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On the Intersections of the Incircle and the Cevian Circumcircle of the Incenter

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Abstract. We give a characterization of the other point of intersection of the incircle with the circle passing through the feet of the internal angle bisectors, different from the Feuerbach point.

1. Introduction

The famous Feuerbach theorem states that the nine-point circle of a triangle is tangent to the incircle and to each of the excircles. Of particular interest is the tangency between the nine-point circle and the incircle, for it is this tangency point among the four that is a triangle center in the sense of Kimberling [5]. Thus, it is this point which was coined as the *Feuerbach point* of the triangle. Besides, its existence, being perhaps one of the first more difficult results that arise in triangle geometry, has been the subject of many discussions over the years, and consequently, many proofs, variations, and related results have appeared in the literature. A celebrated collection of such results is provided by Emelyanov and Emelyanova in [3]. In this note, we shall dwell on a particular theorem, for which they gave a magnificient synthetic proof in [2].

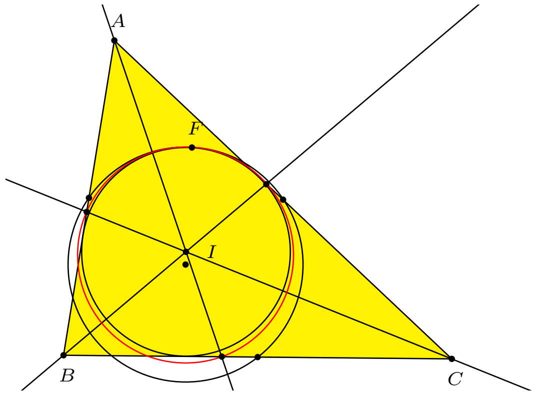

Figure 1

Theorem 1 (Emelyanov and Emelyanova)**.** *The circle through the feet of the internal angle bisectors of a given triangle passes through the Feuerbach point of the triangle.*

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We focus on the second intersection of the incircle with this cevian circumcircle of the incenter. Following an idea of Suceavă and Yiu [7], we give a natural characterization of this point in terms of the reflections of a given line in the sidelines of the cevian triangle of the incircle. We begin with some preliminaries on the *Poncelet point* of a quadrilateral and the *anti-Steiner point* of a line passing through the orthocenter of the triangle.

2. Preliminaries

In essence, the result that lies at the heart of the theory of anti-Steiner point is the following concurrency due to Collings [1].

Theorem 2 (Collings)**.** *If $\mathcal{L}$ is a line passing through the orthocenter H of a triangle ABC, then the reflections of $\mathcal{L}$ in the sides BC, CA, AB are concurrent on the circumcircle of ABC at a point called the* anti-Steiner point *of $\mathcal{L}$*.

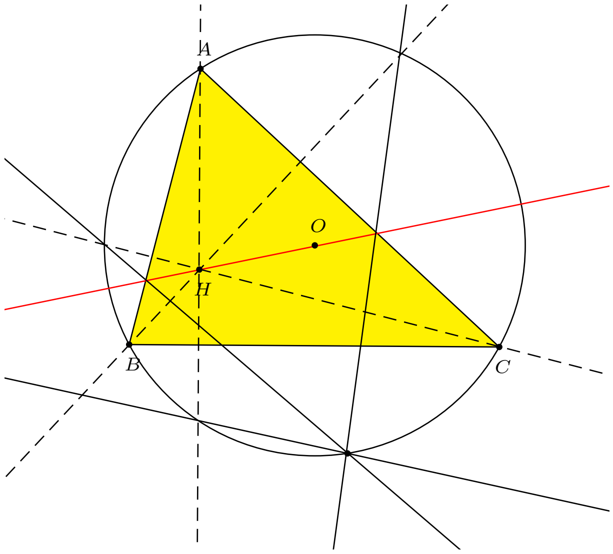

Figure 2

The proof for this is quite straightforward and it consists of a simple angle chasing (see [1] or [4]). It is also well-known that the orthocenter of the intouch triangle lies on the line determined by the circumcenter O and the incenter I of the triangle. This can be proved in many ways synthetically. The most beautiful approach however is by using inversion with respect to the incircle; we refer to [6] for this proof. Given this fact, it is natural now to ask about the anti-Steiner point of OI with reference to the intouch triangle. Suceavă and Yiu did this and obtained the following result.

