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The Isosceles Trapezoid and its Dissecting Similar Triangles

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Abstract. Isosceles trapezoids are dissected into three similar triangles and rearranged to form two additional isosceles trapezoids. Moreover, triangle centers, one from each similar triangle, form the vertices of a centric triangle which has special properties. For example, the centroidal triangles are congruent to each other and have an area one-ninth of the area of the trapezoids; whereas, the circumcentric triangles are not congruent, but still have equal areas.

1. Introduction

If you were asked whether an isosceles trapezoid can be dissected into three similar triangles by a point on the longer base, you would probably reply initially that it is not possible. However, it is sometimes possible and the search for such a point was the gateway to some other very interesting results.

Theorem 1. *If the longer base of an isosceles trapezoid is greater than the sum of the two isosceles sides, then there exists a point on the longer base of the trapezoid which when joined to the endpoints of the shorter base divides the trapezoid into three similar triangles.*

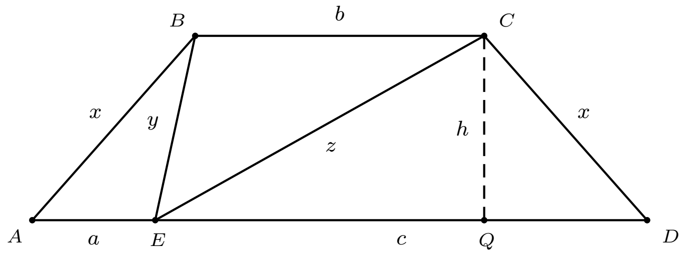

Figure 1.

Proof. To begin our construction we consider isosceles trapezoid $ABCD$ with longer base AD and congruent sides AB and CD as shown in Figure 1. Additionally we let $x = AB = CD$, $b = BC$, $e = AD$, $y = BE$, and $z = CE$.

We propose that the point E can be located on AD by letting

$$AE = \ a = \frac{e}{2} - \sqrt{\left(\frac{e}{2}\right)^2 - x^2},$$

$$ED = \ c = \frac{e}{2} + \sqrt{\left(\frac{e}{2}\right)^2 - x^2}.$$

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Then,

$$\frac{AE}{ED} = \frac{a}{c} = \frac{ac}{c^2} = \frac{x^2}{c^2}.$$

Therefore, $\frac{a}{x} = \frac{x}{c}$ or $\frac{AE}{AB} = \frac{CD}{ED}$. Since $\angle BAE$ and $\angle CDE$ are base angles of the isosceles trapezoid, then triangle BAE is similar to triangle EDC.

Next we consider triangles CQE and CQD where Q is the intersection of a perpendicular dropped from C to base AD. If $CQ = h$, then $QD = \frac{c-b}{2} = \frac{a+c-b}{2}$ so that $EQ = ED - QD = \frac{c-a+b}{2}$. By the Pythagorean Theorem for triangles CQE and CQD respectively, we have $z^2 = h^2 + \left(\frac{c-a+b}{2}\right)^2$ and $x^2 = h^2 + \left(\frac{a+c-b}{2}\right)^2$. By subtracting these equations we obtain

$$z^2 - x^2 = \left(\frac{c-a+b}{2}\right)^2 - \left(\frac{a+c-b}{2}\right)^2 = bc - ac.$$

Since $\frac{a}{x} = \frac{x}{c}$ (see above), we add $x^2 = ac$ to $z^2 - x^2 = bc - ac$ to obtain $z^2 = bc$. Rewriting this as $\frac{z}{b} = \frac{c}{z}$, or equivalently $\frac{EC}{CB} = \frac{DE}{EC}$, and noting that $\angle ECB$ and $\angle DEC$ are alternate interior angles of parallel lines, we have that triangles ECB and DEC are similar. By transitivity, or by repeating the method above, we get that all three triangles are similar to each other. This proves Theorem 1. □

There are some excellent books on dissection, but most involve dissecting a polygon and rearranging the pieces into one or more other polygons. However, none of these references consider isosceles trapezoids and similar triangles. See [1] and [4].

Theorem 2. *Using the notation introduced above we have the following equalities:*
(i) $y^2 = ab$, $x^2 = ac$, $z^2 = bc$;
(ii) $a = \frac{xy}{z}$, $b = \frac{yz}{x}$, $c = \frac{xz}{y}$;
(iii) $xyz = abc$, *and*
(iv) *the area of* $ABCD = \frac{1}{2}h(a+b+c)$.

