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Harmonic Conjugate Circles Relative to a Triangle

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Abstract. We use the term harmonic conjugate conics, for the conics $\mathcal{C}, \mathcal{C}^*$ with equations $\mathcal{C}: \ fx^2 + gy^2 + hz^2 + 2pyz + 2qz + 2rxy = 0$ and $\mathcal{C}^*: \ fx^2 + gy^2 + hz^2 - 2pyz - 2qz - 2rxy = 0$, in barycentric coordinates because if A_1, A_2 are the points where $\mathcal{C}$ meets the sideline BC of the reference triangle ABC, then $\mathcal{C}^*$ meets the same side at the points A'_1, A'_2 that are harmonic conjugates of A_1, A_2 respectively relative to BC and similarly for the other sides of ABC [1]. So we investigate the interesting case where both $\mathcal{C}$ and $\mathcal{C}^*$ are circles.

1. Introduction

We work with barycentric coordinates with reference to a given triangle ABC. A conic $\mathcal{C}$ with matrix

$$M = \begin{pmatrix} f & r & q \\ r & g & p \\ q & p & h \end{pmatrix}$$

and equation

$$fx^2 + gy^2 + hz^2 + 2pyz + 2qzx + 2rxy = 0 \qquad (1)$$

intersects the sideline BC of triangle ABC at the points $A_1 = (0 : y_1 : z_1)$ and $A_2 = (0 : y_2 : z_2)$ with y_i, z_i $(i = 1, 2)$ satisfying $gy^2+2pyz+hz^2 = 0$. Similarly, the conic $\mathcal{C}^*$ with matrix

$$M^* = \begin{pmatrix} f & -r & -q \\ -r & g & -p \\ -q & -p & h \end{pmatrix}$$

and equation

$$fx^2 + gy^2 + hz^2 - 2pyz - 2qzx - 2rxy = 0 \qquad (2)$$

intersects the sideline BC of triangle ABC at the points $A'_1 = (0 : -y_1 : z_1)$ and $A'_2 = (0 : -y_2 : z_2)$. For $i = 1, 2$, the points A_i and A'_i are harmonic conjugates with respect to B and C. Similarly the intersections of $\mathcal{C}$ and $\mathcal{C}^*$ with the other two sides CA, AB are also harmonic conjugates. We call these conics harmonic conjugates relative to triangle ABC (see Figure 1), and it is very interesting to consider their properties and construction if these conics are both circles. If the conic $\mathcal{C}$ is a bicevian conic (passing through the vertices of the cevian triangles of

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two points P, Q), then its harmonic conjugate conic is a pair of lines (the trilinear polars of P and Q).

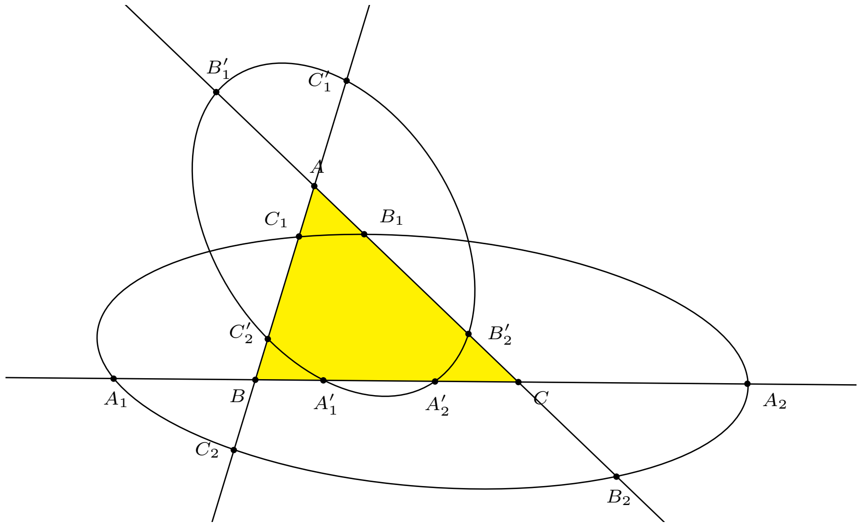

Figure 1. Harmonic conjugate conics

2. Harmonic conjugate circles relative to ABC

Theorem 1. *The harmonic conjugate conic of the circle*

$$a^2yz + b^2zx + c^2xy - (x + y + z)(Px + Qy + Rz) = 0 \qquad (3)$$

is a circle if and only if $(P, Q, R) = m(S_A, S_B, S_C)$ *for some* m.