Theorem 3 (Suceavă and Yiu)**.** *The reflections of the OI-line in the sides of the intouch triangle of ABC concur at the Feuerbach point of ABC.*

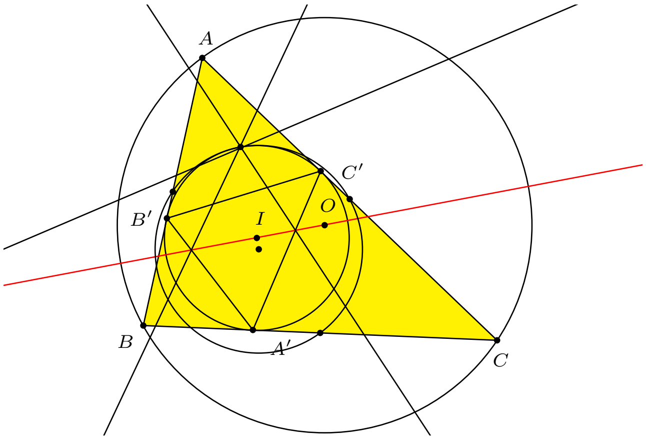

Figure 3

We proceed to give a geometric characterization of the "second" intersection of the cevian circumcenter of the incenter with the incircle, apart from the Feuerbach point.

3. The main result

Theorem 4. *Let I be the incenter of triangle ABC, and H_1 the orthocenter of cevian triangle $A_1B_1C_1$ of I. The anti-Steiner point of the line IH_1* (with respect to $A_1B_1C_1$) *is the "second" intersection of the incircle with the cevian circumcircle of I.*

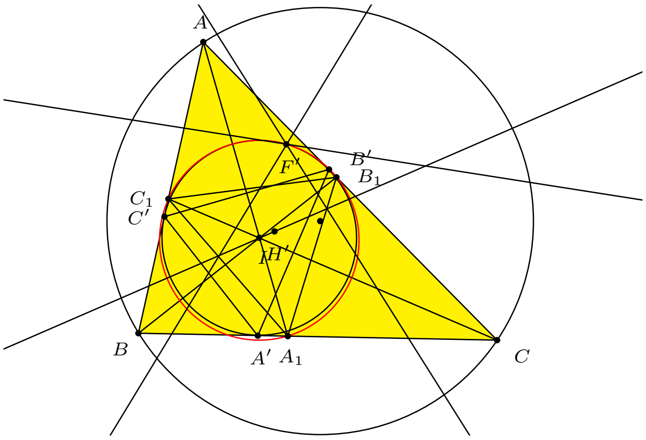

Figure 4

In other words, the anti-Steiner point of the line IH_1 with respect to triangle $A_1B_1C_1$ lies on the incircle of ABC. This is in general different from the Feuerbach point of ABC, unless the incircle and the cevian circumcircle of the incenter are tangent to one another.

We prove Theorem 4 synthetically, with the aid of a few lemmas. Lemma 5 provides more insight on the standard anti-Steiner point configuration.

Lemma 5. *Let P be a point in the plane of a given triangle ABC with orthocenter H. Let A_1, B_1, C_1 be the points where the lines AP, BP, and CP, intersect again the circumcircle. Furthermore, let A_2, B_2, C_2 be the reflections of P across the sidelines BC, CA, and AB, respectively. Then, the circumcircles of triangles ABC, PA_1A_2, PB_1B_2, and PC_1C_2 are concurrent at the anti-Steiner point of the line PH with respect to triangle ABC.*