Proof. The first three follow immediately from the similar dissecting triangles, and (iv) follows directly from the formula for the area of a trapezoid. □

Theorem 3. *Using the notation introduced above, the length of a diagonal, d, is given by*

$$d = \sqrt{ac+ab+bc} = \sqrt{x^2+y^2+z^2}.$$

Proof. By the law of cosines for triangles ABC and CDA, respectively, in Figure 1, we have

$$\begin{aligned} d^2 = {} & AC^2 = x^2 + b^2 - 2xb\cos ABC \\ = {} & x^2 + (a+c)^2 - 2x(a+c)\cos(180^\circ - ABC) \\ = {} & x^2 + (a+c)^2 + 2x(a+c)\cos ABC. \end{aligned}$$

Therefore,

$$\cos ABC = \frac{x^2+b^2-d^2}{2xb} = \frac{x^2+(a+c)^2-d^2}{-2x(a+c)}.$$

After some simplification and Theorem 2(i) this becomes

$$d^2 = x^2 + ab + bc = ac + ab + bc = x^2 + y^2 + z^2.$$

□

Theorem 4 (Generalization of the Pythagorean Theorem)**.** *Using the notation introduced above, $y^2 + z^2 = b(a + c)$.*

Proof. Since the triangles are similar, the angles BEC, BAE and CDE are congruent. By Theorem 2(i),

$$y^2 + z^2 = ab + bc = b(a + c) = b^2,$$

where the last equality holds whenever $\angle BAE = 90°$. □

This result appeared previously in [2].

Next we consider triangles whose vertices are specific triangle centers for each of the three dissecting triangles of Figure 1. Since there are over a thousand identified triangle centers, we restrict our discussion to two of the most well-known; namely, the centroid and circumcenter. We will refer to these new triangles as centroidal and circumcentric, respectively.

2. The Centroidal Triangle

It is well-known that the centroid of a triangle is the intersection of the three medians of a triangle and that the centroid is the center of gravity for the triangle. We denote the centroids of our three similar triangles as G_a, G_b, and G_c as shown in Figure 2.

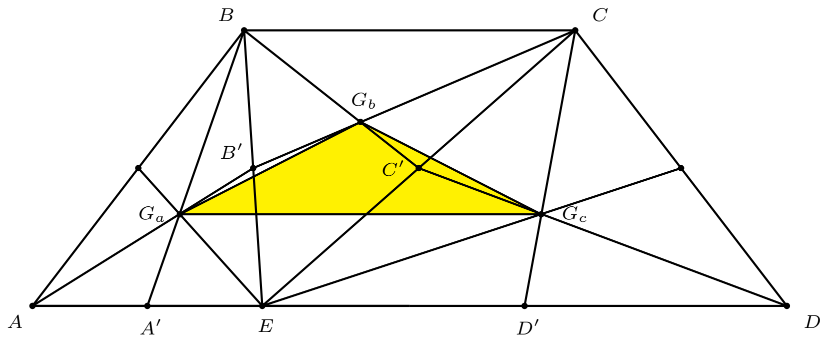

Figure 2. The centroidal triangle

Theorem 5. *Using the notation already introduced,*
(i) *Triangle $G_aG_bG_c$ is isosceles with $G_aG_b = G_cG_b = \frac{1}{3}\sqrt{ab + bc + ca}$,*
(ii) *the base of G_aG_c of triangle $G_aG_bG_c$ is parallel to AD and its length is $G_aG_c = \frac{1}{3}(a + b + c)$, and*
(iii) *the area of triangle $G_aG_bG_c$ is $\frac{1}{9}$ of the area of trapezoid $ABCD$.*

Proof. We consider triangle $G_aG_bG_c$ whose vertices are the respective centroids G_a, G_b, and G_c of triangles BAE, CEB and DEC. Let A', B', C', and D' be the respective midpoints of AE, BE, CE, and DE. By the midsegment, or midline theorem, *the line segment joining the midpoints of two sides of a triangle is parallel to and half the length of the third side*. Therefore, quadrilateral $A'B'C'D'$ has sides parallel to and one-half the corresponding sides of quadrilateral $ABCD$, and the quadrilaterals are similar. In particular, quadrilateral $A'B'C'D'$ is isosceles.