Proof. The matrix of the circle (3) being

$$\begin{pmatrix} -2P & c^2 - P - Q & b^2 - R - P \\ c^2 - P - Q & -2Q & a^2 - Q - R \\ b^2 - R - P & a^2 - Q - R & -2R \end{pmatrix},$$

its harmonic conjugate conic has matrix

$$\begin{pmatrix} -2P & -c^2 + P + Q & -b^2 + R + P \\ -c^2 + P + Q & -2Q & -a^2 + Q + R \\ -b^2 + R + P & -a^2 + Q + R & -2R \end{pmatrix}.$$

This is the conic

$$(2Q+2R-a^2)yz+(2R+2P-b^2)zx+(2P+2Q-c^2)xy-(x+y+z)(Px+Qy+Rz) = 0.$$

It is a circle if and only if

$$2Q + 2R - a^2 : 2R + 2P - b^2 : 2P + 2Q - c^2 = a^2 : b^2 : c^2,$$

i.e.,

$$P : Q : R = b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2 = S_A : S_B : S_C.$$

This is the case if and only if $(P, Q, R) = m(S_A, S_B, S_C)$ for some m. □

Denote by $\mathcal{C}_m$ the circle with equation

$$a^2yz + b^2zx + c^2xy - m(x+y+z)(S_Ax + S_By + S_Cz) = 0.$$

A simple application of the formula in [3, §10.7.2] shows that the center of $\mathcal{C}_m$ is the point

$$O_m = ((1-m)a^2S_A + m\cdot 2S_{BC} : (1-m)b^2S_B + m\cdot 2S_{CA} : (1-m)c^2S_C + m\cdot 2S_{AB}),$$

which divides OH in the ratio

$$OO_m : O_mH = m : 1-m.$$

Proposition 2. *If $m \neq \frac{1}{2}$, the harmonic conjugate circle of $\mathcal{C}_m$ is the circle $\mathcal{C}_{m'}$, where $m' = \frac{m}{2m-1}$.*

Proof. By the proof of Theorem 1, the harmonic conjugate circle of $\mathcal{C}_m$ is the circle

$$(2m(S_B + S_C) - a^2)yz + (2m(S_C + S_A) - b^2)zx + (2m(S_A + S_B) - c^2)xy$$
$$- m(x+y+z)(S_Ax + S_By + S_Cz) = 0,$$

namely,

$$a^2yz + b^2zx + c^2xy - \frac{m}{2m-1}(x+y+z)(S_Ax + S_By + S_Cz) = 0.$$

This is the circle $\mathcal{C}_{m'}$ with $m' = \frac{m}{2m-1}$. □

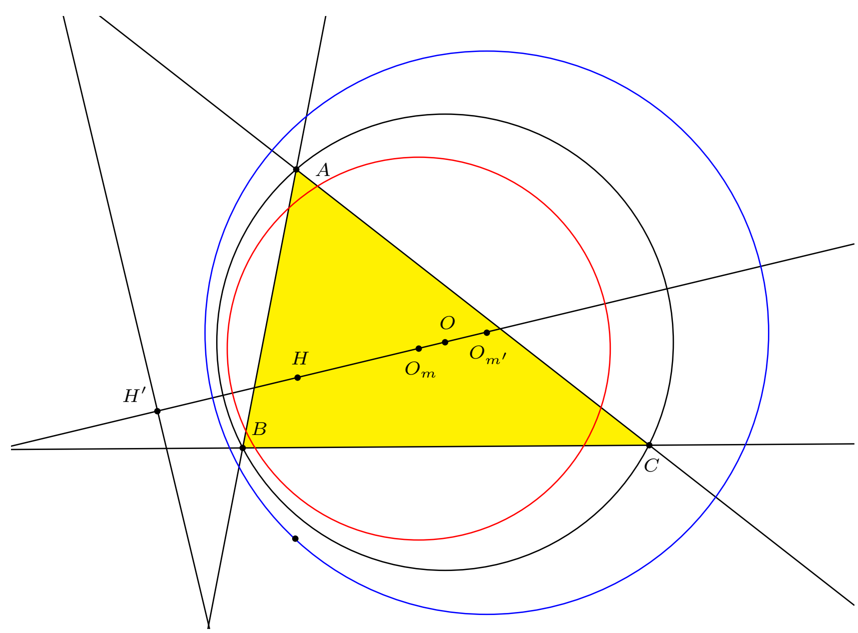

Figure 2. Harmonic conjugate circles

Remark. For $m = \frac{1}{2}$, $\mathcal{C}_m$ is the nine-point circle, the bicevian circle of the centroid and the orthocenter. Its harmonic conjugate conic is the pair of lines consisting of the line at infinity and the orthic axis.