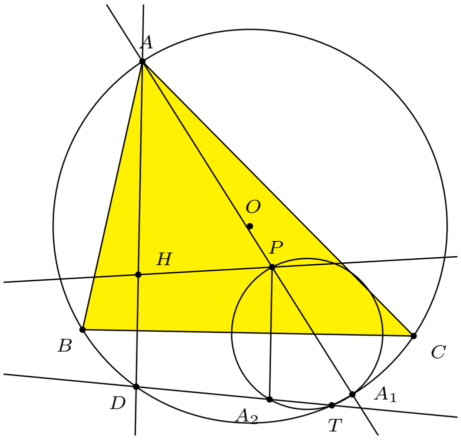

Figure 5

Proof. The line AH cuts the circumcircle of triangle ABC again at the reflection D of H across BC. Thus, the line DA_2 is the reflection of PH with respect to BC and intersects the circumcircle of triangle ABC again at the anti-Steiner point T of PH with respect to ABC. Since the directed angles

$$(TA_1, TA_2) = (TA_1, TD) = (AA_1, AD) = (PA_1, PA_2) \mod 180^\circ,$$

it follows that T lies on the circumcircle of PA_1A_2. Similarly, T lies on the circumcircles of triangles PB_1B_2 and PC_1C_2. $\square$

Lemma 6 is a property of Poncelet points of general quadrilaterals. By definition (see [4]), the Poncelet point T associated with the four points A, B, C, D is the concurrency point of 8 circles: the nine-point circles of triangles ABC, BCD, CDA, DAB, and the pedal circles of the points A, B, C, and D, with respect to the triangles BCD, CDA, DAB, and ABC, respectively.

Lemma 6. *Let P be a point in the plane of triangle ABC and $P_AP_BP_C$ its pedal triangle with respect to ABC. Let A', B', C' be the midpoints of the segments PA, PB, and PC, respectively, and let P_1, P_2, P_3 be the points where the lines PP_A, PP_B, PP_C meet again the pedal circle $P_AP_BP_C$. Then, the lines P_1A', P_2B', and P_3C' concur at a point on the pedal circle $P_AP_BP_C$.*

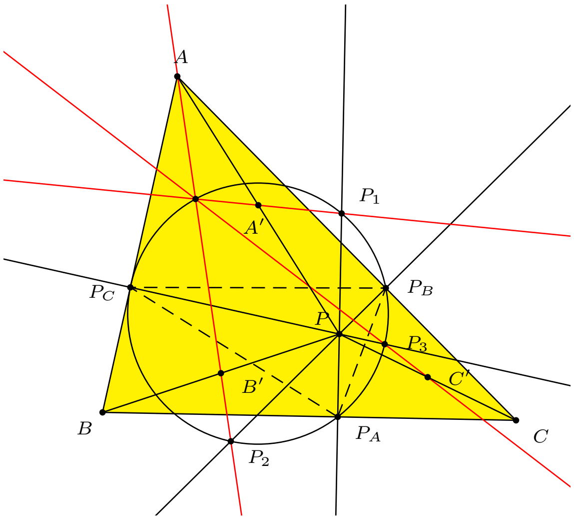

Figure 6

Proof. Let U be the Poncelet point of the quadrilateral $ABCP$. By definition, this point lies on the pedal circle of P with respect to triangle ABC. Now, let D be the second intersection of BC with the pedal circle $P_AP_BP_C$ and let R be the orthogonal projection of A on PC. We have that $URA'C'$ is the nine-point circle of triangle APC. Furthermore, we also get that

$$\begin{aligned}\angle DUC' &= \angle DUP_B - \angle C'UP_B\\ &= 180^\circ - \angle CPP_B - \angle PRP_B\\ &= \angle PAC - \angle CPP_B\\ &= \angle PAC - \angle RAC\\ &= 90^\circ - \angle APC.\end{aligned}$$

Thus,

$$\begin{aligned}\angle DUA' &= \angle DUC' + \angle C'UA'\\ &= 90^\circ - \angle APC + \angle APC\\ &= 90^\circ.\end{aligned}$$

Therefore, since $\angle DUP_1 = 90^\circ$, it follows that U lies on the line P_1A'. Similarly, P_2B' and P_3C' pass through the Poncelet point P. □

Finally, we prove the lemma which lies at the core of the proof of the main Theorem 4.