Since the centroid of a triangle divides each median in a ratio of $2:3$ of the median from the vertex and $1:3$ from the midpoint of the corresponding side, $G_aG_b = \frac{2}{3}A'C'$ for triangle $BA'C'$ and $G_cG_b = \frac{2}{3}B'D'$ for triangle $CB'D'$. Since trapezoid $A'B'C'D'$ has a similarity ratio of $\frac{1}{2}$ with isosceles trapezoid $ABCD$, $G_aG_b = \frac{2}{3}A'C' = \frac{2}{3}\cdot\frac{1}{2}AC = \frac{1}{3}AC$. In the same manner $G_cG_b = \frac{1}{3}BD$. Since diagonals AC and BD are congruent, $G_aG_b = G_cG_b$ and triangle $G_aG_bG_c$ is isosceles. Note that $G_aG_b = G_cG_b = \frac{1}{3}\sqrt{ab+bc+ca}$, which is one-third of the length of the diagonal of the trapezoid.

The base G_aG_c of triangle $G_aG_bG_c$ is parallel to AD and its length is $G_aG_c = \frac{2}{3}A'D' + \frac{1}{3}BC$ in trapezoid $BCD'A'$ so that

$$G_aG_c = \frac{2}{3}\left(\frac{a}{2}+\frac{c}{2}\right) + \frac{1}{3}b = \frac{1}{3}(a+b+c).$$

Finally, the area of triangle $G_aG_bG_c = \frac{1}{2}\times$ base $\times$ height $= \frac{1}{2}\cdot\frac{1}{3}(a+b+c)\cdot\frac{h}{3} = \frac{1}{9}\cdot\frac{1}{2}h(a+b+c) = \frac{1}{9}\times$ area of trapezoid $ABCD$. □

3. The Circumcentric Triangle

Next we consider the circumcenters of each of the three dissecting triangles of Figure 1. A circumcenter is the intersection of the three perpendicular bisectors of the sides of any triangle. The circumradius is the radius of the circumcircle which passes through the three vertices of the particular triangle. For our example in Figure 3, triangle ABE has circumcenter O_a and circumradius $R_a (= AO_a = BO_a = CO_a)$. Similar statements hold for O_b, O_c, R_b, and R_c.

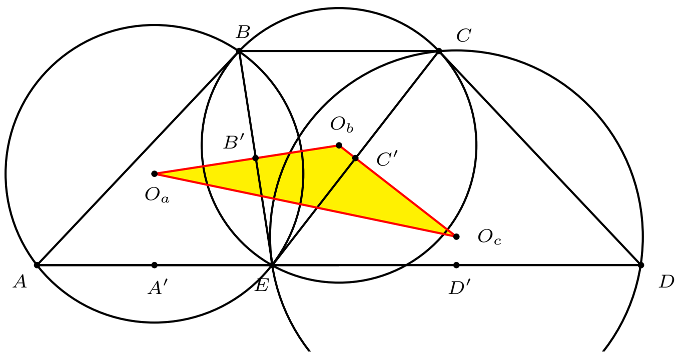

Figure 3. The circumcentric triangle

Theorem 6. *Using the notation already introduced for triangle $O_aO_bO_c$,*
(i) $O_aOb = R_c$ *and* $O_cO_b = R_a$*,*
(ii) $O_aO_c = \sqrt{2R_a^2 + 2R_c^2 - R_b^2}$*, and*
(iii) *the area of triangle* $O_aO_bO_c = \frac{xyz}{8h} = \frac{abc}{8h}$.

Proof. Let A' and B' be the feet of the perpendicular bisectors of two sides of triangle ABE. Since triangle AO_aE is isosceles, AO_aA' and EO_aA' are congruent right triangles. Note that O_a is the vertex of three isosceles subtriangles in triangle ABE, and also a vertex of six right triangles which are congruent in pairs. For convenience we label the angles away from center O_a numerically (see Figure 4) as

$$\begin{aligned}\angle BAE = &\ \gamma = \angle 1 + \angle 2,\\ \angle BEA = &\ \alpha = \angle 2 + \angle 3,\\ \angle ABE = &\ \beta = \angle 1 + \angle 3.\end{aligned}$$

