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Reflection Triangles and Their Iterates

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Abstract. By reflecting each vertex of a triangle in the opposite side one obtains the vertices of the reflection triangle of the given triangle. We analyze the forward and backward orbit of any base triangle under this reflection process and give a complete description of the underlying discrete dynamical system with fractal structure.

1. Introduction

We consider *finite* triangles as well as *infinite* triangles with a finite side, a vertex at infinity and two semi–infinite parallel sides. By reflecting each vertex of a triangle in the opposite side one obtains the vertices of the *reflection triangle* of the given triangle. A degenerate triangle is thus its own reflection triangle – including by convention triangles with two or three coincident vertices. The reverse construction of an antireflection triangle is in general not possible with compass and ruler only [2]. By using interactive geometry software one sees how erratic the behavior of a, say, four times reflected triangle can be with respect to the base triangle (Figure 1).

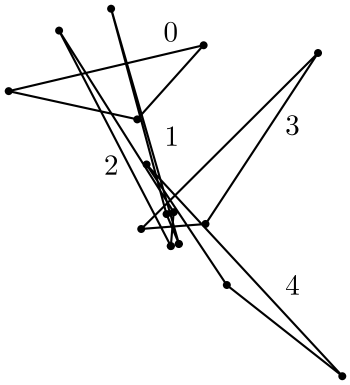

Figure 1

We give a complete description of the dynamical system generated by this reflection process and we reduce the part concerning the non-acute triangles to a symbolic system. Each *proper* triangle (*i.e.*, each finite nondegenerate triangle) is the reflection triangle of 5, 6 or 7 differently placed triangles – 7 when the triangle is equilateral or nearly equilateral (Figure 2). Each degenerate triangle with three

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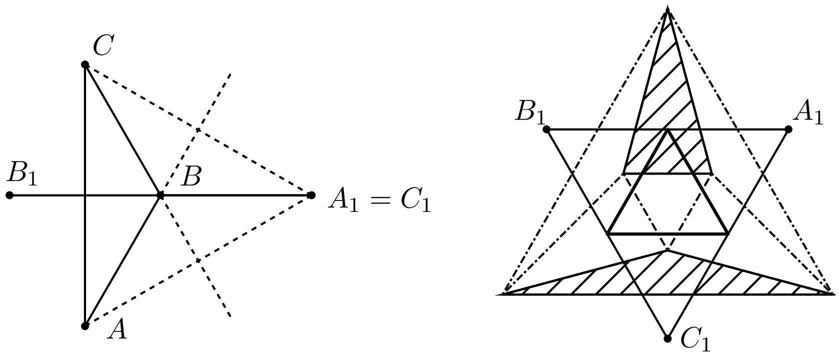

Figure 2. Isosceles triangle with equal 30°–angles and degenerate reflection triangle; the 7 triangles with the same equilateral reflection triangle

distinct vertices is the reflection triangle of exactly 5 triangles. Each nondegenerate infinite triangle is the reflection triangle of exactly 3 infinite triangles. We prove that all finite acute and right–angled triangles tend to an equilateral limit if one iterates this reflection map, and we describe the fractal structure of the triangles having an equilateral or degenerate limit. If one represents the set of triangles up to similarity by the set $\left\{(\alpha,\beta) \mid 0^\circ \leq \beta \leq \alpha \leq 90^\circ - \frac{\beta}{2}\right\}$ in the Euclidean plane, the triangles with equilateral limit form a dense open subset; the triangles with degenerate limit form a countable union of maximal path–connected subsets with empty interior; the triangles without equilateral or degenerate limit form an uncountable totally path–disconnected subset; any neighborhood of a triangle without equilateral limit contains uncountably many triangles with equilateral limit, with degenerate limit, and with neither equilateral nor degenerate limit. We show that there are up to angle similarity four finite and two infinite triangles similar to their reflection triangle (among them the degenerate and equilateral triangles, the *heptagonal triangle* with angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$, and the rectangular infinite triangle). We exhibit the ten 2–cycles – three of them for infinite triangles – and the forty 3–cycles – eight of them for infinite triangles. If one identifies similar triangles, the set of non-acute triangles contains (finitely many) cycles of any fixed finite length – they are always repelling – and uncountably many disjoint divergent forward orbits for both finite and infinite triangles. We exhibit some explicit examples and describe symbolically the periodic and divergent forward orbits. It is possible to design divergent forward orbits with almost any behavior: such an orbit can for example approximate any periodic orbit of non-acute, nondegenerate triangles during any finite number of consecutive reflection steps before leaving this cycle, or it can even be dense in the space of triangles without equilateral limit. If one identifies similar triangles, infinite triangles having a degenerate limit are countably dense in the set of infinite triangles; this is also the case for the backward orbit of any nondegenerate infinite triangle; the backward orbit of a finite triangle without equilateral limit (and not reduced to a single point) is dense in the set of all triangles without equilateral limit.

Properties of finite reflection triangles can be found in [12, 5, 6] and [3, pp. 77–80]. The reflection triangle of a proper triangle Δ is homothetic – in ratio 4 with respect to the centroid of Δ – to the pedal triangle of the nine–point center N [5], *i.e.*, to the triangle with vertices on each side of Δ halfway between the side's midpoint and the altitude's foot. By the Wallace–Simson Theorem [10, p. 137] the reflection triangle of a proper triangle Δ – being similar to the pedal triangle of N – is degenerate if and only if N lies on the circumcircle of Δ: this is the case if and only if the sides a, b, c of Δ satisfy $a^2 + b^2 + c^2 = 5R^2$, R being the circumradius. Thus, by the sine law, the reflection triangle of a proper triangle with angles α, β, γ is degenerate if and only if

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \tfrac{5}{4}. \tag{1}$$

We mainly use a method developed by van IJzeren [9] for solving the problem of finding all triangles with a given finite reflection triangle. We reformulate, extend and fully exploit van IJzeren's results and prove them because the original proof (in Dutch) is partly incomplete and sometimes approximate. The key paper [9] was preceded by another van IJzeren's paper [8] and by publications of Dutch mathematicians on the same subject [2, 11].

2. Van IJzeren coordinates of a triangle

We identify triangles that have the same angles α, β, γ to get the set $\mathcal{T}$ of similarity classes. We then speak of a *(triangle) class* of $\mathcal{T}$ and write $\Delta \in \mathcal{T}$ or $\{\alpha, \beta, \gamma\} \in \mathcal{T}$. It is both natural and convenient to assign angles 0, 0, π to all degenerate triangles (*i.e.*, to triangles with collinear vertices) and to lump them together into a single class $\mathcal{O}$ of $\mathcal{T}$. The classes of infinite triangles are $\Pi_\alpha = \{\alpha, \pi - \alpha, 0\}$, $0 < \alpha < \pi$; these are the classes of triangles having as vertices one point at infinity and two different finite points, and as sides one line segment and two half–lines (which are parallel). Note that $\Pi_\alpha = \Pi_{\pi-\alpha}$ and that $\Pi_{\pi/2}$ contains the infinite rectangular triangles. We denote by I_α the isosceles class of the finite triangles with angles $\{\alpha, \alpha, \pi - 2\alpha\}$, $0 < \alpha < \frac{\pi}{2}$. We often identify $\mathcal{T}$ with $\left\{(\alpha, \beta) \mid 0^\circ \leq \beta \leq \alpha \leq 90^\circ - \frac{\beta}{2}\right\}$ (Figure 4).

For both the class $\Delta = \{\alpha, \beta, \gamma\} \in \mathcal{T}$ and a triangle Δ with these angles, we define the sum $s(\Delta) = \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$, the product $p(\Delta) = \sin^2 \alpha \cdot \sin^2 \beta \cdot \sin^2 \gamma$ and the *van IJzeren map*

$$V(\Delta) = \Delta^* = (s(\Delta), p(\Delta))$$

giving the *van IJzeren coordinates* of Δ. $s(\Delta)$ runs from 0 for a degenerate triangle to $\frac{9}{4}$ for an equilateral triangle; $s(\Delta)$ is > 2, $= 2$ or < 2 if Δ is acute, right–angled or obtuse, respectively. $p(\Delta)$ runs from 0 for a degenerate or infinite triangle to $\frac{27}{64}$ for an equilateral triangle. A given $s(\Delta)$ or $p(\Delta)$ determines the curve of admissible values (α, β) for two acute angles of Δ (Figure 3).

Lemma 1. *The polynomial $u^3 - su^2 + du - p$ has roots $u_1 = \sin^2 \alpha$, $u_2 = \sin^2 \beta$, $u_3 = \sin^2 \gamma$ for some $\{\alpha, \beta, \gamma\} \in \mathcal{T}$ if and only if $s, p \in \mathbf{R}$, $p \geq 0$, $d = \frac{s^2}{4} + p$ and*

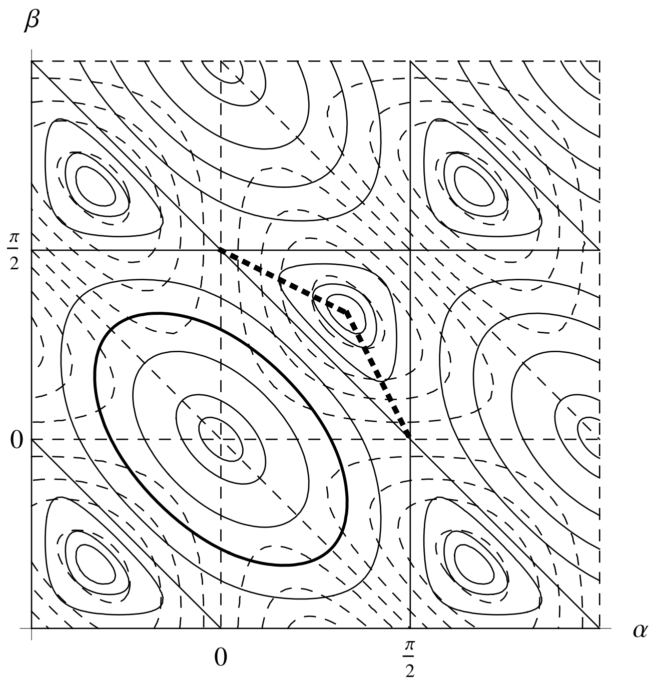

Figure 3. Level curves $\sin^2\alpha+\sin^2\beta+\sin^2(\alpha+\beta)=s$ (plain, thick for $s=\frac{5}{4}$) and $\sin^2\alpha\cdot\sin^2\beta\cdot\sin^2(\alpha+\beta)=p$ (dashed). Points (α,β) corresponding to the two smallest angles α,β of a triangle lie in the square $0\le\alpha,\beta\le\frac{\pi}{2}$ south–west of or on the thick dotted line.

$D(s,p)=(9-4s)^3-(8p+2s^2-18s+27)^2\ge 0$. *$\{\alpha,\beta,\gamma\}$ is then unique and* $s\ge 0$.

Proof. If $d=\frac{s^2}{4}+p$, $\frac{p}{16}D(s,p)$ is the polynomial's discriminant: for $s\in\mathbf{R}$ and $p>0$ one has then $D(s,p)\ge 0$ if and only if the roots are real; for $s\in\mathbf{R}$ and $p=0$ the roots are then 0 and $\frac{s}{2}$ (double) and $D(s,0)=4s^3(2-s)$ is ≥ 0 if and only if $s\in[0,2]$.

($\Rightarrow$) s, d and p are the roots' sum, the sum of products of two roots and the roots' product, respectively. Hence $s,p\in\mathbf{R}$, $p\ge 0$, and $D(s,p)\ge 0$ if $d=\frac{s^2}{4}+p$. We have to prove that $d=\frac{s^2}{4}+p$. If no angle is 0, divide the cosine law by the squared circumdiameter to get

$$2\sin\alpha\sin\beta\cos\gamma=\sin^2\alpha+\sin^2\beta-\sin^2\gamma. \qquad (2)$$

If the triangle is degenerate or infinite, (2) becomes $0=0$ and is true also. Square (2) to get $4u_1u_2(1-u_3)=(s-2u_3)^2$, *i.e.*,

$$4u_1u_2-4p=s^2-4su_3+4u_3^2=s^2-4u_1u_3-4u_2u_3,\ \textit{i.e.},\ 4d-4p=s^2.$$

($\Leftarrow$) The polynomial's roots u_1, u_2, u_3 are real. Since $d=\frac{s^2}{4}+p$, one has $u^3-su^2+du-p=u(u-\frac{s}{2})^2+p(u-1)$: no root can be >1 or <0 if $p>0$; if $p=0$, the roots are 0 and $\frac{s}{2}\in[0,1]$ since $D(s,p)\ge 0$. One can thus write

$u_1 = \sin^2\alpha_1$, $u_2 = \sin^2\beta_1$, $u_3 = \sin^2\gamma_1$ for some $\alpha_1, \beta_1, \gamma_1 \in \left[0, \frac{\pi}{2}\right]$. As above, $4d - 4p = s^2$ if and only if $4u_1u_2(1-u_3) = (s-2u_3)^2 = (u_1+u_2-u_3)^2$, *i.e.*, if and only if $4u_1u_2 - 4u_1u_2u_3 = u_3^2 - 2(u_1+u_2)u_3 + (u_1+u_2)^2$, *i.e.*, if and only if

$$u_3^2 - 2(u_1 + u_2 - 2u_1u_2)u_3 + (u_1 - u_2)^2 = 0. \tag{3}$$

Since $u_1 + u_2 - 2u_1u_2 = u_1(1-u_2) + (1-u_1)u_2$ and $u_1 - u_2 = u_1(1-u_2) - (1-u_1)u_2$, (3) is equivalent to $\big(u_3 - (u_1(1-u_2) + (1-u_1)u_2)\big)^2 = 4u_1(1-u_2)(1-u_1)u_2$, *i.e.*,

$$u_3 = \sin^2\alpha_1\cos^2\beta_1 + \cos^2\alpha_1\sin^2\beta_1 \pm 2\sin\alpha_1\cos\beta_1\cos\alpha_1\sin\beta_1,$$

which is $\sin^2\gamma_1 = \sin^2(\alpha_1 \pm \beta_1)$. If $\sin^2\gamma_1 = \sin^2(\alpha_1+\beta_1)$, take $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \pi - \alpha - \beta$. If $\sin^2\gamma_1 = \sin^2(\alpha_1 - \beta_1)$, suppose $\alpha_1 \geq \beta_1$ without restricting the generality and choose $\gamma = \alpha_1 - \beta_1$, $\beta = \beta_1$ and $\alpha = \pi - \beta - \gamma = \pi - \alpha_1$.

$D(s,p) = -64p^2 + p\left(-32s^2 + 288s - 432\right) - 4s^4 + 8s^3$ shows that $D(s,p) < 0$ for $p \geq 0, s < 0$. Two triangle classes with the same s and the same p have necessarily the same $d = \frac{s^2}{4} + p$ and are equal since they correspond to the same roots $\sin^2\alpha$, $\sin^2\beta$, $\sin^2\gamma$. $\square$

Theorem 2. *The van IJzeren map is a bijection from $\mathcal{T}$ to*

$$\mathcal{T}^* = \left\{(s,p) \mid D(s,p) = (9-4s)^3 - (8p + 2s^2 - 18s + 27)^2 \geq 0, s \geq 0, p \geq 0\right\}$$

with inverse $V^{-1}\colon \mathcal{T}^* \to \mathcal{T}$ given by

$$(s,p) \mapsto \{\arcsin\sqrt{u_1}\,, \arcsin\sqrt{u_2}\,, \pi - \arcsin\sqrt{u_1} - \arcsin\sqrt{u_2}\,\}$$

where $u_1 \leq u_2 \leq u_3$ are the solutions of $u^3 - su^2 + (\frac{1}{4}s^2 + p)u - p = 0$.

For $(s,p) \in \mathcal{T}^*$ the discriminant $\frac{p}{16}D(s,p)$ of the above polynomial in u is 0 if and only if there are multiple roots among $\sin^2\alpha$, $\sin^2\beta$ and $\sin^2\gamma$, *i.e.*, if and only if $(s,p) = \Pi^*_\alpha$ for $p = 0$ or $(s,p) = I^*_\alpha$ for $D(s,p) = 0$ – in addition to $(s,p) = \mathcal{O}^*$ or $\Pi^*_{\pi/2}$ in both cases.

The curve $D(s,p) = 0$, $s \geq 0$, $p \geq 0$, is the *roof* Λ of T^* (Figure 4) and is constituted by $\mathcal{O}^*$, $\Pi^*_{\pi/2}$ and the images of the isosceles classes: the point $\{\alpha, \alpha, \pi - 2\alpha\}^*$, $0 \leq \alpha \leq \frac{\pi}{2}$, or

$$\Lambda(t) = \big(2t(3-2t), 4t^3(1-t)\big),\ 0 \leq t = \sin^2\alpha \leq 1, \tag{4}$$

travels along Λ from the origin $\mathcal{O}^*$ to $\Pi^*_{\pi/2} = (2,0)$.

The points $\Lambda(t)$ given by $t = 0, \frac{2-\sqrt{3}}{4}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{2+\sqrt{3}}{4}$ and 1 are $\mathcal{O}^* = (0,0)$, $I^*_{\pi/12} = (\frac{5-2\sqrt{3}}{4}, \frac{7-4\sqrt{3}}{64}) \approx (0.384, 0.001)$, $I^*_{\pi/6} = (\frac{5}{4}, \frac{3}{64})$, $I^*_{\pi/4} = (2, \frac{1}{4})$, the roof top $I^*_{\pi/3} = (\frac{9}{4}, \frac{27}{64}) = (2.25, 0.421875)$, $I^*_{5\pi/12} = (\frac{5+2\sqrt{3}}{4}, \frac{7+4\sqrt{3}}{64}) \approx (2.116, 0.218)$ and $\Pi^*_{\pi/2} = (2,0)$, respectively.

For $\frac{1}{2} \leq t \leq \frac{3}{4}$ the points $\Lambda(t)$ of the left roof section and $\Lambda(\frac{3}{2} - t)$ of the right roof section have the same abscissa.

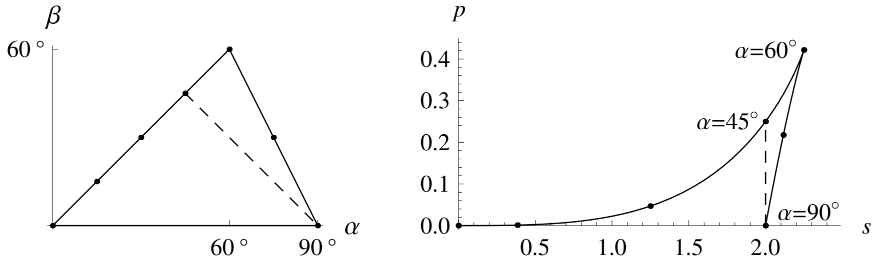

Figure 4. $\mathcal{T}$ as $\{(\alpha,\beta) \mid 0^\circ \leq \beta \leq \alpha \leq 90^\circ - \frac{\beta}{2}\}$ and roof of $\mathcal{T}^*$ with points corresponding to $\mathcal{O}$, to the isosceles classes I_α for $\alpha = 15^\circ, 30^\circ, 45^\circ, 60^\circ$ (the maximal value of β and of p), 75°, and to Π_{90°

$\mathcal{O}^*$ and the images of the classes of infinite triangles $\{\alpha, 0, \pi-\alpha\}$, $0 < \alpha \leq \frac{\pi}{2}$, form the *ground* Γ of $\mathcal{T}^*$ on the s–axis represented by the curve $(s,p) = (2t, 0), 0 \leq t = \sin^2\alpha \leq 1$.

The vertical segment in Figure 4 between $I^*_{\pi/4}$ and $\Pi^*_{\pi/2}$ corresponds to the curve

$$(s,p) = (2, t(1-t)),\ \tfrac{1}{2} \leq t = \sin^2\alpha \leq 1,$$

constituted by the images of the right–angled classes $\{\alpha, \frac{\pi}{2}-\alpha, \frac{\pi}{2}\}$, $\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{2}$. The images of the obtuse triangle classes are to the left of this segment, the images of the acute classes to the right.

3. Coordinates of the reflection triangle

Since the elementary symmetric polynomials $s = u_1+u_2+u_3$, $d = u_1u_2+u_2u_3+u_3u_1$, $p = u_1u_2u_3$ have by Lemma 1 the property $d = \frac{s^2}{4}+p$ if $u_1 = \sin^2\alpha$, $u_2 = \sin^2\beta$ and $u_3 = \sin^2\gamma$ for some $\{\alpha,\beta,\gamma\} \in \mathcal{T}$, every symmetric polynomial in u_1, u_2, u_3 can then be expressed with s and p only:

$$\begin{aligned}
\{\alpha,\beta,\gamma\} \in \mathcal{T} \Rightarrow &\sum_{\text{cyclic}} \sin^2\alpha\sin^2\beta = d = \tfrac{s^2}{4}+p,\\
&\sum_{\text{cyclic}} \sin^4\alpha = s^2-2d = \tfrac{s^2}{2}-2p,\\
&\sum_{\text{cyclic}} \sin^4\alpha\sin^4\beta = d^2-2sp = \left(\tfrac{s^2}{4}+p\right)^2-2sp,\\
&\sum_{\text{cyclic}} \sin^2\alpha\sin^2\beta\left(\sin^2\alpha+\sin^2\beta\right) = sd-3p = \tfrac{s^3}{4}+sp-3p.
\end{aligned} \tag{5}$$

Theorem 3. *If $r(\Delta)$ denotes the reflection triangle (class) of Δ, the map*

$$\rho\colon \mathcal{T}^* \to \mathcal{T}^*,\ (s,p) = \Delta^* \mapsto r(\Delta)^*$$

induced by r is given by $\rho(I^*_{\pi/6}) = (0,0)$ and by

$$\rho(s,p) = \big(\rho_1(s,p), \rho_2(s,p)\big) = \left(\tfrac{(s+16p)(4s-5)^2}{4s+1+64p(4s-7)}, \tfrac{p(4s-5)^6}{(4s+1+64p(4s-7))^2}\right) \text{ otherwise.} \tag{6}$$

Further,

$$D(\rho(s,p)) = D(s,p)\tfrac{(4s-5)^6(4s-1+64p(4s-9))^2}{(4s+1+64p(4s-7))^4} \tag{7}$$

if $\rho(s,p)$ is defined, i.e.*, for all $(s,p) \in \mathbf{R^2}$ not lying on the hyperbola $p = -\frac{4s+1}{64(4s-7)}$. This hyperbola is tangent to the roof at $I^*_{\pi/6}$ and is otherwise exterior to $\mathcal{T}^*$.*

Proof. Consider the proper triangle $\Delta = ABC$ with angles α, β, γ and opposite sides a, b, c and reflect Δ in all its sides to get the reflection triangle $\Delta_1 = A_1B_1C_1$. Let (s,p) and (S,P) be the van IJzeren coordinates of Δ and Δ_1, respectively. Suppose first that Δ_1 is proper and consider the triangle A_1B_1C with angle $\min(3\gamma, |2\pi - 3\gamma|)$ at C. The cosine law, the formula $\cos\gamma - \cos 3\gamma = 4\sin^2\gamma\cos\gamma$ and the sine law give

$$c_1^2 = c^2 + 2ab(\cos\gamma - \cos 3\gamma) = c^2(1 + 8\sin\alpha\sin\beta\cos\gamma) \text{ and thus by (2)}$$

$$R_1^2 \sin^2\gamma_1 = R^2\sin^2\gamma(1 + 4s - 8\sin^2\gamma), \text{ where } R, R_1 \text{ are the circumradii.} \tag{8}$$

The cyclic sum of (8) gives with (5)

$$R_1^2 S = R^2(s(1+4s) - 4(s^2 - 4p)) = R^2(s + 16p). \tag{9}$$

Multiplying $\sum_{\text{cyclic}} \sin^2\alpha_1 \sin^2\beta_1 = \frac{S^2}{4} + P$ by R_1^4 and using (8) for each angle of Δ_1, (5) and (9), one gets

$$\begin{aligned} R_1^4 P = &\ R^4 \sum_{\text{cyclic}} \sin^2\alpha\sin^2\beta(1+4s-8\sin^2\alpha)(1+4s-8\sin^2\beta) - R_1^4\tfrac{S^2}{4} \\ = &\ R^4 p(4s-5)^2. \end{aligned} \tag{10}$$

Note that (10) proves once again (see (1)) that all proper triangles with $s = \frac{5}{4}$ have a degenerate reflection triangle.