Proposition 3. *The centers of a pair of harmonic conjugate circles divide the segment OH harmonically.*

Proof. Let the harmonic conjugate circles be $\mathcal{C}_m$ and $\mathcal{C}_{m'}$, with $m' = \frac{m}{2m-1}$. Their centers are points O_m and $O_{m'}$ satisfying

$$\begin{aligned} OO_{m'} : O_{m'}H &= m' : 1 - m' = \frac{m}{2m-1} : \frac{m-1}{2m-1} \\ &= m : -(1-m) \\ &= OO_m : -O_mH. \end{aligned}$$

Therefore O_m and $O_{m'}$ divide OH harmonically. □

Since $m = m'$ if and only if $m = 0$ or 1, we have the following corollary.

Corollary 4. *The circumcircle and the polar circle (with center H) are the only circles which are their own harmonic conjugate circles.*

Remark. The polar circle is real only when the triangle contains an angle $\geq 90°$. For the construction of the polar circle, see §4.2 below.

3. Construction of coaxial circles

3.1. *Prescribed center.* Given a circle $O(R)$ and a line $\mathcal{L}$ generating a coaxial family of circles, we address the construction problem of the circle in the family with a prescribed center P on the line through O perpendicular to $\mathcal{L}$.

Any intersection of $\mathcal{L}$ and $O(R)$ is common to the circles in the coaxial family. The construction problem is trivial when $\mathcal{L}$ and $O(R)$ intersect.

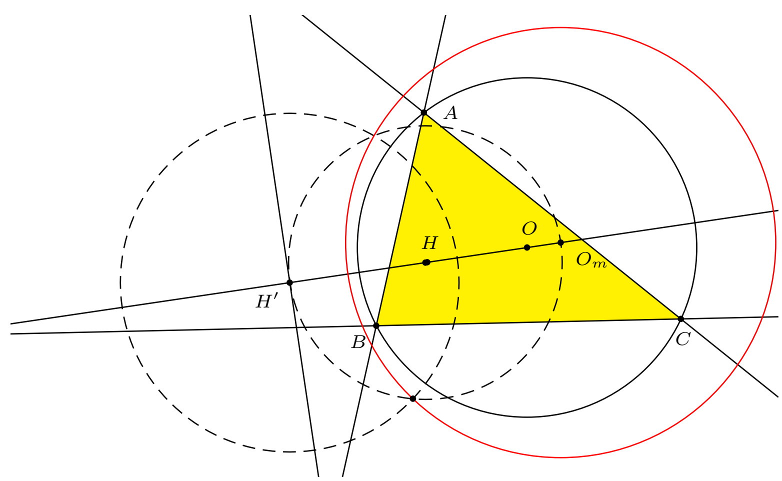

Figure 3. Construction of circles in coaxial family

Suppose $\mathcal{L}$ does not intersect the circle $O(R)$. Let H' be the orthogonal projection of O on the line $\mathcal{L}$. Set up a Cartesian coordinates with origin at H', y-axis along $\mathcal{L}$, and positive x-axis along the half-line $H'O$. If the point O has coordinates $(k_0, 0)$ for $k_0 > R$, the circle $O(R)$ has equation $(x-k_0)^2 + y^2 = R^2$, or

$$x^2 + y^2 - 2k_0x + k_0^2 - R^2 = 0.$$

Construct the circle (H') orthogonal to (O). This circle has radius $\sqrt{k_0^2 - R^2}$.

The real circles in the coaxial family have equations

$$x^2 + y^2 - 2kx + k_0^2 - R^2 = 0, \qquad k^2 \geq k_0^2 - R^2.$$

Given the center $K(k, 0)$, here is a simple construction of the circle.

(i) Suppose $k > 0$. Construct the circle with diameter $H'K$ to intersect the circle (H') at a point P. Then the circle $K(P)$ is the one in the coaxial family with center K (see Figure 3).

(ii) Suppose $k < 0$. Apply (i) to construct the circle in the family with center $(-k, 0)$. Reflect this in the line $\mathcal{L}$ to yield the circle with center $K(k, 0)$.

3.2. *Through a given point.* Given a point P not on the line $\mathcal{L}$, to construct the circle in the coaxial family which contains P, we need only note that this circle, being orthogonal to (H'), should also contain the inversive image P' of P in (H'). The intersection of the perpendicular bisector of PP' and the perpendicular to $\mathcal{L}$ from O is the center K of the circle.