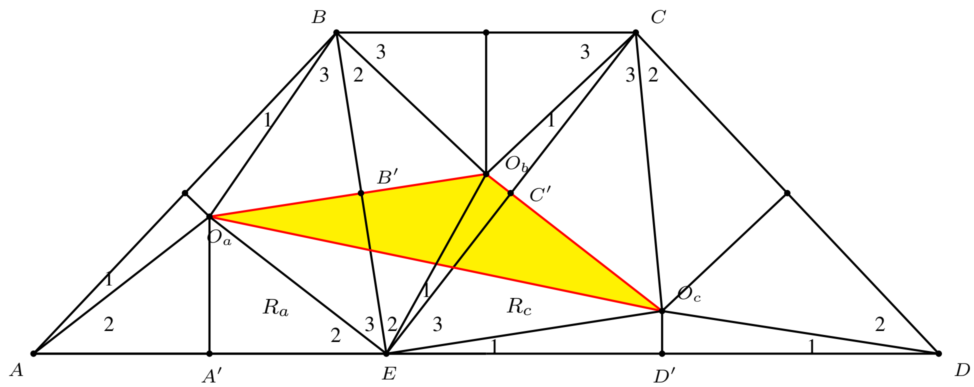

Figure 4. Numbered angles of isosceles and similar triangles

In the same manner corresponding congruent angles are denoted in Figure 4 for the similar triangles CBE and DEC.

In particular, we note that in quadrilateral $B'EC'O_b$ which has two right angles, we have

$$\angle B'O_bC' = 360^\circ - 90^\circ - 90^\circ - \angle 2 - \angle 1 = 180^\circ - \gamma = \alpha + \beta.$$

Also,

$$\angle O_aEO_c = \angle 3 + (\angle 2 + \angle 1) + \angle 3 = (\angle 3 + \angle 2) + (\angle 1 + \angle 3) = \alpha + \beta.$$

Therefore, one pair of opposite angles of quadrilateral $O_aO_bO_cE$ are congruent. Since

$$\begin{aligned}\angle EO_aO_b = &\ \angle EO_aB' = 90^\circ - \angle 3,\\ \angle EO_cO_b = &\ \angle EO_cC' = 90^\circ - \angle 3,\end{aligned}$$

the other pair of opposite angles of quadrilateral $O_aO_bO_cE$ are congruent. Hence quadrilateral $O_aO_bO_cE$ is a parallelogram. Therefore, $O_aO_b = EO_c = R_c$ and $O_cO_b = EO_a = R_a$. This proves (i).

Since the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of the four sides, we have

$$O_aO_c^2 + O_bE^2 = O_aO_b^2 + O_bO_c^2 + O_cE^2 + EO_a^2 = 2R_c^2 + 2R_a^2.$$

Therefore, $O_aO_c^2 = 2R_c^2 + 2R_a^2 - R_b^2$. From this (ii) follows.

If we use the formula $R = \frac{abc}{4\cdot\text{Area}}$ for the circumradius of a triangle with sides of lengths a, b, c (see [3] and [4]), then for triangle ABE

$$R_a^2 = \left(\frac{axy}{4\cdot\frac{1}{2}ah}\right)^2 = \frac{x^2y^2}{4h^2} = \frac{ac\cdot ab}{4h^2} = \frac{a^2bc}{4h^2},$$

with similar results for R_b^2 and R_c^2. Therefore,

$$\begin{aligned} O_aO_c^2 &= 2R_c^2 + 2R_a^2 - R_b^2 = \frac{2a^2bc}{4h^2} + \frac{2abc^2}{4h^2} - \frac{ab^2c}{4h^2}, \\ O_aO_C &= \frac{\sqrt{abc(2a+2c-b)}}{2h}. \end{aligned}$$

Since the opposite sides of a parallelogram are parallel,

$$\begin{aligned} \angle EO_bO_c &= \angle O_bEO_a = \angle 3 + \angle 2 = \alpha, \\ \angle O_bEO_c &= \angle 1 + \angle 3 = \beta. \end{aligned}$$

This implies that $\angle O_bO_cE = \gamma$. Therefore, triangle EO_bO_c is similar to the original three similar dissecting triangles. Since EO_b is a diagonal of parallelogram $O_aO_bO_cE$, similar statements hold for triangle O_aO_bE. Finally,

$$\text{area } O_aO_bO_c = \frac{1}{2}\cdot\text{area of parallelogram } O_aO_bO_cE = \text{area of } EO_bO_c.$$