The product of the three formulas (8) gives together with (5)

$$\begin{aligned} R_1^6 P &= R^6 p(1+4s-8\sin^2\alpha)(1+4s-8\sin^2\beta)(1+4s-8\sin^2\gamma) \\ &= R^6 p\big((1+4s)^3 - 8(1+4s)^2 + 64(1+4s)(\tfrac{s^2}{4} + p) - 512p\big) \\ &= R^6 p\,(4s + 1 + 64p(4s-7))\,. \end{aligned} \tag{11}$$

Use now the relations

$$R_1^2 S \cdot R_1^4 P = S \cdot R_1^6 P \text{ and } \big(R_1^4 P\big)^3 = P \cdot \big(R_1^6 P\big)^2$$

between the left sides of (9)–(11) to combine their right sides in the same way, simplify the powers of R and get $(S,P) = \rho(s,p)$ when Δ and Δ_1 are proper triangles. Since $\rho(0,0) = (0,0)$, the formula is also correct when Δ is degenerate. Theorem 4 will prove the formula when Δ_1 is degenerate and Δ proper. A limit argument establishes the validity of the formula for the infinite case $\Pi_\alpha =$

$\lim_{\varepsilon\to 0+}\{\alpha-\varepsilon, \pi-\alpha-\varepsilon, 2\varepsilon\}$. Theorem 7 gives $r(\Pi_\alpha)$ explicitly and computes its coordinates directly. The proof of (7) follows from (6) by brute computation. □

We denote by ρ^m and r^m, $m \in \mathbf{Z}$, the mth iterate of ρ and r, respectively, and speak of descendants (child, grandchild, ...) or ancestors (parents, grandparents, ...) of a point (s,p) or of a triangle (class). By (7) $(S,P) \in \mathcal{T}^* \setminus \Lambda$ has no parents $(s,p) \in \mathbf{R^2}$ outside $\mathcal{T}^*$ since $D(S,P) > 0$ and $D(s,p) < 0$ are incompatible. Note also that by (7) a non-isosceles parent of I_α (or a parent of $\mathcal{O}^*$, $\Pi^*_{\pi/2}$ that is not on the roof) has coordinates (s,p) with $s = \frac{5}{4}$ (see Theorem 4) or $p = \frac{1-4s}{64(4s-9)}$ (see Theorems 8 and 11).

Several angles play a special role in our story. We denote them by ω indexed by the rounded angle measure in degrees:

$$\omega_{12} = \arcsin\sqrt{\tfrac{3-\sqrt{7}}{8}} \approx 12.148^\circ \qquad \omega_{58} = \arcsin\sqrt{\tfrac{29-6\sqrt{6}}{20}} \approx 57.7435^\circ$$
$$\omega_{21} = \arcsin\sqrt{\tfrac{1}{8}} \approx 20.705^\circ \qquad \omega_{62} = \arcsin\sqrt{\tfrac{1+6\sqrt{6}}{20}} \approx 62.364^\circ$$
$$\omega_{38} = \arcsin\sqrt{\tfrac{3}{8}} \approx 37.761^\circ \qquad \omega_{66} = \arcsin(\sqrt{2}-\tfrac{1}{2}) \approx 66.09^\circ$$
$$\omega_{49} = \arcsin\tfrac{3}{4} \approx 48.59^\circ \qquad \omega_{68} = \arcsin\tfrac{\sqrt{1+\sqrt{6}}}{2} \approx 68.2238^\circ$$
$$\omega_{50} = \arcsin\tfrac{\sqrt{1+\sqrt{2}}}{2} \approx 50.976^\circ \qquad \omega_{71} = \arcsin\sqrt{\tfrac{3+\sqrt{17}}{8}} \approx 70.666^\circ$$
$$\omega_{51} = \arcsin\tfrac{\sqrt{17}-1}{4} \approx 51.332^\circ \qquad \omega_{72} = \arcsin\tfrac{3}{\sqrt{10}} \approx 71.565^\circ$$
$$\omega_{52} = \arcsin\sqrt{\tfrac{5}{8}} = 90^\circ - \omega_{38} \approx 52.2388^\circ$$

4. Degenerate reflection triangles

We provide here some of the details behind (1).

Theorem 4. *The reflection triangle of a nondegenerate triangle Δ is degenerate if and only if $s(\Delta) = \frac{5}{4}$,* i.e.*, if and only if the point (α, β) formed by the two smallest angles of Δ lies on the oval $\sin^2\alpha + \sin^2\beta + \sin^2(\alpha+\beta) = \frac{5}{4}$ through $(\frac{\pi}{6}, \frac{\pi}{6})$ cutting the positive axes at ω_{52}* (Figure 3). *Triangle Δ is then obtuse with obtuse angle between $\frac{2\pi}{3}$ (for $\alpha = \beta = \frac{\pi}{6}$) and $\pi - \omega_{52}$* (infinite triangle).

Proof. Let first ABC be a proper triangle with opposite sides a, b, c, circumcenter O, circumradius R, nine–point center N, centroid G and medians m_a, m_b, m_c, and let X be a point (not necessarily coplanar with ABC). [10, p. 174] proves

$$XA^2 + XB^2 + XC^2 = GA^2 + GB^2 + GC^2 + 3XG^2. \tag{12}$$

By using $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2+b^2+c^2)$ (an immediate consequence of the median theorem [10, p. 68]), taking $X = O$ and using $ON = \frac{3}{2}OG$, (12) becomes

$$3R^2 = \tfrac{1}{3}(a^2+b^2+c^2) + \tfrac{4}{3}ON^2. \tag{13}$$

The homothety $\mathsf{h}(G, \frac{1}{4})$ with center G and ratio $\frac{1}{4}$ transforms $r(ABC)$ into the pedal triangle of N [5]. By the Wallace–Simson Theorem [10, p. 137] $r(ABC)$ is thus degenerate if and only if N lies on the circumcircle, *i.e.*, if and only if (13) becomes $a^2+b^2+c^2=5R^2$, *i.e.*, if and only if $s(ABC)=\frac{5}{4}$ by the sine law. Theorem 7 proves the result for infinite triangles. □

Here is an even shorter proof using an idea of [3, p. 78] (the proof there is flawed): when Δ is a proper triangle, the trilinear vertex matrix of $r(\Delta)$ is

$$\begin{bmatrix} -1 & 2\cos\gamma & 2\cos\beta \\ 2\cos\gamma & -1 & 2\cos\alpha \\ 2\cos\beta & 2\cos\alpha & -1 \end{bmatrix};$$

its determinant is 0 if and only if $r(\Delta)$ is degenerate; the determinant can be written as $4s(\Delta)-5$ since one gets $s(\Delta)=2+2\cos\alpha\cos\beta\cos\gamma$ by expanding $\cos\gamma=-\cos(\alpha+\beta)$.

Theorem 3 tells us that in $\mathbf{R}^2$ the parents $\rho^{-1}(\mathcal{O}^*)$ of $\mathcal{O}^*=(0,0)$ are the origin itself and all the points $(\frac{5}{4},p)$, $p\in\mathbf{R}$: only the origin and the points $(\frac{5}{4},p)$, $0\le p\le\frac{3}{64}$, lie in $\mathcal{T}^*$.

Consider a proper triangle Δ with coordinates (s,p) and its reflection triangle $\Delta_1=A_1B_1C_1$ with sides a_1, b_1, c_1 and coordinates (S,P). (8), (9) and (11) are then also true when Δ_1 is degenerate if one replaces their left side by c_1^2, $a_1^2+b_1^2+c_1^2$ and $a_1^2b_1^2c_1^2$, respectively: thus $a_1^2+b_1^2+c_1^2\neq 0$ and $\frac{a_1^2b_1^2c_1^2}{(a_1^2+b_1^2+c_1^2)^3}=\frac{p(64p(4s-7)+4s+1)}{(16p+s)^3}$. Suppose now that Δ_1 is degenerate, *i.e.*, $s=\frac{5}{4}$, with $c_1=a_1+b_1\neq 0$ and let $x=a_1/c_1\in[0,1]$: then $\frac{a_1^2b_1^2c_1^2}{(a_1^2+b_1^2+c_1^2)^3}=\frac{128p(3-64p)}{(64p+5)^3}=\frac{(x-x^2)^2}{8(x^2-x+1)^3}$ is given as a function of p or x by Figure 5 with maximum $\frac{1}{54}$ for $p=\frac{1}{64}$ and for $x=\frac{1}{2}$, *i.e.*, for a parent with angles $\left\{\frac{\pi}{4},\arcsin\sqrt{\frac{3-\sqrt{7}}{8}},\pi-\arcsin\sqrt{\frac{3+\sqrt{7}}{8}}\right\}=\{45°,\omega_{12},135°-\omega_{12}\}$, and with minimum 0 for $I_{\pi/6}$. The following theorem is proven.

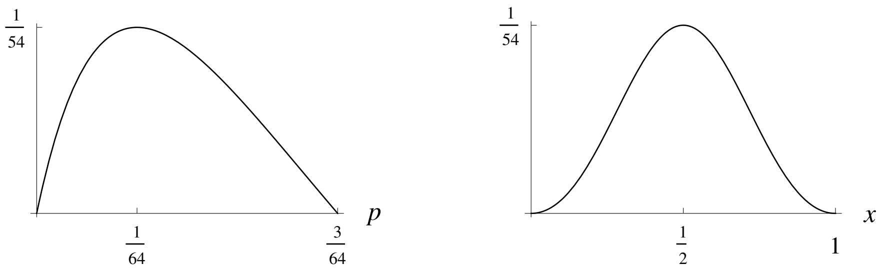

Figure 5. Graphs of $\frac{128p(3-64p)}{(64p+5)^3}$ and $\frac{(x-x^2)^2}{8(x^2-x+1)^3}$

Theorem 5. *A finite degenerate triangle Δ_1 with three different vertices is the reflection triangle of exactly 5 triangles. If the midpoint of the longest side is not a vertex, these 5 triangles are the degenerate triangle itself, a pair of non-similar non-isosceles triangles and their mirror images in the line of Δ_1. If the midpoint* *of the longest side is the third vertex, these* 5 *triangles are the degenerate triangle itself, a non-isosceles triangle with angles* $\{45°, \omega_{12}, 135° - \omega_{12}\}$*, its mirror image in the line of* Δ_1 *and their reflections in the midpoint of the longest side. The corresponding coordinates* (s,p) *of the nondegenerate parents are given by* $s = \frac{5}{4}$ *and by the two (possibly equal) solutions* $p \in \left]0, \frac{3}{64}\right[$ *of* $\frac{128p(3-64p)}{(64p+5)^3} = \frac{(x-x^2)^2}{8(x^2-x+1)^3}$*, where* x *is the ratio of the shortest side of* Δ_1 *to the longest side* (Figure 5).

A finite degenerate triangle with only two different vertices is the reflection triangle of exactly 2 *triangles: itself by convention and an isosceles triangle with equal angles* $\frac{\pi}{6}$*. A point is the reflection triangle of itself only.*

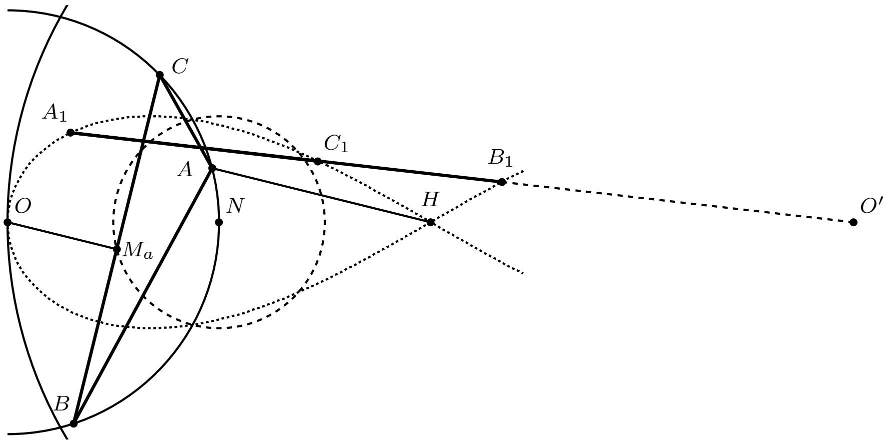

Figure 6. Construction of an inscribed triangle ABC with degenerate reflection triangle $A_1B_1C_1$. The dotted curve is the locus $\mathcal{L}$ of A_1 as function of A.

Here is a construction of all $\Delta \in \mathcal{T}$ with $s = \frac{5}{4}$ that is simpler than the corresponding construction of [5]. Take a point O and a circle $O(R)$ of radius R centered at O (Figure 6). Choose $N \in O(R)$ and reflect O in N to get the orthocenter H. We search for $A, B, C \in O(R)$ with $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Choose any $A \in O(R)$ with $HA \le 2R$, take M_a given by $\overrightarrow{OM_a} = \frac{1}{2}\overrightarrow{AH}$ and construct the chord $a = BC$ with midpoint M_a to get – if not degenerate – a triangle ABC with $s = \frac{5}{4}$. $N\left(\frac{R}{2}\right)$ is then the nine–point circle of ABC. In the four cases where ABC degenerates into a chord (see below), one gets an infinite triangle with angle ω_{52} at the double vertex of ABC by taking a triangle's semi–infinite side along ABC and a finite side on the tangent to $O(R)$. Whether ABC is degenerate or not, one has then also $\overrightarrow{OM_b} = \frac{1}{2}\overrightarrow{BH}$ and $\overrightarrow{OM_c} = \frac{1}{2}\overrightarrow{CH}$.

There is an even simpler determination of M_a: construct the centroid G given by $\overrightarrow{OG} = \frac{2}{3}\overrightarrow{ON}$ and get M_a as the intersection of AG and $N\left(\frac{R}{2}\right)$ on the other side of G.

Let $A_1B_1C_1$ be the degenerate reflection triangle. The line $A_1B_1C_1$ goes through the reflection O' of O in H [5, without proof]: we give here a demonstration by the author, D. Grinberg (personal communication). The Simson line of any point X

of the circumcircle bisects XH [7, p. 46], hence the Simson line of N goes in our case through the midpoint M_{NH} of NH; the homothety $\mathsf{h}(G,4)$ that transforms the pedal triangle of N into the reflection triangle sends thus M_{NH} to a point of the line $A_1B_1C_1$; but this point is on the line ON at distance $\frac{2}{3}R+4\left(\frac{3}{2}R-\frac{2}{3}R\right)=4R$ from O and is thus O'.

Let $\mathcal{L}$ be the locus of A_1 as function of A. The side midpoints of ABC lie on the nine–point circle $N\left(\frac{R}{2}\right)$ inside $O(R)$, and this arc is the locus of M_a as function of A. As A moves on the portion of $O(R)$ inside $H(2R)$, $\Pi_{\omega_{52}}$ is represented at the arc's extremities $E_\pm$ with $\angle NOE_\pm = \pm\arccos\frac{1}{4} \approx \pm 75.523°$ and at $L_\pm$ given by $O(R)\cap N\left(\frac{R}{2}\right)\cap\mathcal{L}$ with $\angle NOL_\pm = \pm 2\arcsin\frac{1}{4}\approx\pm 28.955°$. $I_{\pi/6}$ is represented at $\angle NOA = 0°, \pm 60°$. Any other $\Delta\in\mathcal{T}$ with $s=\frac{5}{4}$ is represented six times (once in each of the intervals delimited by the seven angles above) by a triply covered triangle (with each vertex in turn getting the label A) and its triply covered image under reflection in the line ON. The corresponding six degenerate reflection triangles $A_1B_1C_1$ occupy only two positions symmetrically to the line ON and each vertex in turn is A_1; the situation is similar for the infinite triangle and for the isosceles case: $\mathcal{L}$ contains thus also B_1 and C_1 (on the corresponding altitudes of ABC).

Place the isosceles triangle $\Delta = ABC$ of Figure 2 with equal 30°–angles and its degenerate reflection triangle $A_1B_1C_1$ into Figure 6, with B at N; let then A and B glide towards L_- (and C towards E_+) on the nine–point circle of Figure 6 in such a way that the reflection triangle $A_1B_1C_1$ remains degenerate: the angle α at A grows from 30° to ω_{52}, the coordinates (s,p) of Δ travel on the line $s=\frac{5}{4}$ from $\left(\frac{5}{4},\frac{3}{64}\right)=I^*_{30°}$ on the roof to $\left(\frac{5}{4},0\right)=\Pi^*_{\omega_{52}}$ on the ground and the ratio $x = A_1C_1 : B_1C_1$ runs from 0 to 1 in Figure 5.

The homothety $\mathsf{h}(G,-2)$ sends $N\left(\frac{R}{2}\right)$ to $O(R)$ and thus $L_\pm$ to $E_\mp$ (hence $\{G\}=E_+L_-\cap E_-L_+$). By considering a degenerate triangle ABC with vertices E_+,L_- or E_-,L_+ (infinite triangle's case), one sees that the antipode L'_- of L_- on $N\left(\frac{R}{2}\right)$, being at distance R from H, is the midpoint of HE_+: L_- lies on the circle $L'_-(R)$ with diameter HE_+. The tangents to $\mathcal{L}$ at H form a 60°–angle because they are the tangents to $O(R)$ corresponding to the isosceles ABC representing $I_{\pi/6}$.

In a cartesian coordinate system with origin O and $N=(R,0)$, the locus $\mathcal{L}$ of A_1 as function of $A=(R\cos\varphi, R\sin\varphi)$ is the curve

$$A_1=\left(\frac{2R(7-2\cos\varphi)(1-\cos\varphi)}{5-4\cos\varphi},\frac{2R\sin\varphi(2\cos\varphi-1)}{5-4\cos\varphi}\right),\ |\varphi|\le\arccos\frac{1}{4}. \tag{14}$$

(14) gives also B_1 and C_1 from the polar coordinates of B and C, respectively. The range of the polar angle of B and C is smaller than for A: when A goes from E_- to E_+, B and C start at L_+, go to $E_\pm$ in opposite directions and come back to L_-.

The end points of $\mathcal{L}$ are the midpoints of the segments $O'L_\pm$. Indeed, since A_1 is the upper end point U of $\mathcal{L}$ for the infinite triangle's case $A=E_-$, $B=C=B_1=C_1=L_+=\left(\frac{7}{8}R,\frac{\sqrt{15}}{8}R\right)$, one has $U=\left(\frac{39}{16}R,\frac{\sqrt{15}}{16}R\right)$ by (14). The line

UL_+ is tangent to $\mathcal{L}$ at L_+ since it would be the line of the degenerate reflection triangle in the infinite triangle's case.

Theorem 6. *Let Δ be an proper triangle with degenerate reflection triangle Δ_1. The following properties are equivalent.*
(1) *Δ_1 has two equal sides and three different vertices,* i.e., *Δ has angles $45°$ and ω_{12}.*
(2) *The middle vertex of Δ_1 is halfway between the corresponding vertex of Δ and the orthocenter of Δ,* i.e., *on the nine–point circle of Δ but not on its circumcircle.*

Proof. (2)⇒(1): (14) shows that the upper part of $\mathcal{L}$ cuts $N\left(\frac{R}{2}\right)$ at C_1 if and only if the polar angle of C is $\arccos \frac{7}{8}$ (infinite triangle's case) or $\arccos \frac{3}{4}$: in this second case, $C_1 = \left(\frac{11}{8}R, \frac{\sqrt{7}}{8}R\right)$ is the midpoint of HC. By computing then with (14) the intersections of the line $O'C_1$ and of $\mathcal{L}$, one gets the polar angles $\arccos \frac{5+\sqrt{7}}{8} \approx 17.114°$ for A (say) and $-\arccos \frac{5-\sqrt{7}}{8} \approx -72.886°$ for B, hence $\angle AOB = 90°$ and $\angle COA = \arccos \frac{1+\sqrt{7}}{4}$, thus $\angle ACB = 45°$ and $\angle ABC = \omega_{12}$ by the inscribed angle theorem. C_1 is the midpoint of A_1B_1 by Theorem 5.

