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Some Properties of the Newton-Gauss Line

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Abstract. We present some properties of the Newton-Gauss lines of the complete quadrilaterals associated with a cyclic quadrilateral.

1. Introduction

A complete quadrilateral is the figure determined by four lines, no three of which are concurrent, and their six points of intersection. Figure 1 shows a complete quadrilateral $ABCDEF$, with its three diagonals AC, BD, and EF (compared to two for an ordinary quadrilateral). The midpoints M, N, L of these diagonals are collinear on a line, called the *Newton-Gauss line* of the complete quadrilateral ([1, pp.152–153]). In this note, we present some properties of the Newton - Gauss lines of complete quadrilaterals associated with a cyclic quadrilateral.

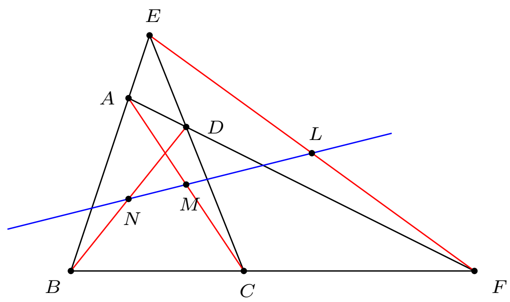

Figure 1.

2. An equality of angles determined by Newton - Gauss line

Given a cyclic quadrilateral $ABCD$, denote by F the point of intersection at the diagonals AC and BD, E the point of intersection at the lines AB and CD, N the midpoint of the segment EF, and M the midpoint of the segment BC (see Figure 2).

Theorem 1. *If P is the midpoint of the segment BF, the Newton - Gauss line of the complete quadrilateral $EAFDBC$ determines with the line PM an angle equal to $\angle EFD$.*

Proof. We show that triangles NPM and EDF are similar.

Since $BE \| PN$ and $FC \| PM$, $\angle EAC = \angle NPM$ and $\frac{BE}{PN} = \frac{FC}{PM} = 2$.

In the cyclic quadrilateral $ABCD$, we have

$$\angle EDF = \angle EDA + \angle ADF = \angle ABC + \angle ACB = \angle EAC.$$

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Therefore, $\angle NPM = \angle EDF$.

Let R_1 and R_2 be the radii of the circumcircles of triangles BED and DFC respectively. Applying the law of sines to these triangles, we have

$$\frac{BE}{FC} = \frac{2R_1 \sin EDB}{2R_2 \sin FDC} = \frac{R_1}{R_2} = \frac{2R_1 \sin EBD}{2R_2 \sin FCD} = \frac{DE}{DF}.$$

Since $BE = 2PN$ and $FC = 2PM$, we have shown that $\frac{PN}{PM} = \frac{DE}{DF}$. The similarity of triangles NPM and EDF follows, and $\angle NMP = \angle EFD$. □

Remark. If Q is the midpoint of the segment FC, the same reasoning shows that that $\angle NMQ = \angle EFA$.

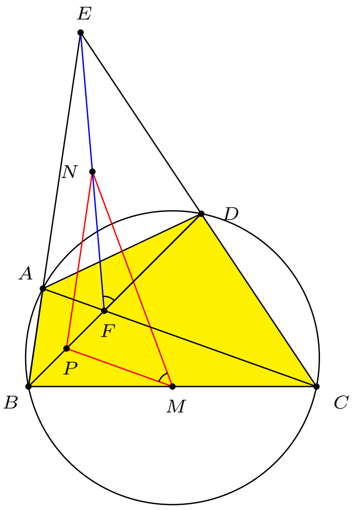

Figure 2

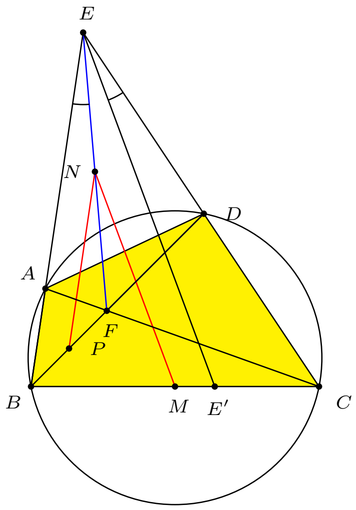

Figure 3

3. A parallel to the Newton-Gauss line

Theorem 2. *The parallel from E to the Newton - Gauss line of the complete quadrilateral $EAFDBC$ and the line EF are isogonal lines of angle BEC.*

Proof. Since triangles EDF and NPM are similar, we have $\angle DEF = \angle PNM$.