Using the basic formula for the area of a triangle we have

$$\begin{aligned} \text{area of } O_aO_bO_cE &= \text{area of } O_aO_bE + \text{area of } EO_bO_c \\ &= \frac{1}{2}\cdot\frac{y}{2}\cdot O_aO_b + \frac{1}{2}\cdot\frac{z}{2}\cdot O_bO_c \\ &= \frac{1}{4}\cdot yR_c + \frac{1}{4}\cdot zR_a. \end{aligned}$$

Recalling the formula $R = \frac{abc}{4\cdot\text{Area}}$ from above, we have

$$\begin{aligned}
\text{area of } O_aO_bO_c &= \frac{1}{8}(yR_c + zR_a) \\
&= \frac{1}{8}\left(y \cdot \frac{czx}{4 \cdot \frac{ch}{2}} + z \cdot \frac{axy}{4 \cdot \frac{ah}{2}}\right) \\
&= \frac{1}{8}\left(\frac{xyz}{2h} + \frac{xyz}{2h}\right) \\
&= \frac{xyz}{8h} = \frac{abc}{8h}.
\end{aligned}$$

□

Corollary 7. *If the dissecting triangles are right triangles, then*
(i) $c = a + b$, *and*
(ii) *the area of triangle $O_aO_bO_c$ is one-eighth the area of trapezoid $ABCD$.*

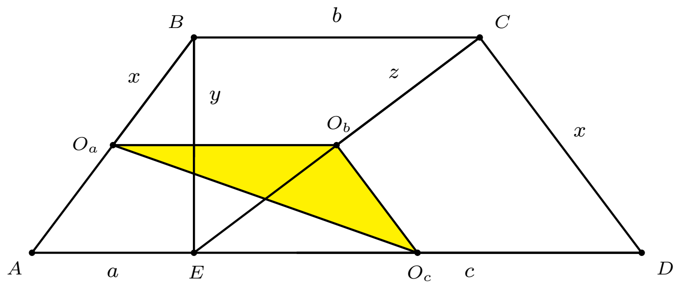

Figure 5. Circumcentric triangle with similar right triangles

Proof. For a right triangle the circumcenter is the midpoint of the hypotenuse of the right triangle. Therefore, $c^2 = x^2 + z^2$ in triangle BEC in Figure 5. Substituting $x^2 = ac$ and $z^2 = bc$ yields $c^2 = ac + bc$. From this the first result follows. Note that

$$\begin{aligned}
\text{area of } O_aO_bO_c = \text{ area of } EO_bO_c &= \frac{1}{4} \cdot \text{area of } ECD \\
&= \frac{1}{4} \cdot \frac{1}{2} \cdot hc = \frac{1}{4} \cdot \frac{1}{2}h(a+b) = \frac{1}{4} \cdot \text{area of } ABCE.
\end{aligned}$$

It also follows that EC separates the trapezoid into two parts with equal area. □

4. The Three IsoscelesTrapezoids

We return to the dissection of §1. Since we started with a dissection problem it surely occurred to the reader that we might be able to rearrange the dissected trapezoid into another configuration. That is indeed the case. The three similar triangles can be rearranged as follows:

Theorem 8. *If the isosceles trapezoid is literally cut apart, then the similar triangles can be rearranged to form two additional isosceles trapezoids which meet the same dissection criteria, have the same area, and have the same diagonal lengths as the original trapezoid.*

Proof. With the trapezoid cut apart and reassembled we get the three cases shown in Figure 6 below. The triangles are numbered #1, #2, and #3 for clarity.