(1)⇒(2): there is only one position on the upper part of $\mathcal{L}$ where both shorter sides of Δ_1 are equal. □

5. Infinite reflection triangles

Theorem 7. *The action of r on a class of infinite triangles is given by $r(\Pi_\alpha) = \Pi_{(2\alpha+\arctan(3\tan\alpha)) \bmod \pi}$* (Figure 8) *and $r(\Pi_\alpha)^* = \left(\frac{s(4s-5)^2}{4s+1}, 0\right)$ for $0 < \alpha \leq \frac{\pi}{2}$, where $s = s(\Pi_\alpha) = 2\sin^2\alpha$.*

Proof. The theorem is true for $\alpha = \frac{\pi}{2}$. Take an acute angle α, consider a triangle with an angle 2α between sides of length 1 and 2 and define δ as the acute or right angle formed by the bisector of 2α and the opposite side. Using the angle bisector theorem and setting $s = 2\sin^2\alpha$ one gets $\sin^2\delta = \frac{9s}{8s+2}$ and thus $\tan\delta = 3\tan\alpha$, *i.e.*, $\delta = \arctan(3\tan\alpha)$. A figure shows that the formula for $r(\Pi_\alpha)$ is exact. Developing $r(\Pi_\alpha)^* = \left(2\sin^2(2\alpha+\delta), 0\right)$ leads to the expression in s. □

Note that $r(\Pi_{\pi/6}) = \Pi_{\pi/3}$. When restricted to the s–axis, ρ is given by $\rho(s,0) = \left(\frac{s(4s-5)^2}{4s+1}, 0\right)$: the fixed points are $(0,0)$, $(\frac{3}{4}, 0) = \Pi^*_{\omega_{38}}$ and $(2,0)$, they lie on the ground Γ and are repelling in $\mathbf{R^2}$. Since an infinite triangle has an infinite reflection triangle, ρ maps Γ to Γ (Figures 7 and 8): $\rho|_\Gamma$ is a triple covering of Γ. Since no point of $\Gamma \setminus \{(0,0)\}$ has parents outside Γ by the formula for ρ and by (7), the backward and forward orbit under ρ of $(s,0)$, $s \in\ ]0,2]$, remains in Γ. $\rho^n|_\Gamma$ is a 3^n–fold covering of Γ with 3^n fixed points for every integer $n \geq 1$ (Figure 7). Since $3^n > 3 + 3^2 + \cdots + 3^{n-1}$ for $n > 1$, $\rho|_\Gamma$ has n–cycles for all $n \geq 1$, *i.e.*, cycles of *minimal* period n. The length of the longest monotonicity interval of the first coordinate of $\rho^n|_\Gamma$ tends to 0 for $n \to \infty$. Each periodic or infinite forward orbit has a countable backward orbit. The following theorem is proven.

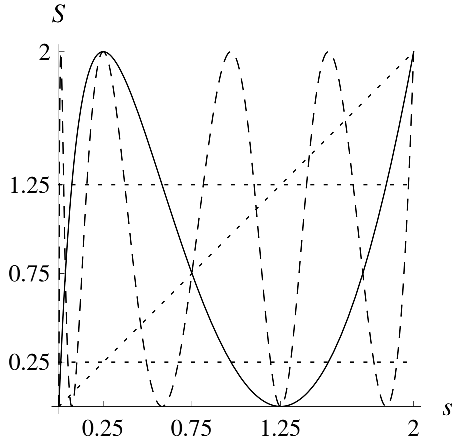

Figure 7. First coordinate of $\rho|_\Gamma$ (plain) and of $\rho^2|_\Gamma$ (dashed)

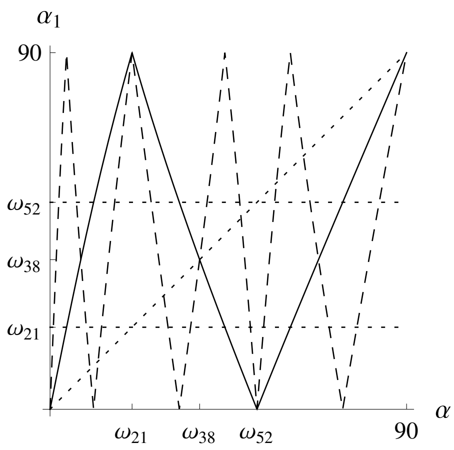

Figure 8. $\alpha_1 = \alpha_1(\alpha)$ given by $\Pi_{\alpha_1} = r(\Pi_\alpha)$ and its iterate (in $^\circ$)

Theorem 8. (1) *The two or three parents* (*in* $\mathbf{R^2}$) *of any* $\Pi^*_\alpha = (S, 0)$, $0 < S \le 2$, *all lie on* $\Gamma \setminus \{(0,0)\}$ *and their abscissae are the solutions of* $\frac{s(4s-5)^2}{4s+1} = S$. *The parents of* $\Pi^*_{\pi/2} = (2,0)$ *are thus itself and* $(\frac{1}{4}, 0) = \Pi^*_{\omega_{21}}$. $\Pi^*_{\omega_{52}} = (\frac{5}{4}, 0)$ *and* $(0,0)$ *are the only parents of* $(0,0)$ *on the s–axis and their abscissae are the solutions of* $\frac{s(4s-5)^2}{4s+1} = S = 0$.
(2) *The backward orbit of any* Π^*_α *under* ρ *lies in* Γ *and is countably dense in* Γ.
(3) $\rho|_\Gamma$ *has a nonzero finite number of* n*–periodic points for all integers* $n \ge 1$.
(4) *There are uncountably many disjoint infinite forward orbits of* $\rho|_\Gamma$.
(5) *Every nondegenerate infinite triangle has exactly* 3 *parents since* $\Pi_{\omega_{21}}$ *generates two inversely similar parents of a given rectangular infinite triangle.*

Figure 7 shows that $\rho|_\Gamma$ has three 2–cycles. Since the abscissa of $\rho^2(s,0) - (s,0)$ is

$$\frac{8s(s-2)(4s-3)(8s^2-12s+1)(256s^4-832s^3+832s^2-260s+13)}{(4s+1)^2(64s^3-160s^2+104s+1)},$$

the points $(\frac{3\pm\sqrt{7}}{4}, 0)$, which are $\Pi^*_{\omega_{12}}$ and $\Pi^*_{45^\circ+\omega_{12}}$, are exchanged by ρ, as are the points $\left(\frac{1}{16}\left(13-\sqrt{13}\pm\sqrt{78-2\sqrt{13}}\right), 0\right)$, *i.e.*, $\Pi^*_{10.08\ldots^\circ}$ and $\Pi^*_{48.24\ldots^\circ}$, and $\left(\frac{1}{16}\left(13+\sqrt{13}\pm\sqrt{78+2\sqrt{13}}\right), 0\right)$, *i.e.*, $\Pi^*_{28.68\ldots^\circ}$ and $\Pi^*_{63.96\ldots^\circ}$; these 2–cycles are repelling in $\mathbf{R^2}$. Notice that ω_{12} already appeared in Theorem 5. The infinite triangle and its grandchild are directly similar when corresponding to the first 2–cycle and inversely similar in the two other 2–cycles. $\rho|_\Gamma$ has eight 3–cycles, they are all repelling in $\mathbf{R^2}$. Four 3–cycles are given by the roots of $16777216s^{12} - 167772160s^{11} + 720371712s^{10} - 1735131136s^9 + 2569863168s^8 - 2413019136s^7 + 1429815296s^6 - 516909056s^5 + 106880256s^4 - 11406272s^3 +$

$543312s^2 - 8820s + 21$, approximately

0.00285317	0.0702027	1.22068
0.0254111	0.553455	1.33684
0.145175	1.7937	1.03778
0.336812	1.91456	1.56253

and the four other 3–cycles consist of the roots of $16777216s^{12} - 163577856s^{11} + 686817280s^{10} - 1625292800s^9 + 2381971456s^8 - 2236841984s^7 + 1345982464s^6 - 504474624s^5 + 110822912s^4 - 12847168s^3 + 670592s^2 - 12028s + 31$, approximately

0.00307391	0.0755414	1.28031
0.028553	0.61172	1.15683
0.172455	1.89595	1.47455
0.409917	1.75352	$0.887586.$

There are no other fixed points or 2– or 3–cycles on the s–axis if one allows $s \in \mathbf{C}$.

6. Fixed points and 2–cycles of ρ

Since

$$\rho(s,p) - (s,p) = \Big(\tfrac{4(-48ps+100p+4s^3-11s^2+6s)}{256ps-448p+4s+1}, \\ -\tfrac{8p(4s-7)(32p-8s^2+16s-9)(64ps-112p+16s^3-60s^2+76s-31)}{(256ps-448p+4s+1)^2}\Big), \quad (15)$$

the 7 fixed points of ρ in $\mathbf{C^2}$ are $\mathcal{O}^*$, $I^*_{\pi/3}$, $\Pi^*_{\pi/2}$, $\{\frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7}\}^* = (\frac{7}{4}, \frac{7}{64})$, $\Pi^*_{\omega_{38}} = (\frac{3}{4}, 0)$, $\big(\frac{6-\sqrt{5}}{4}, \frac{8\sqrt{5}-17}{64}\big) \approx \{0.297, 0.561, 2.284\}^* \approx \{17.027°, 32.132°, 130.84°\}^*$ in $\mathcal{T}^*$ and $\big(\frac{6+\sqrt{5}}{4}, \frac{-17-8\sqrt{5}}{64}\big) \in \mathbf{R^2} \setminus \mathcal{T}^*$. The eigenvalues of the Jacobian matrix of ρ at the fixed points show that $I^*_{\pi/3}$ is attracting in $\mathbf{R^2}$ and that all other fixed points are repelling. The critical points of ρ form the line $s = \frac{5}{4}$ and their image is the origin. A triangle Δ and its reflection triangle are directly similar when Δ is degenerate, equilateral, infinite rectangular or heptagonal, and they are inversely similar when Δ^* is $\Pi^*_{\omega_{38}}$ or $\big(\frac{6-\sqrt{5}}{4}, \frac{8\sqrt{5}-17}{64}\big)$. $\big(\frac{6-\sqrt{5}}{4}, \frac{8\sqrt{5}-17}{64}\big)$ seems to correspond to a new special triangle, whose angles are probably not rational multiples of π. Note that $s\big(\{\frac{\pi}{15}, \frac{\pi}{5}, \frac{11\pi}{15}\}\big)$ is also $\frac{6-\sqrt{5}}{4}$.

Due to the location of the fixed points and to the shape of $\mathcal{T}^*$, which is closed, every forward orbit with both rightward and upward direction right from $s = \frac{7}{4}$ is forced to converge to $I^*_{\pi/3}$: as we will show, this is the case when the class of the base triangle lies in a dense open subset of $\mathcal{T}$ containing among others the classes of acute and right-angled triangles as well as the obtuse isosceles classes that are not $I_{\pi/6}$ or one of its ancestors.

There are 24 2–cycles of ρ in $\mathbf{C^2}$: three have already been described and lie in Γ, seven lie in $\mathcal{T}^* \setminus \Gamma$; the others are extraneous with three in $\mathbf{R^2} \setminus \mathcal{T}^*$ and eleven outside $\mathbf{R^2}$. The seven 2–cycles in $\mathcal{T}^* \setminus \Gamma$ all correspond to 2–cycles of obtuse triangles in $\mathcal{T}$, whose acute angles are approximately

$$
\begin{array}{lcl}
\{8.0763°, 3.79275°\} & \text{and} & \{38.5099°, 17.99879°\} \\
\{31.70115°, 9.19698°\} & \text{and} & \{32.64671°, 21.218476°\} \\
\{38.47736°, 31.19757°\} & \text{and} & \{65.27712°, 13.75689°\} \\
\{8.92974°, 4.0548°\} & \text{and} & \{42.23276°, 19.04471°\} \\
\{28.3017°, 21.20007°\} & \text{and} & \{53.85134°, 16.98919°\} \\
\{28.56877°, 8.60948°\} & \text{and} & \{41.35919°, 23.72889°\} \\
\{28.43994°, 23.62517°\} & \text{and} & \{60.10737°, 12.60168°\}
\end{array}
$$

A triangle is directly similar to its grandchild in the first and in the last two 2–cycles, and inversely similar in the other ones. All these seven 2–cycles are repelling since all eigenvalues of the product $D\rho(s_1, p_1) \cdot D\rho(s_2, p_2)$ of the Jacobian matrices have, for each cycle, a modulus > 1. These 2–cycles are found by factoring the resultants of the two polynomial equations $\rho^2(s, p) = (s, p)$. The first 2–cycle above is given by the real roots s of $65536s^8 - 557056s^7 + 1957888s^6 - 3655680s^5 + 3872768s^4 - 2305408s^3 + 724768s^2 - 108760s + 4631$. The two following 2–cycles and a 2–cycle of $\mathbf{R^2} \setminus \mathcal{T}^*$ are given by the real roots s of $1048576s^{10} - 13107200s^9 + 70713344s^8 - 215482368s^7 + 406921216s^6 - 490459136s^5 + 373159424s^4 - 169643008s^3 + 40513488s^2 - 3790120s + 124099$. For all these four cycles,

$$
\begin{aligned}
p = {} & (1/3373687912782962463932730572 80) \cdot \\
& (569737838757513187116449932928 6144s^{21} - 1548894864401601710504772501462 05696s^{20} \\
& + 196981555615894467829077018253302 1696s^{19} - 155664456710682800898723927916554 48576s^{18} \\
& + 856314627144876256787835950004489 42080s^{17} - 348112463554334373128224482745250 742272s^{16} \\
& + 108350734588874886978138748463167 3077760s^{15} - 263951709209923803704038658735747 9960576s^{14} \\
& + 510111041140536292090774321305741 5839744s^{13} - 787959868256849082489150009826496 3743744s^{12} \\
& + 975501092015866666509543355929071 7143040s^{11} - 966512339039690096528929885529149 8004480s^{10} \\
& + 762172376510086419788583062308698 4560640s^9 - 473693261600146140405367041940343 7375488s^8 \\
& + 228611765030602679588457172054289 0491904s^7 - 838913081019577908008862079371766 857728s^6 \\
& + 227315320515680762946527159936618 376192s^5 - 436538007212937413379450471669442 93120s^4 \\
& + 560270257133815609539380747902469 9136s^3 - 441294571999478960624696851928272 768s^2 \\
& + 190053879695790975456428651547484 04s - 340848826010088138830599778323827) .
\end{aligned}
$$

The two following 2–cycles and a 2–cycle of $\mathbf{R^2} \setminus \mathcal{T}^*$ are given by the real roots s of $1048576s^{10} - 12582912s^9 + 65470464s^8 - 193789952s^7 + 359325696s^6 - 432427008s^5 + 337883648s^4 - 166321920s^3 + 48099088s^2 - 7029296s + 326343$. The last two 2–cycles and the last 2–cycle of $\mathbf{R^2} \setminus \mathcal{T}^*$ are given by the real roots of $1048576s^{10} - 12058624s^9 + 59965440s^8 - 168624128s^7 + 293994496s^6 - 327127040s^5 + 229654528s^4 - 96299264s^3 + 21257456s^2 - 1867864s + 56317$. For all these six cycles,

$$
\begin{aligned}
p = {} & (1/4567428188341362809789303424452351253020672) \cdot \\
& (-3669893123824564952723336254787869325934985 2160s^{23} \\
& + 98481066687047112001267228048588288585922877 8496s^{22} \\
& - 12470437758739421776652337771814086850631568 457728s^{21} \\
& + 99105170498836558716042634538353704493085448 208384s^{20} \\
& - 55455668973319135530858343265214982843132036 7235072s^{19} \\
& + 23233400008287614849438488925482510754779130 95634944s^{18}
\end{aligned}
$$

$- 7564960112634217226649274510083987727875628771311616s^{17}$
$+ 1961255676155016260674915953008358404990950123451187 2s^{16}$
$- 4114028898746633377800580148689773100591690869322547 2s^{15}$
$+ 7055838541380316195894023654936889163702868974449459 2s^{14}$
$- 9956069919422026060931952711421281270178811329118208 0s^{13}$
$+ 11589935616857067400676806343729575114476765830551961 6s^{12}$
$- 11126423723841509264289535056918622777839126723113779 2s^{11}$
$+ 8777917615501788305983785087839821092526977911986585 6s^{10}$
$- 5652355416359476268335433804960642305752577698273689 6s^{9}$
$+ 2939392159275296696302864316150431030580115482804224 0s^{8}$
$- 1215728600680476212100449827557050577608253440945356 8s^{7}$
$+ 3914592300388673052527540455248353181688884203261952s^{6}$
$- 9525558994224064096373092395326080778824959817922 56s^{5}$
$+ 1679907831223641096945404158728443989793641164654 08s^{4}$
$- 2022738340710644889253022923501410415636491246163 2s^{3}$
$+ 1526394055420066271305468814522007678645577112528s^{2}$
$- 638617252921501550082810300507825006473831815 32s$
$+ 112297167156651628900670743147837806149244258 7\big)\,.$

When p is replaced by one of the given polynomials, the corresponding polynomials for s can be indeed factored out in both coordinates of $\rho^2(s,p) - (s,p)$. Two of the 2–cycles of ρ outside $\mathbf{R^2}$ are the cycle

$$(s_\pm, p_\mp) = \left(\tfrac{5+i\pm\sqrt{-1-8i}}{4}, \tfrac{-19-22i\mp\sqrt{-56+202i}}{64}\right)$$

and its complex conjugate cycle; the remaining nine such 2–cycles are given by the non-real roots of the above polynomials in s with the corresponding above formulas for p.

In Section 10 we will prove that there are cycles of any finite length in $\mathcal{T}^* \setminus \Gamma$.

7. Isosceles triangles

Since the reflection triangle of an isosceles triangle is isosceles, ρ maps the roof Λ of Figure 4 to itself. Plug the parametric representation (4) of Λ into formula (6) to obtain $\rho(\Lambda(t))$. An investigation of this function (Figure 9) and its derivative proves that, as I^*_α travels on Λ from the origin to $(2,0)$, $\rho(I^*_\alpha)$ moves continuously as follows: start at the origin, left roof section up for $0 < \alpha \le \frac{\pi}{12}$, right roof section down for $\frac{\pi}{12} \le \alpha \le \frac{\pi}{6}$, right roof section up for $\frac{\pi}{6} \le \alpha \le \frac{\pi}{3}$, a very short down and up round trip on the left roof section near the top for $\frac{\pi}{3} \le \alpha \le \frac{5}{12}\pi$ – with turning (deepest) point

$$I^*_{\omega_{58}} = \left(\tfrac{168\sqrt{6}-187}{100}, \tfrac{3\left(135664\sqrt{6}-326751\right)}{40000}\right) \approx (2.245, 0.417)$$

for $\alpha = \omega_{68}$ – and final descent of the right roof section for $\frac{5}{12}\pi \le \alpha < \frac{\pi}{2}$ with arrival at the bottom $\Pi^*_{\pi/2}$. Not to forget: $\rho(I^*_{\pi/6})$ has been instantly catapulted from $(2,0)$ to the origin!

It is now easy to count the isosceles parents of the isosceles class I_α, $0 < \alpha < \frac{\pi}{2}$ (Figure 10): one if $0 < \alpha < \omega_{58}$, two if $\alpha = \omega_{58}$ and three otherwise. The three isosceles parents of $I_{\pi/3}$, for example, are $I_{\pi/3}$, $I_{\pi/12}$ and $I_{5\pi/12}$ (Figure 2).

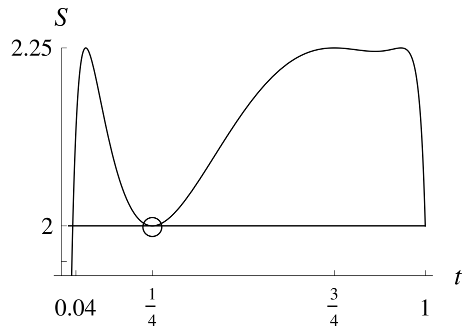

Figure 9. Abscissa $S(t)$ of $\rho(\Lambda(t))$ as a function of $t = \sin^2\alpha$: $S(0) = S(\frac{1}{4}) = 0$, $S(0.04) > 2$ and $S'(t) > 10$ on $[0, 0.04]$. The ordinate of $\rho(\Lambda(t))$ increases and decreases with $S(t)$.

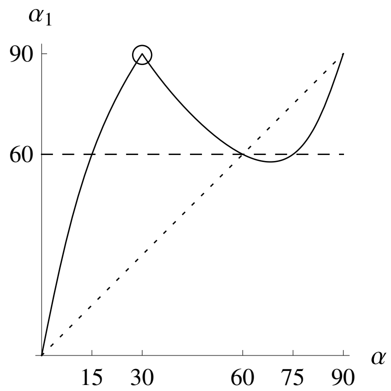

Figure 10. $\alpha_1 = \alpha_1(\alpha)$ given by $I_{\alpha_1} = r(I_\alpha)$ (in °)

If the abscissa of I^*_α is $> \frac{7}{4}$, *i.e.*, if $\alpha > \arcsin\frac{\sqrt{3-\sqrt{2}}}{2} \approx 39.024°$, and if α is different from $\frac{\pi}{3}$, $\rho(I^*_\alpha)$ lies on the roof strictly right from and above I^*_α – as an investigation of $\rho(\Lambda(t)) - \Lambda(t)$ shows (Figure 11). The forward orbit of I^*_α converges then to a fixed point that *must* be the roof top. But an I^*_α with smaller abscissa > 0 will also be stretched over $s = \frac{7}{4}$ by some iterate ρ^n of ρ (Figure 9): the orbit will thus also converge to the top unless $\rho^n(I^*_\alpha)$ transits through $\Pi^*_{\pi/2}$ with immediate transfer to the origin. The latter configuration is only possible if I^*_α belongs to the backward orbit of $I^*_{\pi/6}$: when limited to Λ, this orbit has no bifurcations and is thus an infinite sequence $I^*_{\pi/6}, I^*_{\alpha_{-1}} \approx I^*_{6.33°}, I^*_{\alpha_{-2}} \approx I^*_{1.269°}, \dots$ with $\frac{\pi}{6} > \alpha_{-1} > \alpha_{-2} > \cdots$ tending to 0. The following theorem is proven.

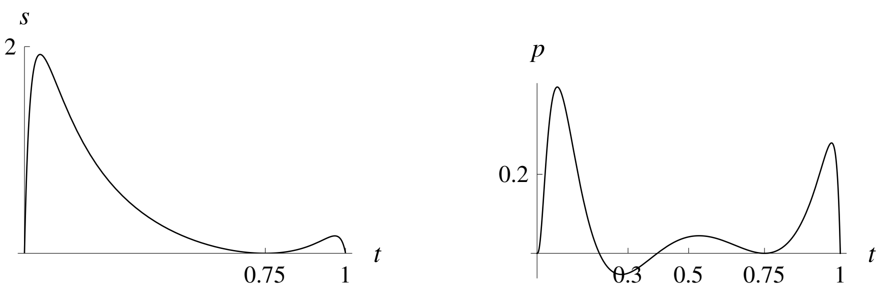

Figure 11. Abscissa s and ordinate p of $\rho(\Lambda(t)) - \Lambda(t)$ as functions of t with non-negative ordinate for $\frac{3-\sqrt{2}}{4} \leq t \leq 1$

Theorem 9. *The iterated reflection class of an isosceles base triangle class I_α converges to an equilateral limit unless I_α belongs to the backward orbit of $I_{\pi/6}$ and converges thus to a degenerate limit in a finite number of steps,* i.e., *unless $I_\alpha = I_{\pi/6}, I_{6.33\ldots^\circ}, I_{1.269\ldots^\circ}, \ldots,$ where $I_{\pi/6} = r(I_{6.33\ldots^\circ}), \ldots$*

See [4] for another proof, which iterates the formula

$$\cos^2 \alpha_1 = \frac{\cos^2 \alpha (4\cos^2 \alpha - 3)^2}{1 + 16\cos^2 \alpha - 16\cos^4 \alpha}$$

for a nondegenerate $r(I_\alpha) = I_{\alpha_1}$ and shows that $\lim_{n\to\infty} \cos \alpha_n = \frac{1}{2}$ unless some α_n is $\frac{\pi}{6}$.