Lemma 7. *Given a triangle ABC with circumcenter O and medial triangle DEF, let P be a point with orthogonal projections P_1, P_2, P_3 on these sides. Let A' be the intersection of the lines EF and P_2P_3, and define B', C' cyclically. Then, the lines P_1A' P_2B' P_3C' concur at the intersection point U of the circumcircles $P_1P_2P_3$ and DEF that is different from the Poncelet point of A, B, C and P. Furthermore, U is the anti-Steiner point of the line OP with respect to the medial triangle DEF.*

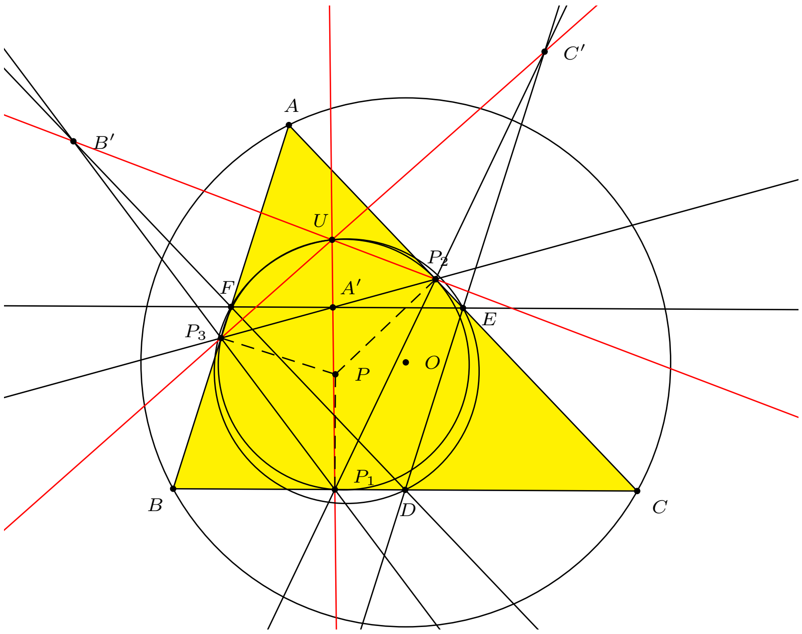

Figure 7

Proof. The orthogonal projection V of A on OP is clearly the second intersection of the circumcircles of the cyclic quadrilaterals PP_2AP_3 and $OEAF$ with diameters AP and AO, respectively. Also, note that V is the Miquel point of the complete quadrilateral bounded by the lines AB, AC, EF, and P_2P_3. Thus, it follows by the standard characterization of Miquel points that V lies on the circumcircle of $FA'P_3$.

On the other hand, let PP_1 intersect the circle AP_2P_3 again at T. Since AP is a diameter of AP_2P_3, $\angle ATP = 90°$, and AT is parallel to EF. In other words, EF is the perpendicular bisector of TP_1, and $\angle TAF = \angle AFE$. We have shown above V lies on the circumcircle of $FA'P_3$. Therefore, $\angle A'VP_3 = \angle AFE$, and A' lies on VT. Furthermore, since A' lies on the radical axis P_2P_3 of the circumcircles AP_2P_3 and $P_1P_2P_3$, it also follows that A' has equal powers with respect to AP_2P_3 and $P_1P_2P_3$. Consequently, if P_1A' cuts the circle $P_1P_2P_3$ again at U, then $TUVP_1$ is an isosceles trapezoid with bases UV and TP_1. Therefore, U is the reflection of V across EF. Finally, since the circumcircles AEF and DEF are symmetric with respect to EF, the point U, which lies on the circumcircle DEF, is the anti-Steiner point of OP with respect to triangle DEF. □

Now we conclude with a proof of Theorem 4.

Let DEF be the intouch triangle of ABC, and $A_0B_0C_0$ the antimedial triangle of DEF. Since the lines B_0C_0, C_0A_0, A_0B_0 are perpendicular to the lines IA, IB, IC respectively, the feet of the internal angle bisectors, A_1, B_1, C_1, are the poles of B_0C_0, C_0A_0, A_0B_0 with respect to the incircle (I). Therefore, by duality, the points A_0, B_0, C_0 are the poles of the lines B_1C_1, C_1A_1, A_1B_1 with respect to (I).