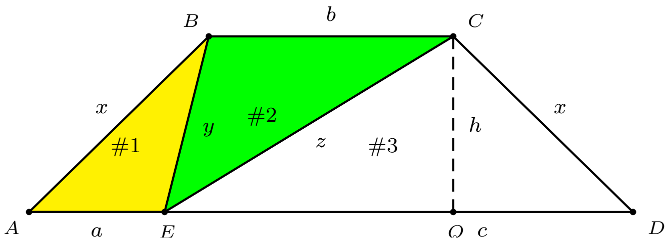

Figure 6(i) Original trapezoid with similar triangles

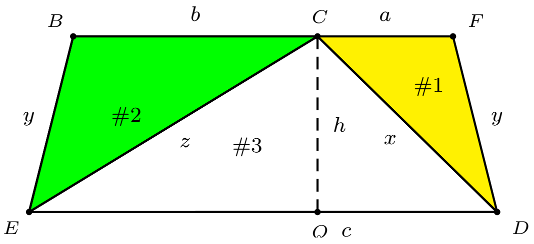

Figure 6(ii) Trapezoid with rearranged triangles

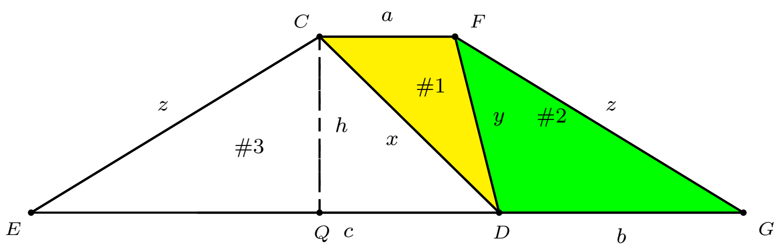

Figure 6(iii) Trapezoid with rearranged triangles

Note that the area of each of the three trapezoids is $\frac{1}{2}h(a+b+c)$ regardless of shape. In Theorem 3 the length of the diagonals for the first trapezoid was given by the formula $d = \sqrt{ab + bc + ca} = \sqrt{x^2 + y^2 + z^2}$. Since the formula is symmetric in the variables, the formulas hold for the latter two cases as well. This can also be seen as a proof without words in Figure 7 where the dotted segments are the diagonals of the three respective trapezoids. Since the diagonals of an isosceles trapezoid are congruent, we have

$$AC = BD, \quad BD = EF, \quad EF = CG.$$

Hence all are equal in length.

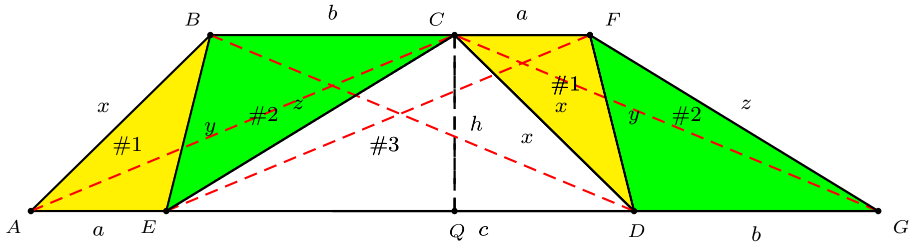

Figure 7. Proof without words: Congruent diagonals

Since many of the formulas derived in the theorems above are symmetric in variables a, b, c, x, y, and z, these particular properties also hold for the two additional trapezoidal arrangements of similar triangles. For example, since two sides of the centroidal triangle of the original trapezoid are given by $\frac{1}{3}\sqrt{ab + bc + ca}$ and the third side by $\frac{1}{3}(a + b + c)$, the three centroidal triangles of all three trapezoids are also isosceles and congruent. Additionally the areas of each of these triangles is one-ninth of the areas of the trapezoids.

Since the sides of the circumcentric triangle of the original trapezoid are given by circumradii R_a, R_c, and $\sqrt{2R_a^2 + 2R_c^2 - R_b^2}$, the circumcentric triangles of the other two trapezoids are not isosceles and are not congruent for the three trapezoidal arrangements. However, the areas of the three circumcentric triangles are the same and are given by $\frac{xyz}{8h} = \frac{abc}{8h}$.

There are some excellent books on dissection, but most involve dissecting a polygon and rearranging the pieces into one or more other polygons. For example, see [1] and [5]. However, none of these references consider isosceles trapezoids and similar triangles as presented in this paper. □

5. More Study

There are some additional questions that might be worth pursuing such as: What properties follow from other centric triangles such as incenters, orthocenters, etc.? Under what conditions are the three Euler lines of the dissecting triangles concurrent or parallel? Under what conditions are the three triangle centers for the dissecting triangles collinear? Will any of the centric triangles be similar to the dissecting triangles? Do comparable properties hold when isosceles trapezoid is replaced by isosceles quadrilateral? Finally, is there a 3-dimensional analog for these properties?

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