8. Parents

ρ maps the point $(\frac{7}{4}, p)$ of the vertical line $s = \frac{7}{4}$ horizontally to the point $(8p + \frac{7}{8}, p)$ of the oblique line $s = 8p + \frac{7}{8}$.

ρ maps the vertical segment $s = 1+\frac{\sqrt{17}}{4} \approx 2.031$, $\frac{-105+28\sqrt{17}-16\sqrt{95-23\sqrt{17}}}{64} \leq p \leq \frac{-105+28\sqrt{17}+16\sqrt{95-23\sqrt{17}}}{64}$, delimited by the roof onto the vertical segment $s = \frac{5+3\sqrt{17}}{8} \approx 2.171$, $\frac{19+5\sqrt{17}}{128} \leq p \leq \frac{181\sqrt{17}-701}{128}$, delimited by the roof between $I^*_{\omega_{71}}$ and $I^*_{\omega_{51}}$. As p grows on the first segment,

$$\rho\big(1 + \tfrac{\sqrt{17}}{4}, p\big) = \left(\tfrac{5+3\sqrt{17}}{8}, P = \frac{(\sqrt{17}-1)^6 p}{\big(64(\sqrt{17}-3)p+\sqrt{17}+5\big)^2}\right) \tag{16}$$

travels on the second segment from the bottom up and back, reaching the left roof section for $p = \frac{4+\sqrt{17}}{64}$ (Figure 12). This gives two acute isosceles parents of $I_{\omega_{71}}$ and one acute non-isosceles parent of $I_{\omega_{51}}$.

We define the *van IJzeren rational function*

$$v(s) = \frac{(4s-5)^2\,\big((4s-5)^2 - 4S(4s-7)\big)\,\big(s(4s-5)^2 - S(4s+1)\big)}{-16\,(16s^2 - 32s - 1)^2} \tag{17}$$

with parameter S, double zero at $s = \frac{5}{4}$ and double poles at $s = 1 + \frac{\sqrt{17}}{4} \approx 2.031$ $\big($if $S \neq \frac{5+3\sqrt{17}}{8}\big)$ and at $s = 1 - \frac{\sqrt{17}}{4} \approx -0.031$ $\big($if $S \neq \frac{5-3\sqrt{17}}{8}\big)$. For

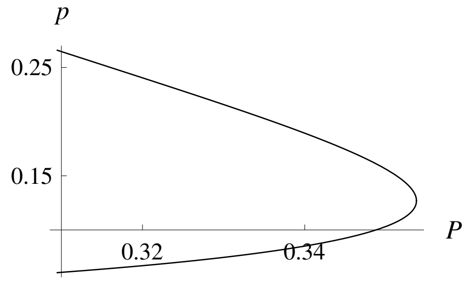

Figure 12. p–values as function of P in (16) and (19)

$S = \frac{5+3\sqrt{17}}{8}$, $v\big(1+\frac{\sqrt{17}}{4}\big) = \frac{1651-251\sqrt{17}}{2176} \approx 0.283$ by continuous extension; the situation is analogous for $S = \frac{5-3\sqrt{17}}{8}$. $v(s)$ is obtained from (6) by solving $S = \rho_1(s,p)$ for p and replacing then p in $\rho_2(s,p)\,(=P)$.

Theorem 10 (Parents)**.** *The parents $\rho^{-1}(\Delta_1^*)$ (in $\mathbf{R^2}$) of any $\Delta_1^* = (S,P) \in \mathcal{T}^* \setminus \{(0,0)\}$ are the points $(s,p) \in \mathbf{R^2}$ with*

$$s \in \left]0, \tfrac{9}{4}\right] \setminus \{\tfrac{5}{4}\},\ s \neq \tfrac{9}{4} \text{ if } (S,P) = (2,0),\ v(s) = P,$$
$$p = \frac{s(4s-5)^2 - S(4s+1)}{-16\big((4s-5)^2 - 4S(4s-7)\big)} \tag{18}$$

or

$$S = \frac{5+3\sqrt{17}}{8},\ s = 1 + \frac{\sqrt{17}}{4},\ \frac{p(4s-5)^6}{(4s+1+64p(4s-7))^2} = P, \tag{19}$$

i.e.,

$$p = \frac{8(\sqrt{17}+1)P + 65\sqrt{17} - 297 \pm \sqrt{128\big(101-29\sqrt{17}\big)P - 38610\sqrt{17} + 160034}}{512\big(3\sqrt{17}-13\big)P}$$

with two values for $P < \frac{181\sqrt{17}-701}{128}$ and one for $P = \frac{181\sqrt{17}-701}{128}$ (Figure 12).

The denominators are never zero. All between three and seven parents of $(S,P) \in \mathcal{T}^* \setminus \Gamma$ lie in $\mathcal{T}^* \setminus \Gamma$ except the rightmost parent $\big(\frac{5}{4} + \sin\alpha, \frac{1+\sin\alpha}{64(1-\sin\alpha)}\big)$ of I_α^* for $\omega_{66} < \alpha < \frac{\pi}{2}$.

Note that the parents of $(S,0) \in \Gamma \setminus \{(0,0)\}$ have already been described – in a simpler way – in Theorem 8.

The children $(S,P) = \rho(s_0,p)$ of the points $(s_0,p) \in \mathcal{T}^*$ with constant abscissa $s_0 \in \left]0,\frac{9}{4}\right] \setminus \left\{\frac{5}{4}, \frac{7}{4}, 1+\frac{\sqrt{17}}{4}\right\}$ constitute a parabola arc $P = v(s_0)$ with end points on $\Gamma \cup \Lambda$. If $s_0 > \frac{1}{4}$, there is one point $(s_0,p_0) \in \mathcal{T}^* \setminus \Lambda$ whose child is on the roof: the parabola arc is then tangent to Λ at $\rho(s_0,p_0)$ (see curve Φ in Figure 34). If $s_0 = \frac{1}{4}$, the parabola arc is tangent to Λ at $(2,0)$.

Choose any $S \in \left]0,\frac{9}{4}\right]$ as $S = 2t(3-2t)$, $0 < t \leq \frac{3}{4}$, and draw the curve $y = v(s)$; choose then any $P \in [P_{\min}, P_{\max}] = \big[\max\big(0, 4(\frac{3}{2}-t)^3(t-\frac{1}{2})\big), 4t^3(1-t)\big]$: by Theorem 10 the parents (s,p) of (S,P) for which $s \neq 1+\frac{\sqrt{17}}{4}$ have the same abscissae as the points with ordinate $y = P$ on the curve $y = v(s),\ s \in \left]0, \frac{9}{4}\right] \setminus \{\frac{5}{4}\}$ with $s \neq \frac{9}{4}$ if $(S, P) = (2, 0)$ – and each such abscissa corresponds to only one parent!

The (not included) start value $t = 0$, the transition values $t = \frac{1}{2},\ \frac{9-\sqrt{17}}{8} \approx 0.61,\ \sqrt{2} - \frac{3}{4} \approx 0.664,\ \frac{29-6\sqrt{6}}{20} \approx 0.715$ and the end value $t = \frac{3}{4}$ delimit open subintervals where the curve $y = v(s)$ has constant characteristic features. These t–values correspond to $S = 0,\ 2,\ \frac{5+3\sqrt{17}}{8} \approx 2.171,\ 12\sqrt{2} - \frac{59}{4} \approx 2.221,\ \frac{168\sqrt{6}-187}{100} \approx 2.245$ and $\frac{9}{4}$. Each of the figures 14–26 has to be read as follows for the corresponding $S \in \left]0, \frac{9}{4}\right]$: the abscissae s of the curve points at the altitude $P > 0$, $P \in [P_{\min}, P_{\max}]$, tell whether the corresponding parents (s, p) of $(S, P) = \Delta_1^* \in \mathcal{T} \setminus \Gamma$ are the coordinates of obtuse, right–angled or acute parents Δ of Δ_1 (except when $(s, p) \notin \mathcal{T}$); filled circles on the boundary $y = P_{\min},\ P_{\max}$ mark the abscissa of an isosceles parent I_α, an empty square indicates a parent (s, p) outside $\mathcal{T}^*$ or the exceptional cases for $S = \frac{5+3\sqrt{17}}{8}$, and the dashed line $s = 1 + \frac{\sqrt{17}}{4}$ goes through the pole. In Figure 14–17 – where $S \in \left]0, 2\right]$ – the parents $(s, 0)$ of $(S, 0)$ are given by $v(s) = 0,\ s \in \left]0, 2\right] \setminus \{\frac{5}{4}\}$: empty circles mark the other zeros. For $S \in \left]0, 2\right]$ and $P \to 0$, the parents (s, p) of (S, P) with $s \to \frac{5}{4}$ tend to $I^*_{\pi/6}$ since $p \to \frac{3}{64}$ by (18).

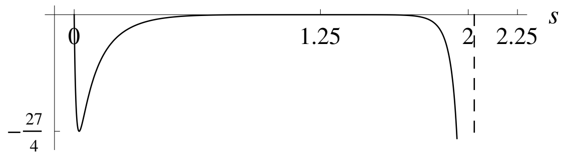

Figure 13. $y = v(s)$ for $S = 0$ with simple root at $s = 0$ and sextuple root at $s = \frac{5}{4}$, which are the abscissae of the parents of $(0, 0)$ in $\mathbf{R^2}$

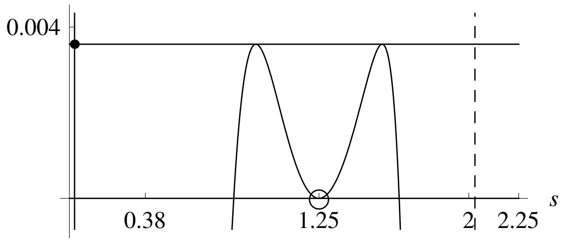

Figure 14. $S = 0.56$, top for $I^*_{\arcsin\sqrt{0.1}} \approx I^*_{18.435^\circ}$, bottom for $\Pi^*_{\arcsin\sqrt{0.28}}$

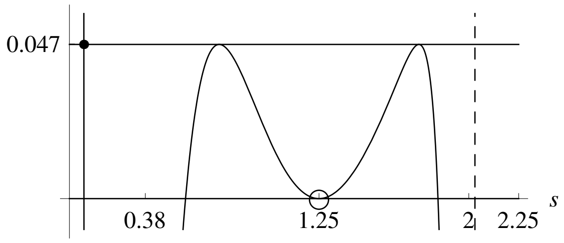

Figure 15. $S = \frac{5}{4}$, top for $I^*_{30^\circ}$, bottom for $\Pi^*_{\omega_{52}}$

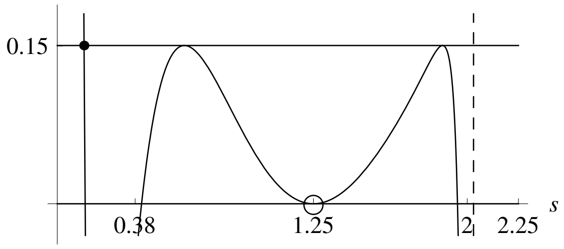

Figure 16. $S = \frac{7}{4}$, top for $I^*_{\arcsin \frac{\sqrt{3-\sqrt{2}}}{2}} \approx I^*_{23.356^\circ}$, bottom for $\Pi^*_{90^\circ - \omega_{21}}$

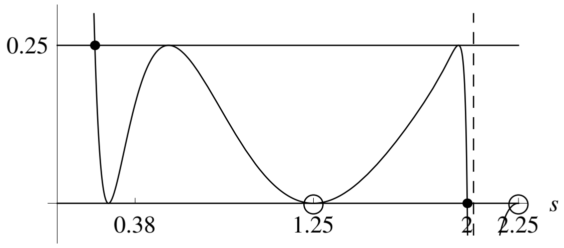

Figure 17. $S = 2$, transition case of the right–angled triangles, top for $I^*_{\pi/4}$, bottom for $\Pi^*_{\pi/2}$. A raising bump culminates at $(\frac{9}{4}, 0)$. The right–angled $\Delta_1 = \{\frac{\pi}{2}, \alpha, \frac{\pi}{2} - \alpha\}$ corresponds to $P = \frac{1}{4}\sin^2 2\alpha$.

Proof of Theorem 10 and of Figures 13–26. Theorem 10 is already proven except for the number of parents of $(S, P) \in \mathcal{T}^* \setminus \Gamma$, their location and the aspect of the curve $y = v(s)$ given by (17). The derivative of $v(s)$ can be factored as $v'(s) =$

$$\frac{(4s-5)\big(192s^3-528s^2+s(128S+100)+136S+125\big)\big(256s^4-1280s^3+2016s^2-1040s+64S+25\big)}{-16(16s^2-32s-1)^3} \tag{20}$$

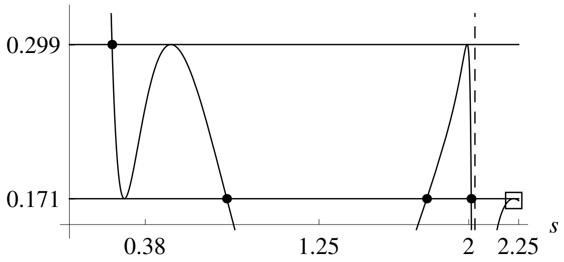

Figure 18. $S = 2.09$, top for $I^*_{\arcsin\sqrt{0.55}} \approx I^*_{47.87^\circ}$, bottom for $I^*_{\arcsin\sqrt{0.95}} \approx I^*_{77.079^\circ}$, $\square$ parent outside $\mathcal{T}^*$

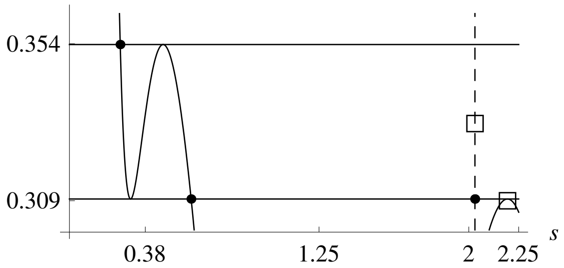

Figure 19. $S = \frac{5+3\sqrt{17}}{8} \approx 2.171$, transition case, top for $I^*_{\omega_{51}}$, bottom for $I^*_{\omega_{71}}$. $\square$ There are *two* parents with $s = 1 + \frac{\sqrt{17}}{4}$ (pole) if $P \in [P_{\min}, P_{\max}[$ and one for $P = P_{\max}$; both such parents of $I_{\omega_{71}}$ are isosceles. $\square$ There is a parent outside $\mathcal{T}^*$.

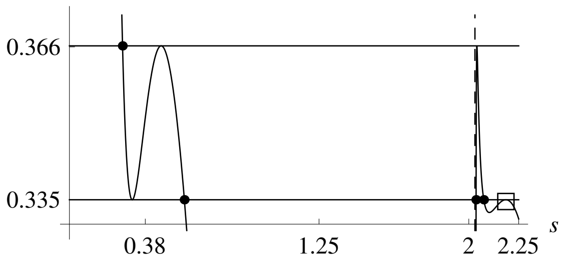

Figure 20. $S = 2.1875$, top for $I^*_{\omega_{52}}$, bottom for $I^*_{90^\circ - \omega_{21}}$, $\square$ parent outside $\mathcal{T}^*$

with 3rd degree factor $q_3(s)$ and 4th degree factor $q_4(s)$ (Figure 27).

For $S = 2t(3 - 2t)$, $t \in \mathbf{R}$, which is invariant under $t \mapsto \frac{3}{2} - t$, one has

$$q_4(s) = (16s^2 - 40s - 16t + 25)(16s^2 - 40s - 16(\tfrac{3}{2} - t) + 25) \tag{21}$$

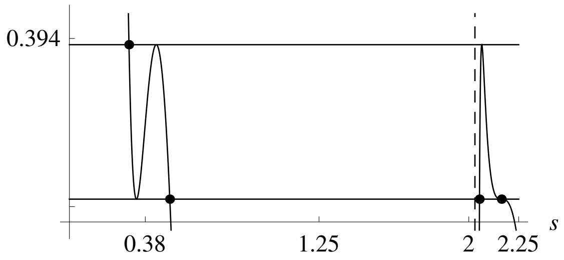

Figure 21. $S = 12\sqrt{2} - \frac{59}{4} \approx 2.221$, transition case, top for $I^*_{\arcsin\sqrt{\sqrt{2}-3/4}} \approx I^*_{54.587^\circ}$, bottom for $I_{\omega_{66}}$. The rightmost root of $v(s) = P_{\min}$ is triple.

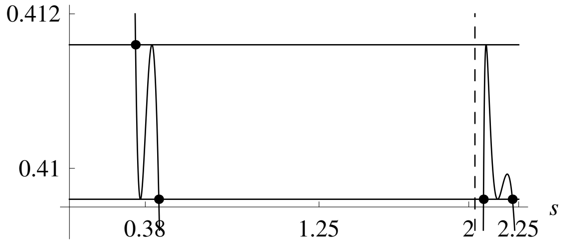

Figure 22. $S = 2.24$, top for $I^*_{\arcsin\sqrt{0.7}} \approx I^*_{56.789^\circ}$, bottom for $I^*_{\arcsin\sqrt{0.8}} \approx I^*_{63.435^\circ}$

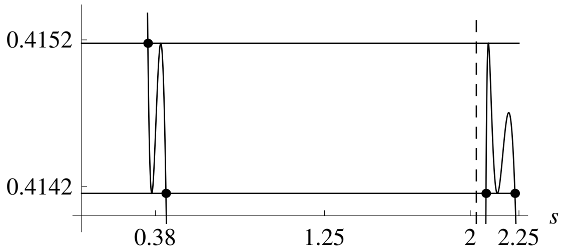

Figure 23. $S = 2.2436$, top for $I^*_{\arcsin\sqrt{0.71}} \approx I^*_{57.417^\circ}$, bottom for $I^*_{\arcsin\sqrt{0.79}} \approx I^*_{62.725^\circ}$

and $v(s) - 4t^3(1-t) =$

$$\frac{\left(16s^2-40s-16t+25\right)^2\left(16s^3+s^2\left(64t^2-96t-40\right)+s\left(-96t^2+176t+25\right)-44t^2-6t\right)}{-16(16s^2-32s-1)^2} \tag{22}$$

with numerator's squared 2nd degree factor $(Q_2(s))^2$ and 3rd degree factor

$$Q_3(s) = 16s^3 + s^2\left(64t^2 - 96t - 40\right) + s\left(-96t^2 + 176t + 25\right) - 44t^2 - 6t. \tag{23}$$

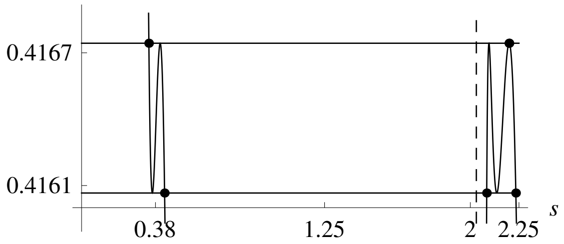

Figure 24. $S = \frac{168\sqrt{6}-187}{100} \approx 2.245$, transition case, top for $I^*_{\omega_{58}}$, bottom for $I^*_{\omega_{62}}$. (S, P) has 7 parents for all $P \in \left]P_{\min}, P_{\max}\right[$.

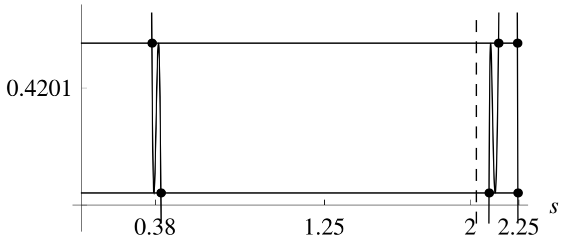

Figure 25. $S = 2.2484$, top for $I^*_{\arcsin\sqrt{0.73}} \approx I^*_{58.694^\circ}$, bottom for $I^*_{\arcsin\sqrt{0.77}} \approx I^*_{61.342^\circ}$

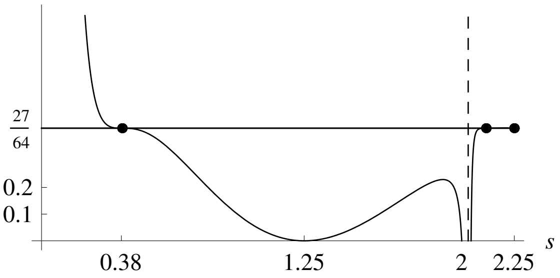

Figure 26. $t = \frac{3}{4}$, $S = \frac{9}{4}$, end case for $I^*_{60^\circ}$. $v(s) - \frac{27}{64}$ has triple roots at $s = \frac{5 \pm 2\sqrt{3}}{4}$ and a simple root at $s = \frac{9}{4}$.