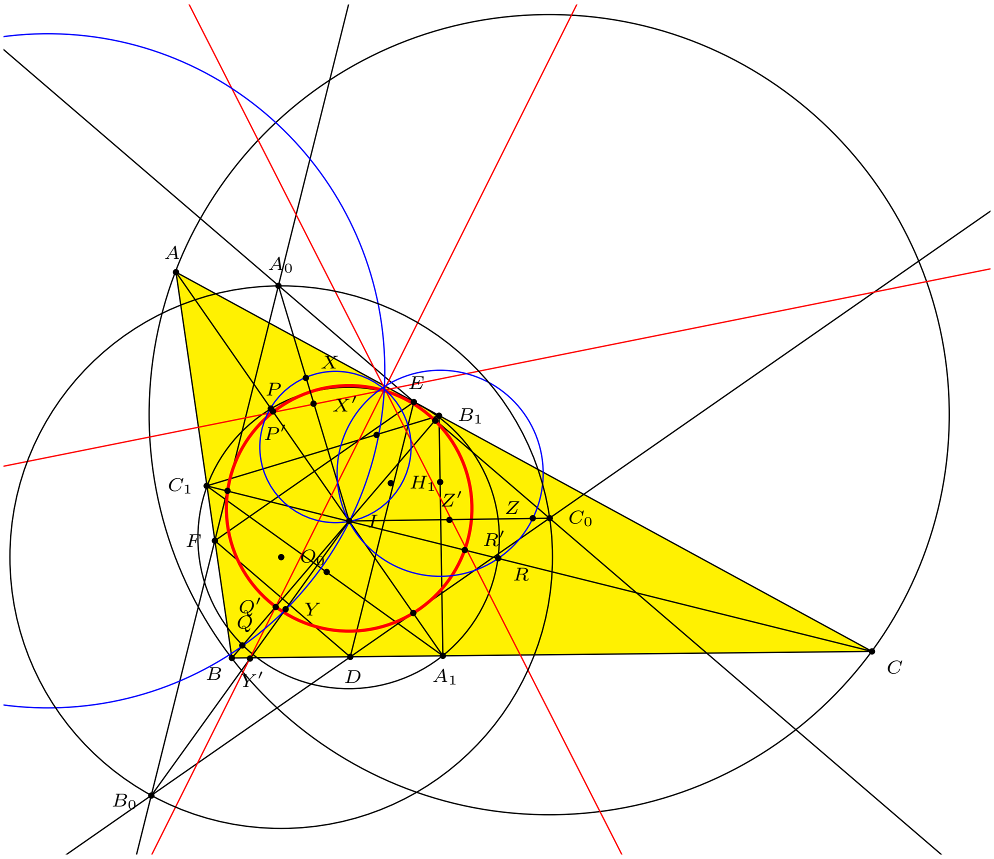

Figure 8

Now, let the segments IA, IB, IC intersect the cevian circumcircle $(A_1B_1C_1)$ of I at P, Q, R respectively, and let X, Y, Z be the reflections of I across the lines B_1C_1, C_1A_1, and A_1B_1, respectively. Inversion with respect to (I) takes ω into the pedal circle ω' of I with respect to triangle $A_0B_0C_0$. Thus, the segments IA, IB, IC cut ω' at the inverse images P', Q', R' of P, Q, R respectively, and the midpoints X', Y', Z' of IA_0, IB_0, IC_0 are the inverse images of X, Y, Z. It follows from Lemma 6 that $P'X'$, $Q'Y'$, $R'Z'$ all meet at the Poncelet point F' of $A_0B_0C_0I$, which, as a matter of fact, lies on ω'. On the other hand, by Lemma 5, the inverses of these lines are the circles (IXP), (IYQ), and (IZR) concurring at the anti-Steiner point of I with respect to triangle $A_1B_1C_1$. Therefore, the intersection points of $(A_1B_1C_1)$ and the incircle (I) are precisely the anti-Steiner point F' of IH_1 with respect to triangle $A_1B_1C_1$ and the Feuerbach point of ABC. Moreover, if O_0 is the circumcenter of triangle $A_0B_0C_0$, then according to Lemma 7, F' is in general different from the anti-Steiner point of IO_0 with respect to triangle DEF. Thus, we conclude that the anti-Steiner point F' of IH_1 with respect to triangle $A_1B_1C_1$ is indeed the intersection of $(I) \cap \omega$, which is different from the Feuerbach point, since by Theorem 3 the anti-Steiner point of IO_0 with respect to DEF is the Feuerbach point of ABC.

This completes the proof of Theorem 4.

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