4. Harmonic conjugate circles for special triangles

4.1. *Equilateral triangles.* If ABC is equilateral with circumcenter O and circumradius R, the only harmonic conjugate circle pairs are concentric circles at O, with radii ρ and ρ' related by

$$\left(\rho^2 - \frac{R^2}{4}\right)\left(\rho'^2 - \frac{R^2}{4}\right) = \left(\frac{3R^2}{4}\right)^2.$$

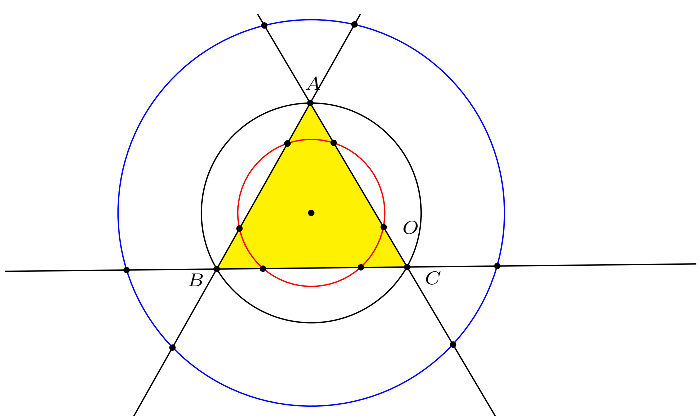

Figure 4. Harmonic conjugate circles of an equilateral triangle

4.2. *Nonacute triangles.* If ABC contains an angle $\geq 90°$, then its orthic axis intersects the circumcircle at real points. [1] Therefore the harmonic conjugate circles pairs can be easily constructed knowing that their centers are harmonic conjugates with respect to OH.

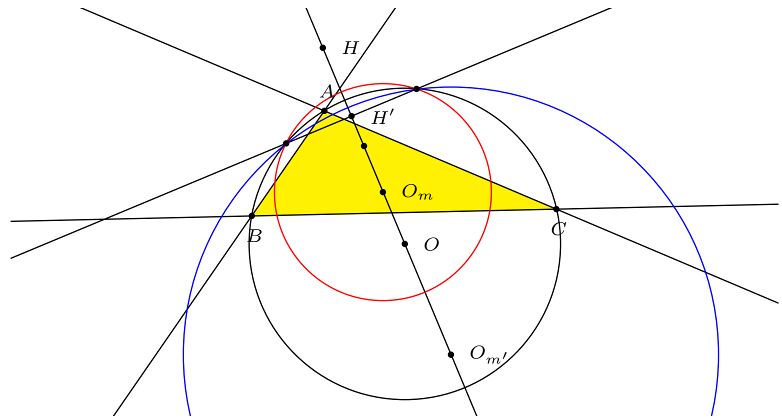

Figure 5. Harmonic conjugate circles of an obtuse triangle

5. Congruent harmonic conjugate circles

There is a unique pair of congruent harmonic conjugate circles. Their centers on the Euler line are symmetric with respect to H'. These two points are therefore the intersection of the Euler line with the circle, center H', orthogonal to the circle with diameter OH.

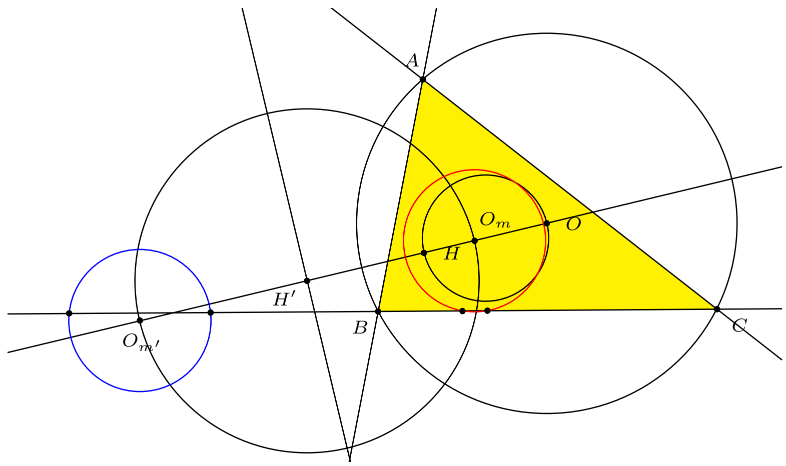

Figure 6. Congruent harmonic conjugate circles

[1]If ABC contains a right angle, then the right angle vectex is on the orthic axis (and the circumcircle).

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