Since for $t \geq 0$

$$Q_2(s) = 16\big(s - \tfrac{5}{4} - \sqrt{t}\,\big)\big(s - \tfrac{5}{4} + \sqrt{t}\,\big), \tag{24}$$

one can factor (21) further for $S = 2t(3 - 2t)$, $t \in \big[0, \frac{3}{2}\big]$:

$$q_4(s) = 256\big(s - \tfrac{5}{4} - \sqrt{t}\,\big)\big(s - \tfrac{5}{4} + \sqrt{t}\,\big)\Big(s - \tfrac{5}{4} - \sqrt{\tfrac{3}{2} - t}\,\Big)\Big(s - \tfrac{5}{4} + \sqrt{\tfrac{3}{2} - t}\,\Big). \tag{25}$$

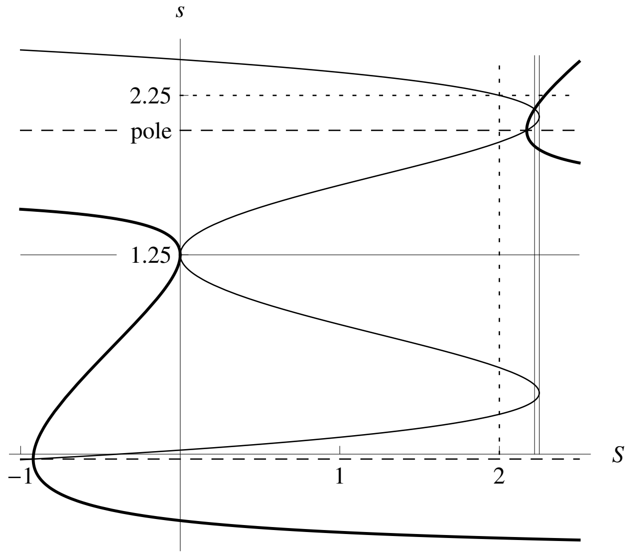

Figure 27. Poles and real zeros of $v'(s)$ as a function of S with constant zero $\frac{5}{4}$, thick curve for the zeros of $q_3(s)$, plain curve for the zeros of $q_4(s)$ and vertical lines at $S = 12\sqrt{2} - \frac{59}{4}$ and $S = \frac{9}{4}$

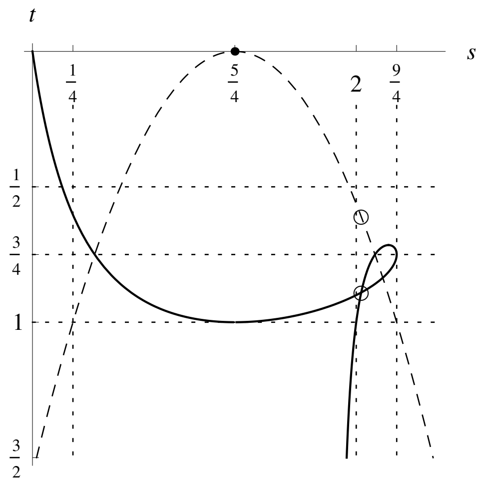

Figure 28. At height t, solutions s of $v(s) = 4t^3(1-t)$ for $S = 2t(3-2t)$, $s \neq 1 + \frac{\sqrt{17}}{4}$: roots of $(Q_2(s))^2$ on the parabola $t = (s - \frac{5}{4})^2$ and roots of $Q_3(s)$ on the bold curve (with one simple and one double root for $t = 0$, $\frac{29-6\sqrt{6}}{20}$, $\frac{3+\sqrt{17}}{8}$ and 1); abscissae of the parents of I^*_α at height $t = \sin^2\alpha$ for $0 < \alpha < \frac{\pi}{2}$, with parents outside $\mathcal{T}^*$ on the right parabola section under the bold curve

For $t \in \left[0, \frac{3}{2}\right]$ and $S = 2t(3-2t) \in \left[0, \frac{9}{4}\right]$, the roots of (22) are thus – except $s = 1+\frac{\sqrt{17}}{4}$ for $t = \frac{3}{4} \pm \frac{\sqrt{17}-3}{8}$ – the roots $s = \frac{5}{4} \pm \sqrt{t}$ of $(Q_2(s))^2$ on the parabola $t = (s - \frac{5}{4})^2$ and the real roots of $Q_3(s)$ (Figure 28): if $t = \sin^2\alpha \in \left]0,1\right[$, these roots, in particular $s = \frac{5}{4} \pm \sin\alpha$, are the abscissae of the parents of I^*_α. The pole $s = 1+\frac{\sqrt{17}}{4}$ is equal to $\frac{5}{4}+\sin\alpha$ for $t = \frac{3}{4} - \frac{\sqrt{17}-3}{8}$ and to a double root of $Q_3(s)$ for $t = \frac{3}{4} + \frac{\sqrt{17}-3}{8}$: $I^*_{\omega_{51}}$ and $I^*_{\omega_{71}}$ have no parent with $s = 1+\frac{\sqrt{17}}{4}$ from these sources. The parent of I^*_α with abscissa $s = \frac{5}{4} + \sin\alpha \neq 1 + \frac{\sqrt{17}}{4}$ has the ordinate $p = \frac{1+\sin\alpha}{64(1-\sin\alpha)}$ according to (18). One gets $D(s,p) = \frac{(2\cos 2\alpha+1)^2(2\cos 2\alpha - 4\sin\alpha+5)}{16(1-\sin\alpha)^2}$, which is < 0 for $\alpha > \omega_{66}$ (parent outside $\mathcal{T}^*$), $= 0$ for $\alpha = \frac{\pi}{3}$ or $\alpha = \omega_{66}$ (isosceles parent of I_α) and > 0 otherwise (non-isosceles parent of I_α): this parent is obtuse for $\alpha < \arcsin\frac{3}{4} = \omega_{49}$, right–angled for $\alpha = \omega_{49}$ and acute for $\omega_{49} < \alpha \leq \omega_{66}$; it is the acute class I_{75° for $\alpha = \frac{\pi}{3}$. The parent with abscissa $s = \frac{5}{4} - \sin\alpha$ has the ordinate $p = \frac{1-\sin\alpha}{64(1+\sin\alpha)}$ and $D(s,p)$ is then $= 0$ for $\alpha = \frac{\pi}{3}$ and > 0 otherwise: the corresponding parent of I_α is always obtuse since $s < 2$. Since the number of real roots of $Q_3(s)$ counted with their multiplicity (Figure 28) coincides with the number of isosceles parents of I_α for all $\alpha \neq \omega_{58}$, we have the following result: with the only exception of the rightmost solution of $v(s) = P_{\max}$ for $S = \frac{168\sqrt{6}-187}{100}$ (giving an isosceles parent of $I_{\omega_{58}}$), double roots of (the denominator of) $v(s) - P_{\max}$ or of $v(s) - P_{\min}$, $P_{\min} > 0$, correspond to non-isosceles parents of the considered isosceles triangle class (unless (s,p) lies outside $\mathcal{T}^*$), and simple or triple roots correspond to isosceles parents. Note that $\frac{168\sqrt{6}-187}{100}$ is the abscissa of the end $I^*_{\omega_{58}}$ of the appendix formed by the roof under the reflection map ρ. For $S \in \left[0, \frac{9}{4}\right]$, the growth of $v(s)$ on $\mathbf{R} \setminus \{1 \pm \frac{\sqrt{17}}{4}\}$ is given by the sign of $v'(s)$ according to (20), (21) and Figure 27. If one considers $S \in \left]0, \frac{9}{4}\right[$, writes it as $S = 2t(3-2t)$ with $t \in \left]0, \frac{3}{4}\right[$ and excludes partly the transition values $t = \frac{1}{2}, \frac{9-\sqrt{17}}{8}, \sqrt{2} - \frac{3}{4}, \frac{29-6\sqrt{6}}{20}$, *i.e.*, $S = 2, \frac{5+3\sqrt{17}}{8}, 12\sqrt{2} - \frac{59}{4}, \frac{168\sqrt{6}-187}{100}$, $v(s)$ has exactly two local extrema (always maxima) at height $P_{\max} = 4t^3(1-t)$ – for $s = \frac{5}{4} \pm \sqrt{t}$ – and exactly two local extrema (a minimum on the left) at height $4(\frac{3}{2}-t)^3(1-(\frac{3}{2}-t))$ – for $s = \frac{5}{4} \pm \sqrt{\frac{3}{2} - t}$. Note that $4(\frac{3}{2}-t)^3(1-(\frac{3}{2}-t)) = P_{\min}$ for $t \in \left[\frac{1}{2}, \frac{3}{4}\right]$ and that t and $\frac{3}{2} - t$ are symmetric with respect to $\frac{3}{4}$ in Figure 28. $\square$

Theorem 11. *The parents in $\mathcal{T}$ of I_α, $\alpha \neq \frac{\pi}{3}$, are – up to the exceptions mentioned below – the two non-isosceles classes $\{\alpha'_\pm, \beta'_\pm, \gamma'_\pm\}$ given by the non-obtuse angles*

$$\begin{aligned}
\alpha'_\pm &= \frac{\pi}{4} \pm \frac{\alpha}{2}, \\
\beta'_\pm &= \operatorname{arccot}\left(2\cos\alpha + 2\sqrt{2 - \left(\frac{1}{2} \pm \sin\alpha\right)^2}\right), \\
\gamma'_\pm &= \operatorname{arccot}\left(2\cos\alpha - 2\sqrt{2 - \left(\frac{1}{2} \pm \sin\alpha\right)^2}\right)
\end{aligned}$$

in $]0,\pi[$ *– with coordinates* $\left(\frac{5}{4} \pm \sin\alpha, \frac{(1\pm\sin\alpha)^2}{64(1-\sin^2\alpha)}\right) \in \mathcal{T}^*$ *– and the isosceles triangle classes with coordinates* (s,p) (automatically on the roof) *corresponding to each real root* s *of* $Q_3(s)$ *given by* (23) *for* $t = \sin^2\alpha$, *with* p *as in Theorem 10.*

For $\alpha = \omega_{66}$ *the triangle class* $\{\alpha'_+, \beta'_+, \gamma'_+\}$ *is isosceles with equal angles* ω_{50} *and corresponds to the triple root* $s = \sqrt{2} + \frac{3}{4}$ *of* $v(s) = P_{\min}$ *for* $S = 12\sqrt{2} - \frac{59}{4}$. *For* $\alpha > \omega_{66}$ *the non-isosceles class* $\{\alpha'_+, \beta'_+, \gamma'_+\}$ *doesn't exist: it corresponds to the parent outside* $\mathcal{T}^*$ *and* $\beta'_+, \gamma'_+ \notin \mathbf{R}$.

Proof. Parts of this theorem have been already demonstrated in the proof of Theorem 10. Theorem 2 for $s = \frac{5}{4} - \sin\alpha$, $p = \frac{1-\sin\alpha}{64(1+\sin\alpha)}$ gives an obtuse parent $\{\alpha', \beta', \gamma'\}$ of I_α with $\sin^2\alpha' = \frac{1-\sin\alpha}{2}$, *i.e.*, $\sin\alpha = \cos 2\alpha' = \sin\left(\frac{\pi}{2} - 2\alpha'\right)$,

$$\sin^2\beta' = \frac{3 + \sin\alpha - 2\sin^2\alpha - \cos\alpha\sqrt{7 + 4\sin\alpha - 4\sin^2\alpha}}{8(1+\sin\alpha)},$$

$$\sin^2\gamma' = \frac{3 + \sin\alpha - 2\sin^2\alpha + \cos\alpha\sqrt{7 + 4\sin\alpha - 4\sin^2\alpha}}{8(1+\sin\alpha)}.$$

Because $\sin^2\gamma' \geq \sin^2\alpha', \sin^2\beta'$ for $0 < \alpha < \frac{\pi}{2}$, α', β' are acute, thus $\alpha' = \frac{\pi}{4} - \frac{\alpha}{2}$, and γ' is obtuse. One gets

$$\cot^2\beta' = \frac{1}{\sin^2\beta'} - 1 = \left(2\cos\alpha + \sqrt{7 + 4\sin\alpha - 4\sin^2\alpha}\right)^2$$

and, with negative parenthesis, $\cot^2\gamma' = \left(2\cos\alpha - \sqrt{7 + 4\sin\alpha - 4\sin^2\alpha}\right)^2$.

For $s = \frac{5}{4} + \sin\alpha$, $p = \frac{1+\sin\alpha}{64(1-\sin\alpha)}$ one gets similarly $\sin^2\alpha' = \frac{1+\sin\alpha}{2}$, *i.e.*, $\sin\alpha = -\cos 2\alpha' = \sin\left(2\alpha' - \frac{\pi}{2}\right)$, $\sin^2\beta' = \frac{3-\sin\alpha-2\sin^2\alpha-\cos\alpha\sqrt{7-4\sin\alpha-4\sin^2\alpha}}{8(1-\sin\alpha)}$ and $\sin^2\gamma' = \frac{3-\sin\alpha-2\sin^2\alpha+\cos\alpha\sqrt{7+4\sin\alpha-4\sin^2\alpha}}{8(1-\sin\alpha)}$ with $\sin^2\gamma', \sin^2\alpha' \geq \sin^2\beta'$ for $0 < \alpha \leq \omega_{66}$ and $\sin^2\gamma' > \sin^2\alpha'$ for $0 < \alpha \leq \omega_{49}$, *i.e.*, when $\{\alpha', \beta', \gamma'\}$ is obtuse or right–angled. Since α' is always acute, $\alpha' = \frac{\pi}{4} + \frac{\alpha}{2}$. One gets $\cot^2\beta' = \left(2\cos\alpha + \sqrt{7 - 4\sin\alpha - 4\sin^2\alpha}\right)^2$ and, with parenthesis changing sign at $\alpha = \omega_{49}$ from < 0 to > 0, $\cot^2\gamma' = \left(2\cos\alpha - \sqrt{7 - 4\sin\alpha - 4\sin^2\alpha}\right)^2$. □

For $\alpha = 0$, a triangle with angles $\{\alpha_\pm, \beta_\pm, \gamma_\pm\} = \{45°, \omega_{12}, 135° - \omega_{12}\}$ is the parent of an isosceles degenerate triangle with three different vertices from Theorem 5. For $\alpha = \frac{\pi}{2}$, $\{\alpha_-, \beta_-, \gamma_-\}$ is the parent $\Pi_{\omega_{21}}$ of $\Pi_{\pi/2}$. The points $(s,p) = \left(\frac{5}{4} \pm \sin\alpha, \frac{(1\pm\sin\alpha)^2}{64(1-\sin^2\alpha)}\right)$ constitute the hyperbola arc $p = \frac{1-4s}{64(4s-9)}$, $\frac{1}{4} \leq s < \frac{9}{4}$, which starts on Γ, is tangent to Λ at $I^*_{\pi/12}$ and $I^*_{5\pi/12}$ and lies outside $\mathcal{T}^*$ between $s = \sqrt{2} + \frac{3}{4}$ and the pole $s = \frac{9}{4}$.

One gets the following non-isosceles parents of isosceles triangles with integer angles (see curve Φ in Figure 34): $\{42°, 12°, 126°\}$ for $I_{6°}$, $\{36°, 12°, 132°\}$ for $I_{18°}$, $\{60°, 15°, 105°\}$ for $I_{30°}$, $\{66°, 18°, 96°\}$ for $I_{42°}$, $\{72°, 24°, 84°\}$ for $I_{54°}$, $\{54°, 48°, 78°\}$ for $I_{66°}$ and $\{18°, 6°, 156°\}$ for $I_{78°}$.

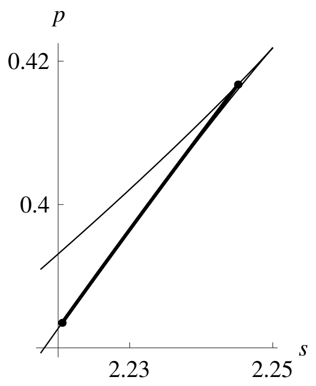

Figure 29. Curve of the coordinates of the hexagenerated triangles

The isosceles parent of the right–angled $I_{\pi/4}$ is I_α with

$$\alpha = \arcsin\sqrt{\frac{1}{12}\left(8 - \frac{13}{\sqrt[3]{73-6\sqrt{87}}} - \sqrt[3]{73-6\sqrt{87}}\right)} \approx 10.1986^\circ.$$

The two isosceles parents of $I_{\omega_{58}}$ have equal angles $\arcsin\sqrt{\frac{11-4\sqrt{6}}{20}} \approx 14.191^\circ$ and ω_{68} (corresponding to the rightmost double root of $v(s) = P_{\max}$ for $S = \frac{168\sqrt{6}-187}{100}$), respectively.

Consider $(S,P) \in \mathcal{T}^*$ neither on the roof nor on the ground. Figures 14–26 show that (S,P) has 5 parents if $S \le 12\sqrt{2} - \frac{59}{4}$ and 7 parents if $S \ge \frac{168\sqrt{6}-187}{100}$, whereas the interval $12\sqrt{2} - \frac{59}{4} < S < \frac{168\sqrt{6}-187}{100}$ assures the mutation from "pentagenerated" to "heptagenerated" non-isosceles classes of $\mathcal{T}$: in this last case, the number of parents of (S,P) depends on P and jumps (over 6 at the level P_6) from 7 near the bottom $P_{\min}$ to 5 near the top $P_{\max}$, and the ordinate $P_6 = P_6(S)$ of the hexagenerated triangle class climbs with growing S. This mutation is achieved at the abscissa $S = \frac{168\sqrt{6}-187}{100}$ of the end $I^*_{\omega_{58}}$ of the appendix formed by the roof under the reflection map ρ. Triangle classes have thus infinitely many or exactly 7, 6, 5, 4, 3 or 2 parents in $\mathcal{T}$ but never only one parent!

For $12\sqrt{2} - \frac{59}{4} < S \le \frac{168\sqrt{6}-187}{100}$, the largest of the three real roots s of

$$q_3(s) = 192s^3 - 528s^2 + s(128S + 100) + 136S + 125$$

in (20) is the abscissa of the first maximum of $v(s)$ left from $\frac{9}{4}$: this gives the ordinate P_6 of the hexagenerated triangle class exactly (Figures 27, 29 and 30).

Figures 14–17 show that finite obtuse or right–angled triangles have only obtuse parents.

All acute triangles with abscissa $> s(I_{\omega_{49}}) = \frac{135}{64} = 2.109375$ have both acute and obtuse parents. The coordinates $(S,P) = r\big(\{\alpha, \frac{\pi}{2} - \alpha, \frac{\pi}{2}\}\big)^*$ of the triangles with right–angled parents form the parabola arc $P = v(2) = \frac{81}{4}(S-2)(\frac{9}{4} - S)$,

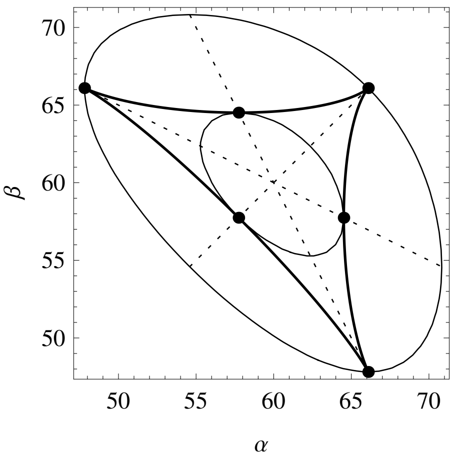

Figure 30. Level curves $s(\Delta) = 12\sqrt{2} - \frac{59}{4}$ and $s(\Delta) = \frac{168\sqrt{6}-187}{100}$ in the $\alpha\beta$–plane for two angles α, β (in $^\circ$) of the triangle Δ: the curve of the hexagenerated triangles separates the pentagenerated from the heptagenerated ones. The *pentagenerated* cusps correspond to $I_{\omega_{66}}$, the three other points to $I_{\omega_{58}}$.

$2 \le S \le \frac{54}{25} = 2.16$, given by (17): S grows with α from 2 for $r(\Pi_{\pi/2}) = \Pi_{\pi/2}$ to 2.16 for $r(I_{\pi/4})$, which is $I_{\arcsin 3/\sqrt{10}} = I_{\omega_{72}}$ since $Q_3(s) = 2$ if and only if $t \in \{\frac{9}{10}, 1\}$. The parabola arc starts and ends on the right roof section and is tangent to the left roof section for $S = \frac{135}{64}$ at $I^*_{\omega_{49}}$. Acute triangles with abscissa > 2.16 have thus no right–angled parents.

A non-isosceles parent of an isosceles class $I_\alpha \in \mathcal{T}$ generates two different parents of a corresponding given isosceles triangle. By considering congruent non-identical triangles as different, we have proven the following result.

Theorem 12. *Let ABC be a proper triangle with vertices A, B, C and angles α, β, γ. Let*

$$S = \sin^2\alpha + \sin^2\beta + \sin^2\gamma,\ P = \sin^2\alpha \cdot \sin^2\beta \cdot \sin^2\gamma.$$

ABC is the reflection triangle of between 5 and 7 parents.
(1) *If ABC is obtuse and non-isosceles, it has exactly 5 parents, which are all obtuse, non-isosceles and pairwise non-similar.*
(2) *If ABC is acute and non-isosceles* (Figure 30)*, it has between 5 and 7 parents depending on S and P. These parents are all non-isosceles and pairwise non-similar:*

(a) $2 < S \le \frac{135}{64} = 2.109375$*: 5 parents, 4 of them obtuse and the last one obtuse, right–angled or acute according as* $P \gtreqless \frac{81}{4}(S-2)(\frac{9}{4}-S)$*;*
(b) $2.109375 \le S \le \frac{54}{25} = 2.16$*: 5 parents, 3 of them obtuse, one acute and the last obtuse, right–angled or acute according as* $P \lesseqgtr \frac{81}{4}(S-2)(\frac{9}{4}-S)$*;*
(c) $2.16 \le S \le 12\sqrt{2} - \frac{59}{4} \approx 2.221$*: 5 parents, 3 of them obtuse and 2 acute;*

(d) $12\sqrt{2}-\frac{59}{4} \leq S \leq \frac{168\sqrt{6}-187}{100} \approx 2.245$*: 5, 6 or 7 parents, 3 of them obtuse, 2 acute and zero, one or two additional acute parents according as* $P \gtreqless P_6 = P_6(S)$ *given by Figure 29;* P_6 *grows with* S *from* $\frac{371}{\sqrt{2}}-\frac{16765}{64} \approx 0.383$ *to* $\frac{3\left(135664\sqrt{6}-326751\right)}{40000} \approx 0.417$.

(e) $\frac{168\sqrt{6}-187}{100} \leq S < \frac{9}{4}$*: 7 parents, 3 of them obtuse and 4 acute.*

(3) *If* ABC *is isosceles with equal angles* α*, it has* 5 *parents except for* $\alpha = \omega_{58}$ (6 *parents*) *and for* $\omega_{58} < \alpha < \omega_{66}$ (7 *parents*)*:*

(a) $0^\circ < \alpha < \omega_{49}$*: one isosceles obtuse parent, a pair of non-similar non-isosceles obtuse parents and their mirror images in the axis of* ABC*;*

(b) $\alpha = \omega_{49}$ ($S = \frac{135}{64}$)*: one isosceles obtuse parent, one non-isosceles obtuse and one non-isosceles right–angled parent and their mirror images;*

(c) $\omega_{49} < \alpha < \omega_{58}$*: one isosceles obtuse parent, one non-isosceles obtuse and one non-isosceles acute parent and their mirror images;*

(d) $\alpha = \omega_{58}$ $\left(S = \frac{168\sqrt{6}-187}{100}\right)$*: one isosceles obtuse and one isosceles acute parent, one non-isosceles obtuse and one non-isosceles acute parent and their mirror images;*

(e) $\omega_{58} < \alpha < \omega_{66}$, $\alpha \neq 60^\circ$*: one isosceles obtuse and two non-similar isosceles acute parents, one non-isosceles obtuse and one non-isosceles acute parent and their mirror images;*

(f) $\alpha = 60^\circ$*: one equilateral parent, three congruent isosceles parents with equal angles* 15° *and three with equal angles* 75° (Figure 2)*;*

(g) $\omega_{66} \leq \alpha < \omega_{72}$ $\left(S = 12\sqrt{2}-\frac{59}{4} \text{ for } \alpha = \omega_{66}\right)$*: one isosceles obtuse and two non-similar isosceles acute parents, one non-isosceles obtuse parent and its mirror image;*

(h) $\alpha = \omega_{72}$*: three isosceles parents (one obtuse, one right–angled and one acute), one non-isosceles obtuse parent and its mirror image;*

(i) $\omega_{72} < \alpha < 90^\circ$*: a pair of non-similar isosceles obtuse parents, one acute isosceles parent, one non-isosceles obtuse parent and its mirror image.*

In order to count and describe the parents of the corresponding coordinates $(S,P) \in \mathcal{T}^* \setminus \Gamma$ in Theorem 12, one has to neglect the mirror images and the repetitions of congruent triangles and to add one exterior parent of I^*_α for $\omega_{66} < \alpha < \frac{\pi}{2}$.