Let E' be the intersection of the side BC with the parallel of NM through E. Because $PN \| BE$ and $NM \| EE'$, $\angle BEF = \angle PNF$ and $\angle FNM = \angle E'EF$. Thus,

$$\angle CEE' = \angle DEF - \angle E'EF = \angle PNM - \angle FNM = \angle PNF = \angle BEF.$$

□

4. Two cyclic quadrilaterals determined the Newton-Gauss line

Let G and H be the orthogonal projections of the point F on the lines AB and CD respectively (see Figure 4).

Theorem 3. *The quadrilaterals $MPGN$ and $MQHN$ are cyclic.*

Proof. By Theorem 1, $\angle EFD = \angle PMN$. The points P and N are the circumcenters of the right triangles BFG and EFG, respectively. It follows that $\angle PGF = \angle PFG$ and $\angle FGN = \angle GFN$. Thus,

$$\begin{aligned}\angle PGN + \angle PMN &= (\angle PGF + \angle FGN) + \angle PMN \\ &= \angle PFG + \angle GFN + \angle EFD \\ &= 180^\circ.\end{aligned}$$

Therefore, $MPGN$ is a cyclic quadrilateral. In the same way, the quadrilateral $MQHN$ is also cyclic. $\square$

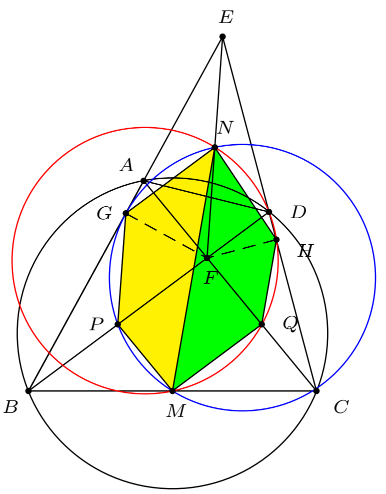

Figure 4

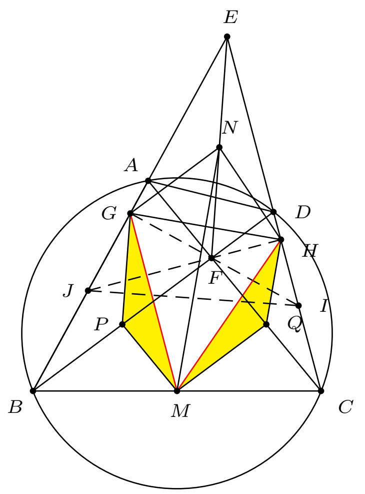

Figure 5

5. Two complete quadrilaterals with the same Newton-Gauss line

Extend the lines GF and HF to intersect EC and EB at I and J respectively (see Figure 5).

Theorem 4. *The complete quadrilaterals $EGFHJI$ and $EAFDBC$ have the same Newton-Gauss line.*

Proof. The two complete quadrilaterals have a common diagonal EF. Its midpoint N lies on the Newton-Gauss lines of both quadrilaterals. Note that N is equidistant from G and H since it is the circumcenter of the cyclic quadrilateral $EGFH$. We show that triangles MPG and HQM are congruent. From this, it follows that M

lies on the perpendicular bisector of GH. Therefore, the line MN contains the midpoint of GH, and is the Newton-Gauss line of $EGFHJI$.

Now, to show the congruence of the triangles MPG and HQM, first note that since M and P are the midpoints of BF and BC, $PMQF$ is a parallelogram. From these, we conclude
(i) $MP = QF = HQ$,
(ii) $GP = PF = MQ$,
(iii) $\angle MPF = \angle FQM$.

Note also that

$$\angle FPG = 2\angle PBG = 2\angle DBA = 2\angle DCA = 2\angle HCF = \angle HQF.$$

Together with (iii) above, this yields

$$\angle MPG = \angle MPF + \angle FPG = \angle FQM + \angle HQF = \angle HQF + \angle FQM = \angle HQM.$$

Together with (i) and (ii), this proves the congruence of triangles MPG and HQM.
□

Remark. Because MPG and HQM are congruent triangles, their circumcircles, namely, $(MPGN)$ and $(MQHN)$ are congruent (see Figure 4).

Reference

[1] R. A. Johnson, *A Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, Houghton Mifflin, Boston, 1929.

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