since $u^2 + v^2 - a^2 - b^2 \neq 0$, we readily see that this is equivalent to $u^2 - v^2 = a^2 - b^2$. In conclusion, the desired locus is the union of the circle $(ABCD)$ and the hyperbola described in the statement.

In the case of a square, the hyperbola becomes $u^2 = v^2$, which means that P belongs to one of the square's diagonals. □

The above proof shows that one can ignore from the beginning points on the sidelines of $ABCD$: if P is on such a sideline, one of the triangles is degenerate. Such degeneracy would make difficult even to understand the statement of some instances of Problem A.

The differences between rectangles and squares are more obvious when examining the case of $X(4)$ (orthocenter) in Problem A.

Theorem 3. *In a plane endowed with a Cartesian coordinate system centered at O, consider a rectangle $ABCD$ with sides parallel to the axes and of length $|AB| = 2a$, $|BC| = 2b$, and diagonals intersecting at O. For a point P in the plane, not on the sidelines of $ABCD$, denote by E, F, G, H the orthocenters of triangles ABP, BCP, CDP, DAP, respectively. Let $\mathcal{C}$ denote the geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral. Then:*

(a) *If $a \neq b$, then $\mathcal{C}$ is the union of the hyperbola $u^2 - v^2 = a^2 - b^2$ and the sextic*

$$\begin{aligned}(u^2+v^2+a^2-b^2)^2(v^2-b^2) - (u^2+v^2-a^2+b^2)^2(u^2-a^2)\\ = 4a^2u^2(v^2-b^2) - 4b^2v^2(u^2-a^2) \qquad (1)\end{aligned}$$

from which points on the sidelines of $ABCD$ are excluded.
(b) *If $ABCD$ is a square, then $\mathcal{C}$ consists of the points on the diagonals or on the circumcircle of the square which are different from the vertices of the square.*

Proof. To take advantage of computations already performed, we use the fact that the coordinates of the orthocenter are the sums of the coordinates of the vertices with respect to a coordinate system centered in the circumcenter. Having in view the previous proof and the restriction on P, we readily find

$$E\left(u, \frac{a^2-u^2}{v+b}-b\right), \quad F\left(\frac{b^2-v^2}{u-a}+a, v\right),$$
$$G\left(u, \frac{a^2-u^2}{v-b}+b\right), \quad H\left(\frac{b^2-v^2}{u+a}-a, v\right).$$

First, using the coordinates of E, F, G, H, a short calculation shows that

$$E = G \Longleftrightarrow F = H \Longleftrightarrow u^2 - v^2 = a^2 - b^2.$$

It follows that the points on the hyperbola $u^2 - v^2 = a^2 - b^2$ which are different from A, B, C, D belong to the locus. Now, consider a point P not on this hyperbola. Then $E \neq G$, $F \neq H$, and lines EG, FH intersect at P. Thus, E, F, G, H are concyclic if and only if $\overrightarrow{PE} \cdot \overrightarrow{PG} = \overrightarrow{PF} \cdot \overrightarrow{PH}$. This yields the equation (1) of the sextic.

It is worth noting that the points of intersection (other than A, B, C, D) of any two of the circles with diameters AB, BC, CD, DA are in $\mathcal{C}$, either on the hyperbola or on the sextic.

If $ABCD$ is a square of sidelength 2, the equations of these curves simplify to $u^2 = v^2$ and

$$(u^2 - v^2)\big((u^2 + v^2)^2 - 4\big) = 0,$$

giving the diagonals and the circumcircle of the square. □

Figures 1 to 3 illustrate Theorem 3. These graphs convey the idea that the position of real points on the sextic depends on the rectangle's shape.