9. Convergence to an equilateral or degenerate limit

After continuous extension, all level curves of ρ_1 and of ρ_2 given by (6) are tangent to Λ at $I^*_{\pi/6} = (\frac{5}{4}, \frac{3}{64})$. By (15) one has $\rho_1(s,p) = s$ for $(s,p) \neq I^*_{\pi/6}$ if and only if $p = \frac{s(s-2)(4s-3)}{4(12s-25)}$: this curve lies in $\mathcal{T}^*$ if and only if $s \in \{0\} \cup \left[\frac{3}{4}, 2\right] \cup \{\frac{9}{4}\}$ and is tangent to Λ at $I^*_{\pi/6}$. One has $\rho_2(s,p) = p$ for $(s,p) \neq I^*_{\pi/6}$ if and only if $p = 0$ or $s = \frac{7}{4}$ or $p = \frac{1}{4}(s-1)^2 + \frac{1}{32}$ or $p = \frac{(4s-5)^3+4s+1}{-64(4s-7)}$: both last curves are tangent to Λ at $I^*_{\pi/6}$ from outside $\mathcal{T}^*$ and the parabola has no other point in $\mathcal{T}^*$.

The arrows $\nearrow, \nwarrow, \swarrow, \searrow$ in Figure 31 show the constant quadrant of the vector $\rho(s,p) - (s,p)$ in each of the zones of $\mathcal{T}^*$ delimited by the curves $\rho_1(s,p) = s$

and $\rho_2(s,p) = p$, whose intersections are the fixed points of ρ. Note that zone VII is the thin region bounded below by $\rho_1 = s$ and above by the curved branch of $\rho_2 = p$ and by the roof. Since $\rho(s,p)$ lies strictly eastwards and northwards from (s,p) for all $(s,p) \in \mathcal{T}^*$ with $2 \le s < \frac{9}{4}$, $p > 0$, the sequence $\big(\rho^n(s,p)\big)_{n\in\mathbf{N}}$ for such an (s,p) converges to or reaches $I^*_{\pi/3}$ with strictly increasing coordinates. The first part of the following theorem is proven.

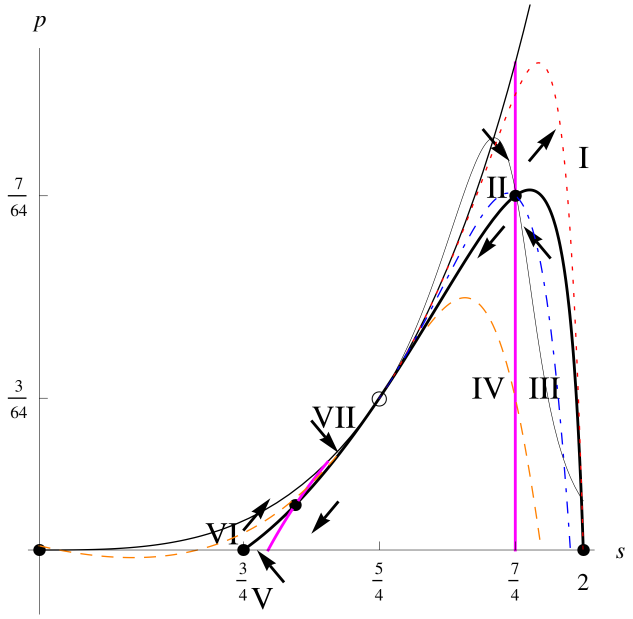

Figure 31. Quadrant of the vector $\rho(s,p) - (s,p)$ depending on the zone of $(s,p) \in \mathcal{T}^*$ with curves $\rho_1 = s$ (black, thick), $\rho_1 = p$ (magenta, thick), $\rho_1 = \frac{5}{4}$ (dashed, orange) and, for $s \ge \frac{5}{4}$, $\rho_1 = \frac{7}{4}$ (dot–dashed, blue), $\rho_1 = 2$ (dotted, red), $\rho_2 = p_{\text{top}}$ (thin)

Theorem 13. (1) *An acute or right–angled proper triangle has always an acute reflection triangle and its iterated reflection triangle converges to an equilateral limit with strictly growing coordinates.*
(2) *An acute or right–angled triangle becomes equilateral after a finite number of reflection steps if and only if its class is an isosceles acute ancestor of I_{60° given by the infinite sequence of successive parents I_{60°, I_{75°, $I_{84.6588\ldots^\circ}$, $I_{88.205\ldots^\circ}, \ldots$ whose equal angles grow towards 90°.*

Proof. (2) Each I_α with $\alpha \ge 75^\circ$ has exactly one acute or right–angled parent: an isosceles one with equal angles $> \alpha$ (Figure 10). $\square$

Theorem 14. $(s,p) \in \mathcal{T}^* \setminus \big(\Gamma \cup \big\{(\frac{7}{4}, \frac{7}{64})\big\}\big)$ *converges to $I^*_{\pi/3}$ under iteration of ρ (with strictly growing coordinates except possibly for the first reflection step) when*
(1) (s,p) *is in zone I of Figure 31 with boundary,* i.e., $s \ge \frac{7}{4}$ *and* $\rho_1(s,p) \ge s$, *or*
(2) $\rho_1(s,p) \ge \frac{7}{4}$ *and* $\rho_2(s,p) \ge p_{\text{top}}$ *where* $p_{\text{top}} \approx 0.11118$ *is the ordinate of the*

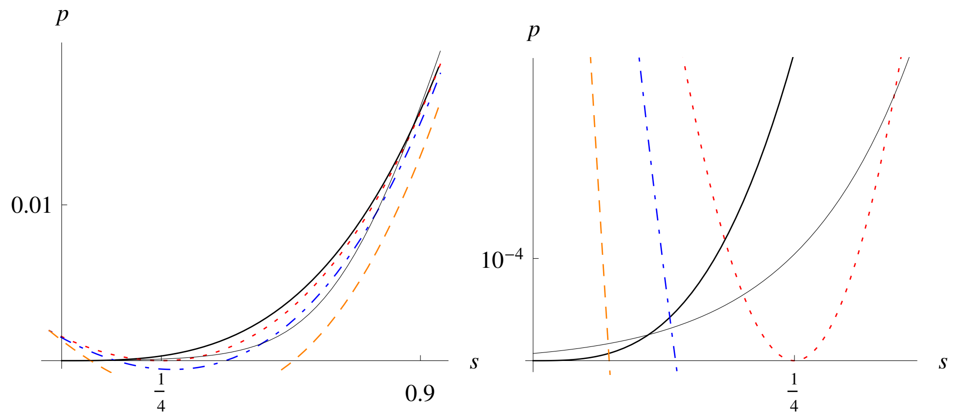

Figure 32. Detailed views of the left part of Figure 31. In the right figure, the intersection points of the roof (black, thick) with $\rho_1(s,p) = 2$ (dotted, red) and $\rho_1(s,p) = \frac{5}{4}$ (dashed, orange) give the coordinates of the isosceles parents of $I_{\pi/4}$ and $I_{\pi/6}$, respectively. $(\frac{1}{4}, 0)$ is the parent $\Pi^*_{\omega_{21}}$ of $\Pi^*_{\pi/2}$.

maximum point of the curve $\rho_1(s,p) = s$ (Figures 31 and 32)*, or*
(3) $\rho_1(s,p) \geq 2$.

Note that $576p_{\text{top}}$ *is the middle root of* $p^3 - 294p^2 + 13209p + 97200$.

Proof. We only have to prove that the corner of zone I near the heptagonal fixed point $(\frac{7}{4}, \frac{7}{64})$ is mapped by ρ to zone I and not to zone III, and this is true: the points with $\rho_1(s,p) = s$, $s > \frac{7}{4}$, $p > 0$, are mapped upwards by ρ and $\frac{\partial \rho_2}{\partial p}(s,p) = \frac{(5-4s)^6(64p(4s-7)-4s-1)}{-(64p(4s-7)+4s+1)^3}$ is strictly positive in the rectangle $\frac{7}{4} < s < 2$, $\frac{7}{64} < p < \frac{1}{8}$ containing the maximum point of the curve $\rho_1(s,p) = s$. $\square$

Theorem 14 gives Figure 33 where each (α, β) is identified with the triangle class $\{\alpha, \beta, 180° - \alpha - \beta\} \in \mathcal{T}$. The large points are the fixed points of r; the small points are $I_{30°}$, its isosceles parent and grandparent and the parent $\Pi_{\omega_{21}}$ of $\Pi_{90°}$. The squares mark the right–angled $I_{45°}$ and its isosceles parent on the thin dotted curve $\rho_1(s,p) = 2$. The curve $s = \frac{5}{4}$ is dot–dashed and goes through $I_{30°} = (30°, 30°)$; its parent curves are dashed: one of them goes through the parent $(60°, 15°)$ of $I_{30°}$ corresponding to $(s,p) = (\frac{7}{4}, \frac{3}{64})$. There are points with $\rho_1(s,p) \geq 2$ in zones I, II, VII and VI of Figure 31; there are points with $\rho_1(s,p) \geq \frac{7}{4}$ and $\rho_2(s,p) \geq p_{\text{top}}$ in zones I, II, III and VI. Note that every neighborhood of the heptagonal fixed point $\left(\frac{360°}{7}, \frac{180°}{7}\right)$ contains triangle classes with equilateral limit. Because the leftmost roots of $v(s) = P$ for $S = \frac{5}{4}$ are almost equal for all P, the inner branch of parents of $s = \frac{5}{4}$ that passes through $I_{6.33...°}$ in Figure 33 is nearly a level curve of s; furthermore, the nearby arc of the curve $\rho_1(s,p) = 2$ that joins the pair of points representing $\Pi_{\omega_{21}}$ is almost parallel to the square diagonal $s = 2$: the fractal structure is born.

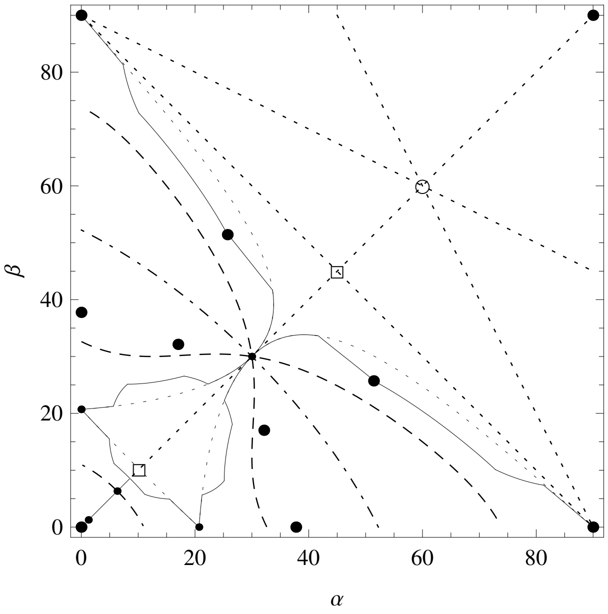

Figure 33. Convergence to an equilateral limit is ensured when two angles (α, β) of the base triangle are in the zone enclosed by or north–east from the plain curve, or on this curve, filled points excepted.

We now describe the set of triangles with equilateral or degenerate limit systematically. We denote by $\mathcal{A}_n$ and $\mathcal{D}_n$ the set of classes in $\mathcal{T}$ that become acute and degenerate after exactly n applications of the reflection map r, $n \in \mathbf{N}$. $\mathcal{A}_n^*$ and $\mathcal{D}_n^*$ are the corresponding subsets of $\mathcal{T}^*$: $\mathcal{A}_n^*$, $n \geq 1$, consists of the points $(s,p) \in \mathcal{T}^*$ for which the first coordinates of $\rho^{n-1}(s,p)$ and of $\rho^n(s,p)$ are ≤ 2 and > 2, respectively. Since $\mathcal{O}^*$ is a repelling fixed point of ρ, the basins of attraction of $I_{\pi/3}^*$ and $\mathcal{O}^*$ in $\mathcal{T}^*$ are the disjoint unions $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}_n^*$ and $\mathcal{D}^* = \bigcup_{n \geq 0} \mathcal{D}_n^*$, respectively. Figure 34 shows the "wing" $\bigcup_{n=1}^3 \mathcal{A}_n$ with skeleton $\bigcup_{n=1}^3 \mathcal{D}_n$, where (α, β) represents $\{\alpha, \beta, 180° - \alpha - \beta\} \in \mathcal{T}$.

The boundary curve of $\mathcal{A}_n^*$, $n \geq 1$, consists of the following points:

(1) the points $(s,p) \in \mathcal{T}^*$ for which the first coordinate of $\rho^{n-1}(s,p)$ or of $\rho^n(s,p)$ is 2
(2) the members of $\rho^{-k}\left(I_{\pi/6}^*\right)$ (lying thus on $\mathcal{D}_{k+1}^*$) for $0 \leq k \leq n-1$
(3) the members of $\rho^{-n}\left(\Pi_{\pi/2}^*\right) = \bigcup_{k=0}^n \rho^{-k}\left(\Pi_{\pi/2}^*\right)$ (lying thus on Γ)
(4) the roof section between the roof member of $\rho^{-(n-1)}\left(I_{\pi/4}^*\right)$ and its roof parent.

$\mathcal{A}_n^*$, $n \geq 1$, is composed of $5^{n-1} + 1$ maximal simply connected subsets. $\mathcal{T}^* \setminus \bigcup_{k=0}^n \overline{\mathcal{A}_k^*}$, $n \geq 0$, consists of $\frac{5^n+3}{4}$ maximal simply connected components – the overline denoting set closure. For $n \geq 1$, $\frac{3^n-1}{2}$ such components are juxtaposed arches whose feet are the $\frac{3^n+1}{2}$ members of $\rho^{-n}\left(\Pi_{\pi/2}^*\right)$ on Γ. Starting

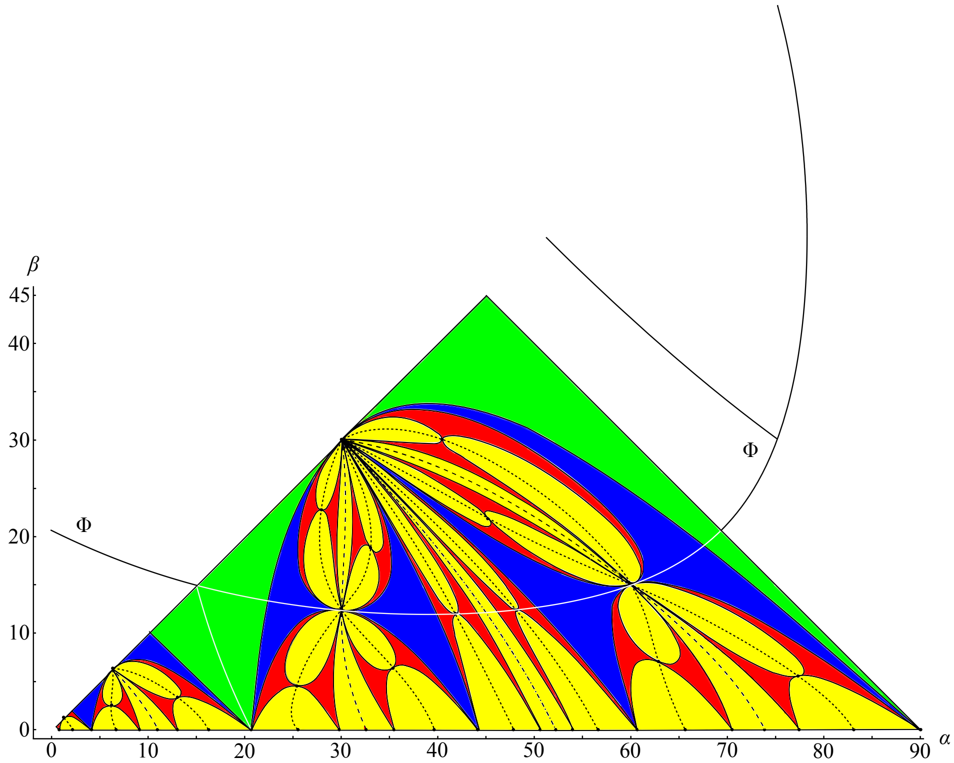

Figure 34. Wing of the obtuse triangles that become acute after one ($\mathcal{A}_1$, green), two (blue) or three reflection steps (red) with curves of the triangles that become degenerate after one ($\mathcal{D}_1$, dot–dashed), two (dashed) or three reflection steps (dotted)

from the rightmost $\Pi^*_{\pi/2}$, the members of $\rho^{-n}\left(\Pi^*_{\pi/2}\right)$ and the $\frac{3^n-1}{2}$ members of $\bigcup_{k=0}^{n-1}\rho^{-k}\left(\Pi^*_{\omega_{52}}\right)$ alternate on Γ. Each member of $\rho^{-k}\left(\Pi^*_{\omega_{52}}\right)$ – lying between the leftmost member of $\rho^{-m}\left(\Pi^*_{\pi/2}\right)$ and the leftmost member of $\rho^{-(m+1)}\left(\Pi^*_{\pi/2}\right)$, say – is the starting point of one of the 3^k curves of $\mathcal{D}^*_{k+1}$: after continuous extension at some points of $\bigcup_{\ell=0}^{k}\rho^{-\ell}\left(I^*_{\pi/6}\right)$, this curve ends at the roof member of $\rho^{-m}\left(I^*_{\pi/6}\right)$. One has further $\overline{\mathcal{D}^*_n} = \mathcal{D}^*_n \cup \bigcup_{k=0}^{n-1}\rho^{-k}\left(I^*_{\pi/6}\right)$ and $\overline{\mathcal{A}^*_n} \cap \overline{\mathcal{D}^*_n} = \bigcup_{k=0}^{n-1}\rho^{-k}\left(I^*_{\pi/6}\right)$, $n \geq 1$.

Figure 34 shows the frontier line Φ of the non-isosceles parents of the isosceles triangle classes given by Theorem 11. The same classes are represented two times on the branches issued from the left bifurcation, three times after the right bifurcation. Φ cuts the dot–dashed middle curve $\mathcal{D}_1$ at $(45^\circ, \omega_{12})$: a triangle with these angles is the parent of an isosceles degenerate triangle (*i.e.*, a segment and its midpoint). The intersection point of Φ with the line $\beta = \alpha$ corresponding to the roof is the isosceles parent I_{15° of I_{60° and its left end is the parent $\Pi_{\omega_{21}}$ of Π_{90°. Φ intersects the line $s = 2$ at the right–angled parent $(90^\circ - \omega_{21}, \omega_{21})$ of

$I_{\omega_{49}}$, the right bifurcation point is the isosceles parent I_{75° of I_{60° and the end of the following left branch is the isosceles parent $I_{\omega_{50}}$ of the end class $I_{\omega_{66}}$.

Consider Figure 34 filled with $\mathcal{A}$ and $\mathcal{D}$. Let $\mathcal{P}_n$, $n \geq 1$, be the closure of the component of $\mathcal{A}_n$ with a boundary segment on the "roof" $\beta = \alpha$ together with the underlying arch bounded by the α–axis. Let $\mathcal{S}_n$, $n \geq 1$, be the following subset of $\mathcal{P}_1$: the closure of both pairs of components of $\mathcal{A}_{n+1}$ connecting the α–axis with I_{30° on both sides of the middle curve $\mathcal{D}_1$ together with both underlying arches and enclosed bubbles. Let $\mathcal{S}_n^{lb}$, $\mathcal{S}_n^{lt}$, $\mathcal{S}_n^{rb}$ and $\mathcal{S}_n^{rt}$ be the left bottom, left top, right bottom and right top parts of $\mathcal{S}_n$ delimited by Φ left and right from $\mathcal{D}_1$.

Every class of proper non-acute and non-isosceles triangles has exactly 5 parents, every class of infinite triangles except Π_{90° has exactly 3 parents, and every I_α, $0^\circ < \alpha \leq 45^\circ$, has exactly one isosceles and two non-isosceles parents. Here are these mappings.

The reflection map r is a bijective fractal blow–up of $\mathcal{P}_{n+1}$, $n \geq 1$, to $\mathcal{P}_n$, *i.e.*, every component, boundary or point of $\mathcal{A}_k$, $\mathcal{D}_k$, $\ldots$ in $\mathcal{P}_{n+1}$ is blown up bijectively for all $k \geq n+1$ (with appropriate orientation–preserving distortion and translation) to the geographically corresponding component, boundary or point of $\mathcal{A}_{k-1}$, $\mathcal{D}_{k-1}$, $\ldots$ in $\mathcal{P}_n$. r is a bijective fractal blow–up or blow–down to $\mathcal{P}_n$ of $\mathcal{S}_n^{rb}$ and of $\mathcal{S}_n^{lb}$ flipped about a vertical axis. And r is a bijective fractal blow–up or blow–down to $\mathcal{P}_n$ without α–axis of $\mathcal{S}_n^{rt} \setminus \{I_{30^\circ}\}$ flipped about the line $\beta = \alpha$ and of $\mathcal{S}_n^{lt} \setminus \{I_{30^\circ}\}$ after a half–turn. Note that the top of $\mathcal{S}_n^{rt}$ and of $\mathcal{S}_n^{lt}$ has first to be stretched after I_{30° has been removed!

Every point of $\mathcal{P}_n$, $n \geq 1$, has thus one parent in $\mathcal{P}_{n+1}$ and all its other parents in $\mathcal{P}_1$, more precisely in $\mathcal{S}_n$. If one identifies the set of classes of infinite triangles with the interval $[0^\circ, 90^\circ]$ of the α–axis, the action of r on the infinite classes consists of three bijective fractal blow–ups to $[0^\circ, 90^\circ]$: one of $[0^\circ, \omega_{21}]$, one of $[\omega_{21}, \omega_{52}]$ after a flip and one of $[\omega_{52}, 90^\circ]$.

