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More Integer Triangles with $R/r = N$

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Abstract. Given an integer-sided triangle with an integer ratio of the radii of the circumcircle and incircle, a simple method is presented for finding another triangle with the same ratio.

In a recent paper, MacLeod [1] discusses the problem of finding integer-sided triangles with an integer ratio of the radii of the circumcircle and incircle. He finds sixteen examples of integer triangles for values of this ratio between 1 and 999. It will be shown that, with one exception, another triangle with the same ratio can be found for each.

Macleod shows that the ratio, N, for a triangle with sides a, b, and c is given by

$$\frac{2abc}{(a+b-c)(a+c-b)(b+c-a)} = N. \tag{1}$$

Define $\alpha = a+b-c$, $\beta = a+c-b$, and $\gamma = b+c-a$. Then

$$\frac{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}{4\alpha\beta\gamma} = N. \tag{2}$$

Let α' and β' be found from any one of MacLeod's triangles. Then (2) may be used to find γ'. But notice that (2) is then a quadratic equation for γ:

$$(\alpha'+\beta')(\alpha'+\gamma)(\beta'+\gamma) = 4N\alpha'\beta'\gamma. \tag{3}$$

One root is the known value, γ', while the other root gives a new triangle with the same value for N. Note that the sum of the two roots is $-\alpha' - \beta' + \frac{4N\alpha'\beta'}{\alpha'+\beta'}$. Since one root is γ', the other is given by

$$\gamma = -\alpha' - \beta' - \gamma' + \frac{4N\alpha'\beta'}{\alpha'+\beta'}.$$

For $N = 2$, $a = b = c = 1$; so $\alpha' = \beta' = \gamma' = 1$ and $\gamma = 1$. No new triangle results.

For $N = 26$, $a = 11$, $b = 39$, $c = 49$; so $\alpha' = 1$, $\beta' = 21$, $\gamma' = 77$ and $\gamma = \frac{3}{11}$. Scaling by a factor of 11 gives $\alpha' = 11$, $\beta' = 231$, and $\gamma' = 3$. The sides of the resulting triangle are $a' = 121$, $b' = 7$, and $c' = 117$.

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The first few values and the last value of N given by Macleod along with the original triangles and the new ones are shown in the table below.

N	a	b	c	a'	b'	c'
1	1	1	1	1	1	1
26	11	39	49	7	117	121
74	259	475	729	27	1805	1813
218	115	5239	5341	763	12493	13225
250	97	10051	10125	1125	8303	9409
866	3025	5629	8649	93	73177	73205

Table 1. Macleod triangles and the corresponding new ones (sides arranged in ascending order).

Reference

[1] A. J. MacLeod, Integer triangles with $R/r = N$, *Forum Geom.*, 10 (2010) 149–155.

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