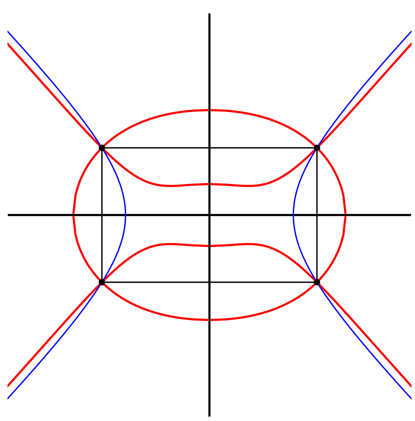

Figure 1. Locus of P in Theorem 3: Rectangle with lengths $\frac{16}{5}$ and 2

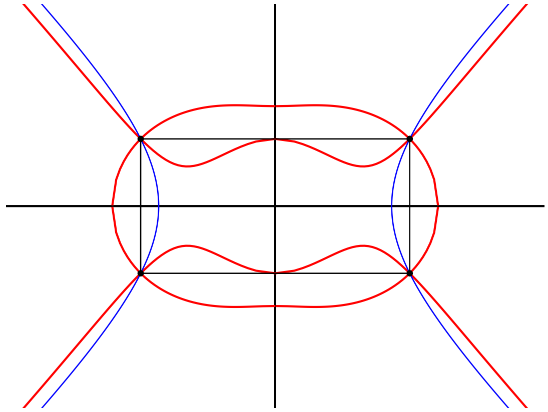

Figure 2. Locus of P in Theorem 3: Rectangle with sides 4 and 2

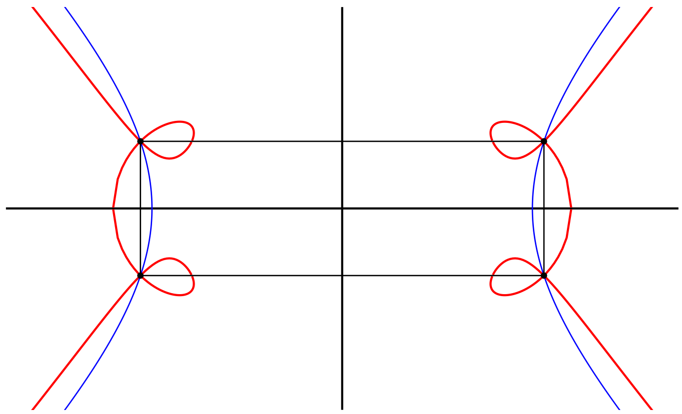

Figure 3. Locus of P in Theorem 3: Rectangle with sides 6 and 2

Solving Problem A for rectangle instead of square would rise great difficulties. As seen above, we might loose the support of geometric intuition, the answer might be phrased in terms of algebraic equation whose complexity would make a complete analysis very laborious. It would be very difficult to decide whether various algebraic curves appearing from computation have indeed points in the sought-for geometrical locus, or even if the locus is nonempty. Therefore we chose to state Problem A for square only. This hypothesis assures that in any instance of Problem A there are points P with the desired property. Namely, if P is on one of the diagonals of the square, that diagonal is a symmetry axis for the configuration. Thus $EFGH$ is an isosceles trapezium, which is certainly cyclic. The argument is valid for P different from the vertices of the square. Therefore, the diagonals (with the possible exception of square's vertices) are included in the geometrical locus.

The proof of Theorem 3 is more difficult than the previous ones. Yet it is much easier than the proof of the next result, which gives a partial answer to the problem obtained by specializing Problem A to incenters.

Theorem 4. *For a point P in the interior of a square $ABCD$, one denotes by E, F, G, and H the incenter of triangle ABP, BCP, CDP, DAP, respectively. The geometrical locus of points P for which $EFGH$ is a cyclic quadrilateral consists of the diagonals AC and BD.*

This settles a conjecture put forward more than 25 years ago by Daia [2]. The only proof we are aware of has been just published [1] and is similar to the proof of Theorem 3. In order to decide whether E, F, G, H are concyclic, Ptolemy's theorem rather than the intersecting chords theorem was used. The crucial difference is the complexity of expressions yielding the coordinates of the incenters in terms of the coordinates of the additional point P. This difference is huge, the required computations can not be performed without computer assistance. For instance, after squaring twice the equality stated by Ptolemy's theorem, one gets $16f^2 = 0$, where f is a polynomial in 14 variables having 576 terms. The algorithms employed to manipulate such large expressions belong to the field generally known as symbolic computation. Even powerful computer algebra systems like MAPLE and SINGULAR running on present-day machines needed several hours and a large amount of memory to complete the task. The output consists of several dozens of polynomial relations satisfied by the variables describing the geometric configuration. In order to obtain a geometric interpretation for the algebraic translation of the conclusion it was essential to use the hypothesis that P sits in the interior of the square. In algebraic terms, this means that the real roots of the polynomials are positive and less than one. In the absence of such an information, it is not at all clear that the real variety describing the asked geometrical locus is contained in the union of the two diagonals.

Full details of the proof for Theorem 4 are given in [1]. As the quest for elegance and simplicity is still highly regarded by many mathematicians, we would like to have a less computationally involved proof to Theorem 4. It is to be expected that a satisfactory solution to this problem will be not only more conceptual but also more enlightening than the approach sketched in the previous paragraph. Moreover, it is hoped that the new ideas needed for such a proof will serve to remove the hypothesis "P in the interior of the square" from the statement of Theorem 4.

3. Conclusions

Treating only four instances, this article barely scratches the surface of a vast research problem. Since as of June 2011 the Encyclopedia of Triangle Centers lists more than 3600 items, it is apparent that a huge work remains to be done.

The approach employed in the proofs of Theorems 2 and 3 seems promising. It consists of two phases. First, one identifies the set containing the points P for which the four centers fail to satisfy the hypothesis of the intersecting chords theorem. This set is contained in the desired locus unless the corresponding quadrilateral $EFGH$ is a non-isosceles trapezium. Next, for points P outside the set one uses the intersecting chords theorem.

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