For a global description of the reflection map r we identify $\mathcal{T}$ in Figure 35 with $\left\{(\alpha, \beta) \mid 0^\circ \leq \beta \leq \alpha \leq 90^\circ - \frac{\beta}{2}\right\}$ and consider the set $\mathcal{T}_1$ of the non-acute, nondegenerate classes and the set $\mathcal{T}_2$ of the non-obtuse classes – $\mathcal{T}_1$ and $\mathcal{T}_2$ sharing the set $\mathcal{T}_\perp$ of the right–angled classes. The zones i–v and 1–7 of $\mathcal{T}$ are delimited by the following plain curves:

(1) the curve $\mathcal{D}_1$ of the nondegenerate classes that degenerate at the first stage,
(2) the curve of the parents of the right–angled classes, whose 5 segments without I_{30° – between zones 1 and v, 2 and iv, 3 and iii, 5 and ii, 4 and i, respectively – are each mapped bijectively to $\mathcal{T}_\perp$ or $\mathcal{T}_\perp \setminus \{\Pi_{90^\circ}\}$,
(3) the curve Φ of the non-isosceles parents of the isosceles classes,
(4) the curve between zone 6 and zone 7 (from $I_{\omega_{50}}$ to $I_{\omega_{68}}$) that corresponds to the rightmost parents of the hexagenerated points of $\mathcal{T}^*$ and whose dotted child curve (from $I_{\omega_{66}}$ to $I_{\omega_{58}}$) is the thick line of hexagenerated triangles of Figure 30.

The reflection map r can be described as follows if one considers zones i–iv without $\mathcal{D}_1$: the curve $\mathcal{D}_1$ and $(0,0)$ are mapped to $(0,0)$; zone i, zone iv flipped about a vertical axis and zone v without origin are each scaled bijectively and fractally to

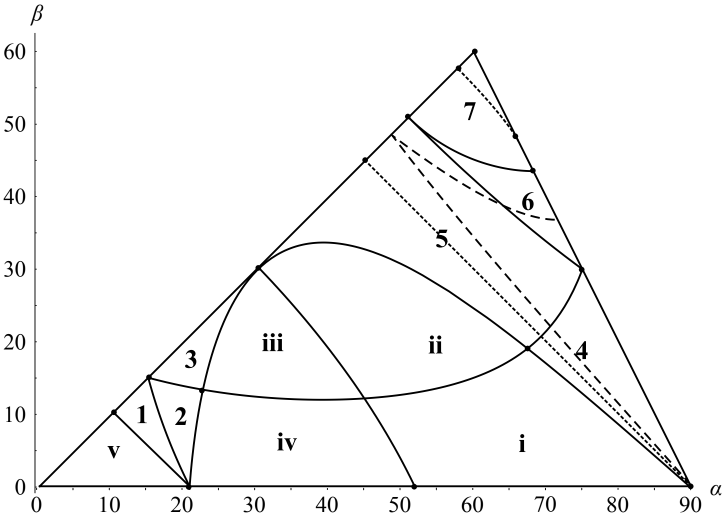

Figure 35. Decomposition of the reflection map r into bijective submappings

$\mathcal{T}_1$; zone ii flipped about the line $\beta = \alpha$ and zone iii after a half–turn are each scaled bijectively and fractally to $\mathcal{T}_1$ without α–axis. Zone 1, zone 2 flipped about a vertical axis and zone 4 are each scaled bijectively and fractally to $\mathcal{T}_2$; zone 3 without $I_{30°}$ after a half–turn and zone 5 without $I_{30°}$ flipped about the line $\beta = \alpha$ are each scaled bijectively and fractally to $\mathcal{T}_2$ without $\Pi_{90°}$. Note that the upper border section of zone 5 from $I_{75°}$ to $I_{30°}$ is mapped to the whole right "roof" section from $I_{60°}$ to $\Pi_{90°}$. Zone 6 after a half–turn and zone 7 flipped about a vertical axis are each scaled bijectively and fractally to the heptagenerated tip of $\mathcal{T}_2$.

The triangle classes $r\big(\{\alpha, 90° - \alpha, 90°\}\big)$ with right–angled parents constitute the dashed curve $r(\mathcal{T}_\perp)$ of Figure 35 joining with decreasing α the fixed point $\Pi_{90°}$ to $r(I_{45°}) = I_{\omega_{72}}$ over $r\big(\{90° - \omega_{21}, \omega_{21}, 90°\}\big) = I_{\omega_{49}}$. Their coordinates (S, P) form the parabola arc $P = \frac{81}{4}(S - 2)(\frac{9}{4} - S)$, $2 \leq S \leq 2.16$.

A class of non-isosceles finite triangles in Figure 35 has 5, 6 or 7 parents when it is located below, on or above the upper dotted curve, respectively; it has (exactly) one right–angled parent if and only if it is on $r(\mathcal{T}_\perp)$; exactly 3, 4 or 5 of its parents are obtuse when it is located on or above the upper section of $r(\mathcal{T}_\perp)$, below this section but not below the bottom section, or below $r(\mathcal{T}_\perp)$, respectively. The preceding sentence is also true for finite isosceles *triangles*, except that triangles in the class $I_{\omega_{66}}$ have only 5 parents (instead of 6) and that triangles in the class $I_{\omega_{49}}$ have two right–angled parents (instead of one acute and one right–angled) and three obtuse parents.

10. Periodic orbits

We use the notations of Section 9.

Theorem 15. $\rho\big|_{\mathcal{T}^*\setminus\Gamma}$ *has n–periodic points for all integers $n \geq 1$.*

Proof. Consider the bottom half $\mathcal{S}_n^{lt\downarrow}$ of $\mathcal{S}_n^{lt}$ delimited by Φ and by the upper parent curve of Φ. r^n is a bijective continuous mapping from $\mathcal{S}_n^{lt\downarrow}$ to the top $r^n\big(\mathcal{S}_n^{lt\downarrow}\big)$ of $\mathcal{T}_1$ delimited by Φ, and the inverse mapping is continuous also. Since $r^n\big(\mathcal{S}_n^{lt\downarrow}\big)$ is homeomorphic to a closed disk and since $r^n\big(\mathcal{S}_n^{lt\downarrow}\big) \supset \mathcal{S}_n^{lt\downarrow}$, $\mathcal{S}_n^{lt\downarrow}$ contains a fixed point of r^n by the Brouwer Theorem. For $n \geq 2$, $\bigcup_{k=1}^{n-1} r^k\big(\mathcal{S}_n^{lt\downarrow}\big)$ doesn't intersect $\mathcal{S}_n^{lt\downarrow}$: all fixed points of r^n in $\mathcal{S}_n^{lt\downarrow}$ have thus order n. $\square$

The same argument is valid for $\mathcal{S}_n^{rt\downarrow}$. For $\mathcal{S}_n^{lb}$ and $\mathcal{S}_n^{rb}$ the fixed point can be a class of infinite triangles (we will show that it is always such a class). For $n = 1$ there is exactly one fixed point of r in $\mathcal{S}_1^{lt\downarrow}$, $\mathcal{S}_1^{rt\downarrow}$, $\mathcal{S}_1^{lb}$ and $\mathcal{S}_1^{rb}$: the triangle class with coordinates $\big(\frac{6-\sqrt{5}}{4}, \frac{8\sqrt{5}-17}{64}\big)$, the heptagonal class, $\Pi_{\omega_{38}}$ and Π_{90°, respectively. $(39.952203015767141115\ldots^\circ, 18.346346518943955680\ldots^\circ)$ in $\mathcal{S}_3^{lt\downarrow}$ is for example a 3–periodic triangle class. All computations in this section were done with 1000–digit precision.

The following construction generates all cycles for classes of finite triangles in $\mathcal{T}_1$, as we will show in Section 11: take any fractal ancestor copy $\mathcal{C}$ of $\mathcal{P}_1 \setminus \alpha$–axis that is included in $\mathcal{P}_1$ and not bordered by the α–axis; the outer layer of $\mathcal{C}$ belongs to $\mathcal{A}_{n+1}$ for some unique $n \geq 1$; cut away the part of $\mathcal{C}$ beyond the ancestor curve of Φ through $\mathcal{C}$ that is as far as possible from and n generations older than the ancestor curve of Φ bordering $\mathcal{C}$ (this ancestor may be Φ here); denote by $\mathcal{R}$ the rest of $\mathcal{C}$; take the smallest integer $N \geq 1$ with $r^N(\mathcal{R}) \supset \mathcal{R}$: one has $N \leq n$ since $r^n(\mathcal{R}) \supset \mathcal{R}$; r^N is then a bijective continuous mapping from $\mathcal{R}$ to $r^N(\mathcal{R}) \supset \mathcal{R}$ with continuous inverse and there is at least one N–cycle as in the proof of Theorem 15 since $\bigcup_{k=1}^{N-1} r^k(\mathcal{R})$ doesn't intersect $\mathcal{R}$ if $N \geq 2$. This N–cycle is unique and the same cycle is generated in this way by infinitely many different fractal copies of $\mathcal{P}_1 \setminus \alpha$–axis in $\mathcal{P}_1$ (see Section 11).

$$\begin{array}{c}(25.478876347440316089\ldots^\circ, 3.6818528532788970876\ldots^\circ)\\(62.431567122689586325\ldots^\circ, 12.276789619498866686\ldots^\circ)\\(32.460249346540695688\ldots^\circ, 24.998279789538063086\ldots^\circ)\end{array}$$

is a 3–cycle not leaving $\mathcal{S}_1$.

$$\begin{array}{c}(37.865926747917574986\ldots^\circ, 18.061811244908607526\ldots^\circ)\\(10.468235814868372615\ldots^\circ, 4.8401011494351450701\ldots^\circ)\\(48.638604189899250723\ldots^\circ, 22.211186045240131467\ldots^\circ)\end{array}$$

is a 3–cycle of triangle classes in order in $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$ and $\mathcal{S}_1^{rt}$.

$$(42.090874141099660640\ldots^\circ, 15.557122843876427568\ldots^\circ)$$
$$(1.2635523114915185243\ldots^\circ, 0.8247788078196525102\ldots^\circ)$$
$$(6.3075862480243139879\ldots^\circ, 4.1172394455012728648\ldots^\circ)$$
$$(30.390568589226577771\ldots^\circ, 19.803092968967591208\ldots^\circ)$$

is a 4–cycle of triangle classes in order in $\mathcal{S}_3^{lt}$, $\mathcal{P}_3$, $\mathcal{P}_2$ and $\mathcal{S}_1^{lt}$.

$$(37.247939372886625265\ldots^\circ, 19.189939461450692321\ldots^\circ)$$
$$(10.723421490339811872\ldots^\circ, 4.2741308209904622975\ldots^\circ)$$
$$(49.920751710266512618\ldots^\circ, 19.633287363391045768\ldots^\circ)$$
$$(30.697646461742403045\ldots^\circ, 17.370185973399543132\ldots^\circ)$$

is a 4–cycle of triangle classes in order in $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$, $\mathcal{S}_1^{rt}$ and $\mathcal{S}_1^{lt}$.

$$(37.630255649010598209\ldots^\circ, 18.570369773326372964\ldots^\circ)$$
$$(10.420573639194774736\ldots^\circ, 4.5115591822140415293\ldots^\circ)$$
$$(48.550547727001821453\ldots^\circ, 20.765781310885329500\ldots^\circ)$$
$$(32.363595430957208503\ldots^\circ, 16.384331092939721789\ldots^\circ)$$
$$(30.729181801658592737\ldots^\circ, 17.688152298022029834\ldots^\circ)$$

is a 5–cycle of triangle classes in order in $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$, $\mathcal{S}_1^{rt}$, $\mathcal{S}_1^{lt}$ and $\mathcal{S}_1^{lt}$.

$$(37.930269796367360642\ldots^\circ, 18.102923174699484745\ldots^\circ)$$
$$(10.135362642153623417\ldots^\circ, 4.6659841044983596966\ldots^\circ)$$
$$(47.278732653265572140\ldots^\circ, 21.526719537744220795\ldots^\circ)$$
$$(32.908073875879027270\ldots^\circ, 15.212876460421699178\ldots^\circ)$$
$$(27.941680542770112113\ldots^\circ, 18.655538982479742580\ldots^\circ)$$
$$(48.659125226707857104\ldots^\circ, 22.220242130215287975\ldots^\circ)$$

is a 6–cycle of triangle classes in order in $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$, $\mathcal{S}_1^{rt}$, $\mathcal{S}_1^{lt}$, $\mathcal{S}_1^{lt}$ and $\mathcal{S}_1^{rt}$.

$$(39.305662309899846302\ldots^\circ, 17.677017538458936691\ldots^\circ)$$
$$(5.7747047491290930782\ldots^\circ, 2.8485972409982163053\ldots^\circ)$$
$$(28.121014496985812289\ldots^\circ, 13.853288022731393651\ldots^\circ)$$
$$(36.786251566382858823\ldots^\circ, 31.096467455697263241\ldots^\circ)$$
$$(66.202454138266987877\ldots^\circ, 11.299882269350171350\ldots^\circ)$$
$$(38.901818026182387037\ldots^\circ, 25.434990337954490686\ldots^\circ)$$
$$(30.886718722856714101\ldots^\circ, 7.0225504408166614203\ldots^\circ)$$

is a 7–cycle of triangle classes in order in $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$, $\mathcal{S}_1^{lt}$, $\mathcal{S}_1^{rt}$, $\mathcal{S}_1^{rb}$, $\mathcal{S}_1^{rt}$ and $\mathcal{S}_1^{lb}$.

$$(38.468777685667500548\ldots^\circ, 18.102890974997997195\ldots^\circ)$$
$$(8.0151057516993356943\ldots^\circ, 3.7150704462974721546\ldots^\circ)$$
$$(38.254172619328622821\ldots^\circ, 17.649186686577651211\ldots^\circ)$$
$$(9.7328922219345150314\ldots^\circ, 4.7538361797984130640\ldots^\circ)$$
$$(45.519097683522135284\ldots^\circ, 22.022284341558206040\ldots^\circ)$$
$$(31.297303214442445113\ldots^\circ, 13.020261718008364724\ldots^\circ)$$
$$(28.711664232298528730\ldots^\circ, 25.939377664641886290\ldots^\circ)$$
$$(66.695344715752296964\ldots^\circ, 8.3394888580526813797\ldots^\circ)$$

is a 8–cycle of triangle classes in order in $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$, $\mathcal{S}_2^{lt}$, $\mathcal{P}_2$, $\mathcal{S}_1^{rt}$, $\mathcal{S}_1^{lt}$, $\mathcal{S}_1^{lt}$ and $\mathcal{S}_1^{rb}$.

As for $\mathcal{S}_n^{lb}$ and $\mathcal{S}_n^{rb}$, any fractal ancestor copy $\mathcal{C}$ of $\mathcal{P}_1$ that is bordered by the α–axis and included in $\mathcal{P}_1$ is covered for the first time by $r^n(\mathcal{C})$ for some $n \geq 1$; r^n is then a bijective continuous mapping from $\mathcal{C}$ to $r^n(\mathcal{C}) \supset \mathcal{C}$ with continuous inverse and – since $\bigcup_{k=1}^{n-1} r^k(\mathcal{C})$ doesn't intersect $\mathcal{C}$ for $n \geq 2$ – there is at least one n–cycle. We show in Section 11 that this n–cycle is unique and consists of classes of infinite triangles and that each cycle of such classes can be generated in this way by infinitely many different fractal copies of $\mathcal{P}_1$ bordered by the α–axis in $\mathcal{P}_1$.

Theorem 16. *$\mathcal{T} \setminus \big(\mathcal{A} \cup \mathcal{D}\big)$ is totally path–disconnected if $\mathcal{T} = \big\{(\alpha, \beta) \mid 0° \leq \beta \leq \alpha \leq 90° - \frac{\beta}{2}\big\}$.*

Proof. Otherwise some fixed continuous curve between two different points of $\mathcal{T} \setminus \big(\mathcal{A} \cup \mathcal{D}\big)$ would be included in each member of an infinite nested family of shrinking fractal ancestor copies of $\mathcal{P}_1$ or of $\mathcal{P}_1 \setminus \alpha$–axis whose diameters tend to 0, a contradiction. □

11. Reflection triangles as symbolic dynamics

We use the notations of Section 9. Referring to Figure 36, which is based on Figure 8, we code a class Π_α of infinite triangles by the infinite sequence $x = x_1x_2x_3\dots$ of digits $x_k \in \{0,1,2\}$ giving the position of α in "base 3" with respect to the fractal subdivision of $[0°, 90°]$ induced by the monotonicity intervals of $\rho|_\Gamma$ and its iterates. If x is eventually periodic we overline the period's digits. We identify the ends $0\overline{2}$ and $1\overline{0}$ as well as $1\overline{2}$ and $2\overline{0}$. For a class x of infinite triangles or for the zero sequence x coding $\mathcal{O}$, the reflection class $r(x)$ is then given by a left shift when $x_1 = 0$ or 2 and a left shift with permutation $0 \leftrightarrow 2$ in x when $x_1 = 1$. Note that $r^n(x) = x_{n+1}\dots$ or $r^n(x) = (x_{n+1}\dots)_{0\leftrightarrow 2}$ according as $x_1 \dots x_n$ contains an even or odd number of 1's.

One has $\mathcal{O} = \overline{0}$, $\Pi_{\omega_{21}} = 1\overline{0}$, $\Pi_{\omega_{38}} = \overline{1}$, $\Pi_{\omega_{52}} = 2\overline{0}$, and $\Pi_{\pi/2} = \overline{2}$. The lexicographic order of two sequences is the same as the order $\alpha < \beta \leq \frac{\pi}{2}$ for the corresponding infinite triangles Π_α and Π_β. The parents of x are $0x$, $2x$ and $1x_{0\leftrightarrow 2}$ (if one neglects the parents of $x = \mathcal{O}$ that are classes of finite triangles). A sequence is an ancestor of $\overline{0}$ ($\overline{0}$ included) if and only if it contains an even number of 1's with end $\overline{0}$ or an odd number of 1's with end $\overline{2}$. A sequence is an ancestor of $\overline{2}$ ($\overline{2}$ included) if and only if it contains an even number of 1's with end $\overline{2}$ or an odd number of 1's with end $\overline{0}$. The three 2–cycles are generated by $\overline{02} = \Pi_{\omega_{12}}$, $\overline{0121}$ and $\overline{1012}$. (See the discussion after Theorem 8.)

x generates a periodic orbit if and only if the sequence is periodic: if $r^n(x) = x$ for some $n \in \mathbf{N}\setminus\{0\}$, one has indeed $x = \overline{x_1 \dots x_n}$ or $x = \overline{x_1 \dots x_n(x_1 \dots x_n)_{0\leftrightarrow 2}}$ according as $x_1 \dots x_n$ contains an even or odd number of 1's; conversely, if $x = \overline{x_1 \dots x_n}$, one has $r^n(x) = x$ or $r^n(x) = x_{0\leftrightarrow 2}$ and thus $r^n(x) = x$ or $r^{2n}(x) = x$: the orbit is periodic. x generates an infinite forward orbit if and only if the sequence never becomes periodic.

We code a class of non-acute nondegenerate triangles (*i.e.*, a class of $\mathcal{T}_1$) by a nonempty sequence $z = w_1y_1w_2y_2\dots$ of digits $w_k \in \mathbf{N} \setminus \{0\}$ and $y_k \in$

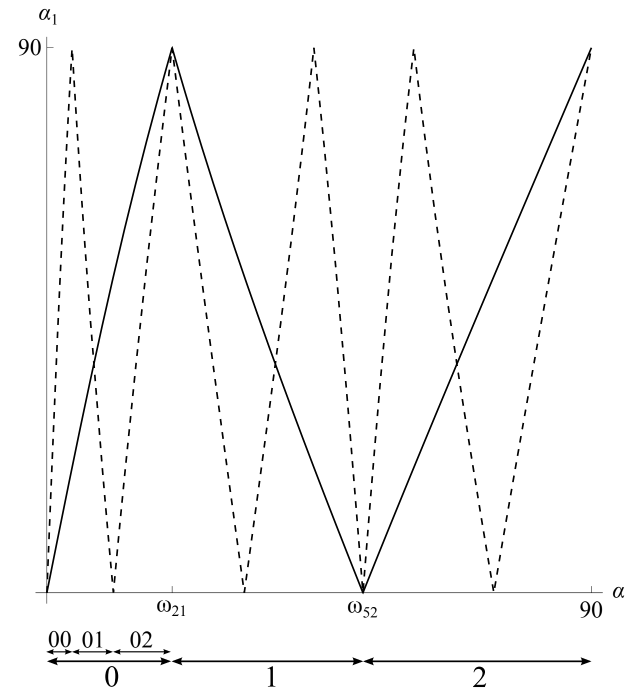

Figure 36. Fractal subdivision of $[0°, 90°]$ induced by $\Pi_{\alpha_1} = r(\Pi_\alpha)$ and its iterates

$\{D, E, i, ii, iii, iv\}$ with the following property: if z is a finite sequence, it ends with E – for "exterior" – or with D – for "becoming degenerate" – and these are the only occurrences of D and E. At each zooming stage k (see Figures 34 and 35), w_k numerates the side–by–side fractal copies of $\mathcal{P}_1$ or $\mathcal{P}_1 \setminus \alpha$–axis (starting from the border in $\mathcal{A}_k$) and y_k locates the triangle class in this copy: D if the triangle class is on the midline that becomes eventually degenerate, E if it is in one of the two components of $\mathcal{A}$ bordering this copy and i, ii, iii or iv if it is in the bottom right, top right, top left or bottom left inside quarter (without midline), respectively. The triangle classes of $\mathcal{S}_2^{lb}$ correspond for example to sequences beginning with $1iv2\ldots$

All triangle classes on the same midline section are thus coded identically, as are the triangle classes in the components of $\mathcal{A}$ bordering the same copy. The infinite sequences z containing neither ii nor iii code the classes of infinite triangles that don't become degenerate. An infinite sequence z containing only $y_k \in \{i, iv\}$ for all $k > k_0$ after a last $y_{k_0} \in \{ii, iii\}$ is identified with the finite sequence obtained by putting $y_{k_0+1} = D$: for example $1i1ii\overline{1i1iv} = 1i1ii1D$. The preceding sentence is also true if one interchanges i, iv with ii, iii. An infinite sequence z containing only $y_k \in \{ii, iii\}$ is identified with $w_1 D$. Triangle classes ending in E or D have two representations when they are on the curve separating quarter i from ii or iii from iv at the last stage: $(60°, 15°)$ is for example $1i1D$ or $1ii1D$.

Classes of infinite triangles ending in $\overline{1i}$ – except $z = \overline{1i}$ – or in $\overline{1iv}$ have a second representation ending in $\overline{1iv}$ or $\overline{1i}$, respectively.

We consider the following involutive permutations of $\{i, ii, iii, iv\}$: σ_i is the identity, σ_{ii} interchanges $i \leftrightarrow ii$ and $iii \leftrightarrow iv$, σ_{iii} interchanges $i \leftrightarrow iii$ and $ii \leftrightarrow iv$ and σ_{iv} interchanges $i \leftrightarrow iv$ and $ii \leftrightarrow iii$. These permutations form a dihedral group $C_2 \times C_2$ under composition – with $\sigma_{iii} \circ \sigma_{ii} = \sigma_{ii} \circ \sigma_{iii} = \sigma_{iv}$, and cyclically. The reflection class $r(z)$ is then given by the following transformation of $z = w_1 y_1 w_2 y_2 \cdots \in \mathcal{T}_1$:

(1) if $w_1 > 1$, $r(z) = (w_1 - 1) y_1 w_2 y_2 \ldots$,
(2) $r(1E) =$ acute triangle outside $\mathcal{T}_1$,
(3) $r(1D) =$ degenerate triangle $(0°, 0°)$ outside $\mathcal{T}_1$,
(4) $r(1 y_1 w_2 y_2 \ldots) = \sigma_{y_1}(w_2 y_2 \ldots)$ for $y_1 \in \{i, ii, iii, iv\}$ except when all $y_k \in \{ii, iii\}$ (then $1 y_1 w_2 y_2 \cdots = 1D$).

The parents of z are $(w_1 + 1) y_1 \ldots$, $1y\, \sigma_y(z)$ for $y = i, iv$, and – if z codes a class of proper triangles – $1y\, \sigma_y(z)$ for $y = ii, iii$. An infinite triangle tends to $\Pi_{\pi/2}$ under iteration of r if and only if its code ends in $\overline{1i}$ or in $\overline{1iv}$. The fixed points of r in $\mathcal{T}_1$ are the heptagonal class $\overline{1ii1i}$, the triangle class $\overline{1iii1i}$ with coordinates $\left(\frac{6-\sqrt{5}}{4}, \frac{8\sqrt{5}-17}{64}\right)$, $\Pi_{\omega_{38}} = \overline{1iv1i}$ and $\Pi_{\pi/2} = \overline{1i}$.

If $r^n(z)$ causes a left shift of $2m$ digits, $m \geq 1$, one has

$$r^n(z) = \sigma_y\big((w_{m+1} - \nu) y_{m+1} w_{m+2} y_{m+2} \ldots\big)$$

where $y \in \{i, ii, iii, iv\}$ is given by $t_1 = y_1$, $t_{k+1} = \sigma_{\sigma_{t_k}(y_{k+1})}(t_k)$ for $1 \leq k \leq m-1$, $y = t_m$ and where ν is an integer $\in [0, w_{m+1} - 1]$; one has $n = \nu + \sum_{k=1}^{m} w_k$.

Theorem 17. (1) *The following situations are equivalent:*

(a) $n \geq 1$, *$r^n(z) = z$ and $r^n(z)$ causes a left shift of $2m$ digits.*
(b) $m \geq 1$,

$$z = (w_1 - \nu) y_1 w_2 y_2 \ldots w_m y_m \overline{\sigma_y(w_1 y_1 w_2 y_2 \ldots w_m y_m)\, w_1 y_1 w_2 y_2 \ldots w_m y_m} \tag{26}$$

for some integer $\nu \in [0, w_1 - 1]$, $n = \sum_{k=1}^{m} w_k$ and y is given by $t_1 = y_1$, $t_{k+1} = \sigma_{\sigma_{t_k}(y_{k+1})}(t_k)$ for $1 \leq k \leq m-1$, $y = t_m$.

The forward (periodic) orbit of z is then generated by

$$r^{n-\nu}(z) = \overline{w_1 y_1 w_2 y_2 \ldots w_m y_m\, \sigma_y(w_1 y_1 w_2 y_2 \ldots w_m y_m)} \tag{27}$$

and the sequence z is periodic if and only if $\nu = 0$.
(2) *For each integer $N \geq 1$ the number of N–periodic orbits is finite and nonzero for both finite and infinite triangles.*
(3) *A triangle class of $\mathcal{T}_1$ belongs to the backward orbit of a periodic orbit of $\mathcal{T}_1$ if and only if its sequence z is eventually periodic and contains infinitely many $y_k \in \{i, iv\}$.*
(4) *A triangle class of $\mathcal{T}_1$ belongs to an infinite divergent forward orbit if and only if its sequence z is infinite, never becomes periodic and contains either no $y_k \in \{ii, iii\}$ or both infinitely many $y_k \in \{ii, iii\}$ and infinitely many $y_k \in \{i, iv\}$.*

(5) *Every periodic orbit in $\mathcal{T}_1$ is repelling. An infinite forward orbit in $\mathcal{T}_1$ is thus never asymptotically periodic.*

Proof. (1) By setting $r^n(z) = z$ in the paragraph preceding the theorem.

(2) There is at least one N–periodic orbit for classes of both finite and infinite triangles by Theorems 15 and 8 and there are finitely many sequences with the necessary form (27) for $n = N$.

(3) The ancestors of a generator (27) of a periodic orbit are eventually periodic sequences.

Conversely, if the sequence z is eventually periodic, $r^{k_0}(z)$ is periodic for infinitely many k_0. Fix such a k_0 with $r^{k_0}(z) = \overline{w_1 y_1 w_2 y_2 \ldots w_M y_M}$. If $n = \sum_{k=1}^{M} w_k$ the sequences $r^{k_0+\ell n}(z)$, $\ell \in \mathbf{N}$, are then $\sigma_{\tilde{y}_\ell}\big(r^{k_0}(z)\big)$ for some $\tilde{y}_\ell \in \{i, ii, iii, iv\}$: since there are two equal $\tilde{y}_\ell$ in $\{\tilde{y}_0, \ldots, \tilde{y}_4\}$ some descendant $r^{k_0+\ell n}(z)$ of z with $0 \le \ell \le 3$ generates a periodic orbit.

(4) follows from the preceding results.

(5) We already know that the fixed point $\Pi_{\pi/2}$ is repelling. Consider another periodic orbit: it is generated as in (27) by a sequence

$$z_0 = \overline{w_1 y_1 w_2 y_2 \ldots w_m y_m \, \sigma_y(w_1 y_1 w_2 y_2 \ldots w_m y_m)}$$

with $r^n(z_0) = z_0$ for $n = \sum_{k=1}^{m} w_k$ that corresponds to the triangle class $\Delta_0 \neq \Pi_{\pi/2}$. Let $\varepsilon > 0$ be the distance between Δ_0 and $\mathcal{A}_n$ in $\mathcal{T}_1$. We consider $\Delta \neq \Delta_0$ in the ε–neighborhood of Δ_0 with sequence z. If z is finite, some descendant of Δ will be degenerate or acute, *i.e.*, some $r^{\ell_0 n}(\Delta)$ will be outside the ε–neighborhood of $r^{\ell_0 n}(\Delta_0) = \Delta_0$. If $\Delta \notin \mathcal{A} \cup \mathcal{D}$ the first $2m$ digits of z and z_0 coincide. Let $k_0 \in \,]\ell_0 m, (\ell_0 + 1)m]$ with $\ell_0 \in \mathbf{N} \setminus \{0\}$ be the index of the first different digit. Then $r^{\ell_0 n}(z)$ and $r^{\ell_0 n}(z_0) = z_0$ differ in one of the first $2m$ digits: $r^{\ell_0 n}(\Delta)$ is outside the ε–neighborhood of $r^{\ell_0 n}(\Delta_0) = \Delta_0$, the periodic orbit is repelling. $\square$

Note that the forward orbit of z in (27) may have less than n points, even if $y \neq i$: for example

$$\overline{1i1ii1ii1i1i1ii \, \sigma_y(1i1ii1ii1i1i1ii)} = \overline{1i1ii1ii1i} = \overline{1i1ii \, \sigma_{\tilde{y}}(1i1ii)}$$

since $y = \tilde{y} = ii$.

We now analyze the construction of Section 10. Suppose without restricting the generality that a cycle is generated by (26). This cycle contains a triangle class of $\mathcal{P}_1$ beginning with $1 y_1 w_2 y_2 \ldots w_m y_m w_1$. Take the fractal copy $\mathcal{C}$ of $\mathcal{P}_1$ with this address (conversely, the address $1 y_1 w_2 y_2 \ldots w_m y_m w_1$ of any fractal copy $\mathcal{C}$ of $\mathcal{P}_1$ in $\mathcal{P}_1$ can be chosen as begin of (26)). Suppose that the cycle is a N–cycle with $2M$ shifts: note that $N = \sum_{k=1}^{M} w_k$ divides $n = \sum_{k=1}^{m} w_k$. The given cycle (26) is then generated by

$$z = 1 y_1 w_2 y_2 \ldots w_M y_M \, \overline{\sigma_{\tilde{y}}(w_1 y_1 w_2 y_2 \ldots w_M y_M) \, w_1 y_1 w_2 y_2 \ldots w_M y_M}$$

and is exactly the cycle generated by $\mathcal{C}$ in the construction of Section 10 since the addresses of $r^k(\mathcal{C})$, $0 \le k \le N-1$, are correct: there is thus only one such cycle. The same cycle is also generated by

$$1\,\underbrace{y_1 w_2 y_2 \dots w_m y_m \sigma_y(w_1 y_1 w_2 y_2 \dots w_m y_m)\, w_1 y_1 w_2 y_2 \dots w_m y_m}_{\text{head}}\,\overline{\sigma_y(w_1\text{head}) w_1 \text{head}}$$

and the segment $\sigma_y(w_1 y_1 w_2 y_2 \dots w_m y_m)\, w_1 y_1 w_2 y_2 \dots w_m y_m$ can be concatenated any finite number of times in the head: this gives addresses of infinitely many nested $\mathcal{C}$ generating the same cycle. The more concatenations of this segment the head contains, the more the starting $\mathcal{C}$ and its first $N-1$ descendants $r(\mathcal{C}),\ \dots,\ r^{N-1}(\mathcal{C})$ converge to the orbit points.

The three 2–cycles for classes of infinite triangles are generated by $\overline{2i} = \Pi_{\omega_{12}}$, $\overline{2iv2i}$ and $\overline{1i1iv1iv1i}$. In the same order as in Section 6 the seven 2–cycles for classes of finite triangles are generated by $\overline{2iii2i}$, $\overline{1iii1ii1iv1i}$, $\overline{1i1ii1ii1i}$, $\overline{2ii2i}$, $\overline{1iii1iv1ii1i}$, $\overline{1ii1iii1iv1i}$ and $\overline{1iii1iii1i1i}$.

Table 1 contains the fundamental periods of periodic generators of the 40 different 3–cycles. The explicit 3–cycles of Section 10 are generated in order by $\overline{3iii3i}$, $\overline{1i1iii1ii1ii1iv1i}$ and $\overline{2ii1iv2iii1i}$. The explicit 8–cycle is generated by

$$\overline{2iii2iv1ii1iv1iv1ii\,\sigma_{iv}(2iii2iv1ii1iv1iv1ii)} = (8.015\dots^\circ, 3.715\dots^\circ).$$

Theorem 18. *Under the reflection map r, there are in $\mathcal{T}_1$ uncountably many disjoint infinite forward orbits of classes of both finite and infinite triangles.*

Proof. The infinite sequence $z = \underbrace{1i}_{1\times}\,1ii1i\,\underbrace{1i1i}_{2\times}\,1ii1i\,\underbrace{1i1i1i}_{3\times}\dots$ codes a class of finite triangles with unique representation and generates an infinite forward orbit in $\mathcal{P}_1$: z begins indeed with one copy of $1i$, $r^3(z)$ with two copies, $r^7(z)$ with three copies, $r^{12}(z)$ with four copies and so on. The backward orbit of this (and of every) infinite forward orbit is countable. One can thus replace the occurrence numbers $1, 2, 3, 4, \dots$ of $1i$ in the successive groups (separated by $1ii1i$) by the successive digits of uncountably many irrational numbers in such a way that all generated forward orbits, which are infinite, are disjoint. By replacing ii by iv in z one gets an infinite orbit of infinite triangles. Note that one can also consider the infinite triangle $x = 021102021102020211\dots$ □

$z = w_1 y_1 w_2 y_2 \dots$ with $w_k = k$ for all $k \geq 1$ and $(y_k)_{k\geq 1} = \overline{i, ii, iii, iv}$ generates an infinite forward orbit of classes of finite triangles, too (with accumulation point $\mathcal{O}$).

Theorem 19. *$\mathcal{A}$ is a dense open subset of $\mathcal{T} = \{(\alpha,\beta) \mid 0^\circ \leq \beta \leq \alpha \leq 90^\circ - \frac{\beta}{2}\}$. Any neighborhood of a point of $\mathcal{T} \setminus \mathcal{A}$ intersects countably many periodic orbits and uncountably many disjoint divergent forward orbits; the rest of the neighborhood consists of uncountably many points of $\mathcal{D}$, uncountably many points of $\mathcal{A}$ and countably many other points that become eventually periodic.*

Proof. A point of $\mathcal{A}_n$, $n \in \mathbf{N}$, has some neighborhood in $\mathcal{A}_{n-1} \cup \mathcal{A}_n$ if one sets $\mathcal{A}_{-1} = \mathcal{A}_0$. The rest follows from the fact that every neighborhood of a point of $\mathcal{T} \setminus \mathcal{A}$ contains (infinitely many) fractal copies of $\mathcal{P}_1 \setminus \alpha$–axis: take such a copy and let $w_1 y_1 \dots w_M$ be its address; this head can be prolonged to get the given number

$\overline{1i1i1ii1ii1ii1i}$	:	$(77.992, 5.4261)$,	$(61.422, 14.969)$,	$(32.443, 31.271)$	
$\overline{1i1i1iii1iii1iii1i}$	:	$(77.137, 5.2765)$,	$(59.569, 14.357)$,	$(28.774, 27.798)$	
$\overline{1i1i1iv1iv1iv1i}$	:	$(78.072, 0.0000)$,	$(62.116, 0.0000)$,	$(24.228, 0.0000)$	$\longleftarrow$
$\overline{1i1ii1i}$	:	$(77.538, 5.7639)$,	$(60.431, 15.811)$,	$(32.623, 31.329)$	
$\overline{1i1ii1iii1iii1iv1i}$	:	$(64.111, 12.369)$,	$(35.593, 26.667)$,	$(24.946, 3.6423)$	
$\overline{1i1ii1iv1iv1iii1i}$	:	$(76.703, 5.5669)$,	$(58.628, 15.057)$,	$(28.720, 27.668)$	
$\overline{1i1iii1i}$	:	$(63.351, 7.1712)$,	$(31.121, 13.087)$,	$(29.183, 26.185)$	
$\overline{1i1iii1ii1ii1iv1i}$	:	$(62.432, 12.277)$,	$(32.460, 24.998)$,	$(25.479, 3.6819)$	
$\overline{1i1iii1iv1iv1ii1i}$	:	$(61.950, 7.8678)$,	$(28.710, 13.651)$,	$(35.158, 30.232)$	
$\overline{1i1iv1i}$	:	$(69.448, 0.0000)$,	$(41.773, 0.0000)$,	$(26.919, 0.0000)$	$\longleftarrow$
$\overline{1i1iv1ii1ii1iii1i}$	:	$(63.532, 6.4266)$,	$(31.070, 11.546)$,	$(29.295, 26.539)$	
$\overline{1i1iv1iii1iii1ii1i}$	:	$(62.337, 6.7728)$,	$(28.805, 11.523)$,	$(34.391, 30.234)$	
$\overline{1ii1i1iii1iv1iii1i}$	:	$(45.654, 26.671)$,	$(50.356, 18.460)$,	$(27.406, 16.857)$	
$\overline{1ii1i1iv1iii1iv1i}$	:	$(53.737, 20.669)$,	$(39.419, 24.424)$,	$(25.777, 6.4984)$	
$\overline{1ii1iii1i}$	:	$(39.242, 25.752)$,	$(34.431, 7.9491)$,	$(28.909, 15.731)$	
$\overline{1ii1iii1ii1i1iv1i}$	:	$(36.334, 27.318)$,	$(36.353, 5.6518)$,	$(29.022, 10.379)$	
$\overline{1ii1iv1i}$	:	$(53.953, 18.520)$,	$(33.152, 22.790)$,	$(26.294, 5.8906)$	
$\overline{1ii1iv1ii1i1iii1i}$	:	$(47.415, 21.434)$,	$(32.805, 15.325)$,	$(28.339, 18.663)$	
$\overline{1iii1i1iv1ii1iv1i}$	:	$(31.077, 14.761)$,	$(32.071, 23.138)$,	$(34.174, 7.1741)$	
$\overline{1iii1ii1iii1i1iv1i}$	:	$(31.634, 24.738)$,	$(35.739, 5.4231)$,	$(30.966, 10.501)$	
$\overline{2i1i}$	:	$(17.076, 0.0000)$,	$(76.815, 0.0000)$,	$(59.165, 0.0000)$	$\longleftarrow$
$\overline{2i1ii2ii1i}$	:	$(14.181, 2.6145)$,	$(64.753, 11.579)$,	$(36.335, 25.097)$	
$\overline{2i1iii2iii1i}$	:	$(13.882, 2.5866)$,	$(63.564, 11.512)$,	$(34.082, 23.968)$	
$\overline{2i1iv2iv1i}$	:	$(15.623, 0.0000)$,	$(71.266, 0.0000)$,	$(46.083, 0.0000)$	$\longleftarrow$
$\overline{2ii1i}$	:	$(10.110, 5.3306)$,	$(46.937, 24.481)$,	$(43.827, 17.928)$	
$\overline{2ii1ii2i1i}$	:	$(8.2127, 6.0759)$,	$(38.467, 28.345)$,	$(50.539, 10.136)$	
$\overline{2ii1iii2iv1i}$	:	$(8.3357, 5.7834)$,	$(39.101, 27.004)$,	$(42.786, 9.3457)$	
$\overline{2ii1iv2iii1i}$	:	$(10.468, 4.8401)$,	$(48.639, 22.211)$,	$(37.866, 18.062)$	
$\overline{2iii1i}$	:	$(6.3707, 4.0384)$,	$(30.697, 19.423)$,	$(40.263, 15.637)$	
$\overline{2iii1ii2iv1i}$	:	$(6.4310, 4.7667)$,	$(30.826, 22.812)$,	$(43.381, 9.8110)$	
$\overline{2iii1iii2i1i}$	:	$(6.2608, 4.8616)$,	$(30.025, 23.283)$,	$(49.365, 10.426)$	
$\overline{2iii1iv2ii1i}$	:	$(6.2109, 4.2180)$,	$(29.925, 20.291)$,	$(44.884, 15.582)$	
$\overline{2iv1i}$	:	$(6.8623, 0.0000)$,	$(33.576, 0.0000)$,	$(49.511, 0.0000)$	$\longleftarrow$
$\overline{2iv1ii2iii1i}$	:	$(6.2470, 1.7815)$,	$(30.485, 8.6796)$,	$(36.590, 21.684)$	
$\overline{2iv1iii2ii1i}$	:	$(6.1133, 1.7573)$,	$(29.860, 8.5706)$,	$(38.323, 22.175)$	
$\overline{2iv1iv2i1i}$	:	$(6.4721, 0.0000)$,	$(31.739, 0.0000)$,	$(54.842, 0.0000)$	$\longleftarrow$
$\overline{3i}$	:	$(2.2468, 0.0000)$,	$(11.207, 0.0000)$,	$(53.139, 0.0000)$	$\longleftarrow$
$\overline{3ii3i}$	:	$(1.7190, 0.7883)$,	$(8.5745, 3.9322)$,	$(40.696, 18.552)$	
$\overline{3iii3i}$	:	$(1.6847, 0.7779)$,	$(8.4041, 3.8806)$,	$(39.952, 18.346)$	
$\overline{3iv3i}$	:	$(2.1646, 0.0000)$,	$(10.798, 0.0000)$,	$(51.375, 0.0000)$	$\longleftarrow$

Table 1. Periodic generators of the forty 3–cycles with their approximate angles (in °) and the approximate angles of their child and grandchild in order ($\longleftarrow$ denotes classes of infinite triangles)

of points of the desired type in the same copy; the given neighborhood cannot contain uncountably many eventually periodic triangles outside $\mathcal{A} \cup \mathcal{D}$ since their total number in $\mathcal{T}$ is countable. $\square$

One can construct codes z with almost any behavior under iteration of the reflection map, as for the sequences of pedal triangles [1]. We design for example a code z whose forward orbit is dense in $\mathcal{T} \setminus \mathcal{A}$: write all words $w_1 y_1 \dots w_. y_.$ of finite length with digits $w \in \mathbf{N} \setminus \{0\}$ and $y \in \{i, ii, iii, iv\}$; order these words by lexicographic order of the w's and then of the y's for each sum 1, 2, 3, … of the w's; concatenate the words and submit each of them in order to an appropriate permutation $\sigma_{i,ii,iii,iv}$ such that the original word will appear as head of the corresponding descendant of z.

Theorem 20. *The backward orbit of a class of proper triangles of $\mathcal{T} \setminus \mathcal{A}$ is dense in $\mathcal{T} \setminus \mathcal{A}$.*

Proof. Consider a class of proper triangles $\Delta_0 \in \mathcal{T} \setminus \mathcal{A}$ and suppose that $\Delta_0 \in \mathcal{P}_N \setminus \alpha$–axis. Fix a neighborhood of $\Delta \in \mathcal{T} \setminus \mathcal{A}$ and choose a fractal copy $\mathcal{C}$ of $\mathcal{P}_1 \setminus \alpha$–axis in this neighborhood. Take $n \geq 1$ such that r^n maps $\mathcal{C}$ bijectively to $\mathcal{P}_1 \setminus \alpha$–axis (such a n exists) and take the copy $\mathcal{C}' \subset \mathcal{C}$ that is the inverse image of $\mathcal{S}_N^{lb} \setminus \alpha$–axis under this mapping. r^{n+1} maps then $\mathcal{C}'$ bijectively to $\mathcal{P}_N \setminus \alpha$–axis: there is thus some $\Delta' \in \mathcal{C}'$ with $r^{n+1}(\Delta') = \Delta_0$. $\square$

Note that the backward orbit of the degenerate class contains the backward orbit of $I_{\pi/6}$ – and of every class Δ of proper triangles with $s(\Delta) = \frac{5}{4}$ – and is thus also dense in $\mathcal{T} \setminus \mathcal{A}$. If Δ_0 is a class of proper triangles outside $\mathcal{A} \cup \mathcal{D}$ with code z_0, the code of some ancestor of Δ_0 in a fixed neighborhood of $\Delta \in \mathcal{T} \setminus \mathcal{A}$ can be constructed as follows: take a fractal copy $\mathcal{C}$ of $\mathcal{P}_1 \setminus \alpha$–axis with address $w_1 y_1 \dots w_M$ in this neighborhood; take this address as head of a code z whose tail is z_0 and fill the space between head and tail with one i, ii, iii or iv in such a way that z_0 will appear as a descendant of z: the triangle class with code z is an ancestor of Δ_0 in the given neighborhood of Δ.

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