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The Perpendicular Bisector Construction, the Isoptic point, and the Simson Line of a Quadrilateral

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Abstract. Given a noncyclic quadrilateral, we consider an iterative procedure producing a new quadrilateral at each step. At each iteration, the vertices of the new quadrilateral are the circumcenters of the triad circles of the previous generation quadrilateral. The main goal of the paper is to prove a number of interesting properties of the limit point of this iterative process. We show that the limit point is the common center of spiral similarities taking any of the triad circles into another triad circle. As a consequence, the point has the isoptic property *i.e.*, all triad circles are visible from the limit point at the same angle. Furthermore, the limit point can be viewed as a generalization of a circumcenter. It also has properties similar to those of the isodynamic point of a triangle. We also characterize the limit point as the unique point for which the pedal quadrilateral is a parallelogram. Continuing to study the pedal properties with respect to a quadrilateral, we show that for every quadrilateral there is a unique point (which we call the Simson point) such that its pedal consists of four points on a line, which we call the Simson line, in analogy to the case of a triangle. Finally, we define a version of isogonal conjugation for a quadrilateral and prove that the isogonal conjugate of the limit point is a parallelogram, while that of the Simson point is a degenerate quadrilateral whose vertices coincide at infinity.

1. Introduction

The perpendicular bisector construction that we investigate in this paper arises very naturally in an attempt to find a replacement for a circumcenter in the case of a noncyclic quadrilateral $Q^{(1)} = A_1B_1C_1D_1$. Indeed, while there is no circle going through all four vertices, for every triple of vertices there is a unique circle (called the *triad circle*) passing through them. The centers of these four triad circles can be taken as the vertices of a new quadrilateral, and the process can be iterated to obtain a sequence of noncyclic quadrilaterals: $Q^{(1)}, Q^{(2)}, Q^{(3)}, \dots$.

To reverse the iterative process, one finds the isogonal conjugates of each of the vertices with respect to the triangle formed by the remaining vertices of the quadrilateral.

It turns out that all odd generation quadrilaterals are similar, and all even generation quadrilaterals are similar. Moreover, there is a point that serves as the center of spiral similarity for any pair of odd generation quadrilaterals as well as for any

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pair of even generation quadrilaterals. The angle of rotation is 0 or π depending on whether the quadrilateral is concave or convex, and the ratio r of similarity is a constant that is negative for convex noncyclic quadrilaterals, zero for cyclic quadrilaterals, and ≥ 1 for concave quadrilaterals. If $|r| \neq 1$, the same special point turns out to be the limit point for the iterative process or for the reverse process.

The main goal of this paper is to prove the following theorem.

Theorem 1. *For each quadrilateral $Q^{(1)} = A_1B_1C_1D_1$ there is a unique point W that has any (and, therefore, all) of the following properties:*

(1) *W is the center of the spiral similarity for any two odd (even) generation quadrilaterals in the iterative process;*
(2) *Depending on the value of the ratio of similarity in the iterative process, there are the following possibilities:*
  (a) *If $|r| < 1$, the quadrilaterals in the iterated perpendicular bisectors construction converge to W;*
  (b) *If $|r| = 1$, the iterative process is periodic (with period 2 or 4); W is the common center of rotations for any two odd (even) generation quadrilaterals;*
  (c) *If $|r| > 1$, the quadrilaterals in the reverse iterative process (obtained by isogonal conjugation) converge to W;*
(3) *W is the common point of the six circles of similitude $CS(o_i, o_j)$ for any pair of triad circles o_i, o_j, $i, j \in \{1, 2, 3, 4\}$, where $o_1 = (D_1A_1B_1)$, $o_2 = (A_1B_1C_1)$, $o_3 = (B_1C_1D_1)$, $o_4 = (C_1D_1A_1)$.*
(4) *(isoptic property) Each of the triad circles is visible from W at the same angle.*
(5) *(generalization of circumcenter) The (directed) angle subtended by any of the quadrilateral's sides at W equals to the sum of the angles subtended by the same side at the two remaining vertices.*
(6) *(isodynamic property) The distance from W to any vertex is inversely proportional to the radius of the triad circle determined by the remaining three vertices.*
(7) *W is obtained by inversion of any of the vertices of the original quadrilateral in the corresponding triad-circle of the second generation:*

$$W = \mathit{Inv}_{o_1^{(2)}}(A) = \mathit{Inv}_{o_2^{(2)}}(B) = \mathit{Inv}_{o_3^{(2)}}(C) = \mathit{Inv}_{o_4^{(2)}}(D),$$

where $o_1^{(2)} = (D_2A_2B_2)$, $o_2^{(2)} = (A_2B_2C_2)$, $o_3^{(2)} = (B_2C_2D_2)$, $o_4^{(2)} = (C_2D_2A_2)$.
(8) *W is obtained by composition of isogonal conjugation of a vertex in the triangle formed by the remaining vertices and inversion in the circumcircle of that triangle.*
(9) *W is the center of spiral similarity for any pair of triad circles (of possibly different generations). That is, $W \in CS(o_i^{(k)},\ o_j^{(l)})$ for all i, j, k, l.*
(10) *The pedal quadrilateral of W is a (nondegenerate) parallelogram. Moreover, its angles equal to the angles of the Varignon parallelogram.*

Many of these properties of W were known earlier. In particular, several authors (G. T. Bennett in an unpublished work, De Majo [11], H. V. Mallison [12]) have considered a point that is defined as the common center of spiral similarities. Once the existence of such a point is established, it is easy to conclude that all the triad circles are viewed from this point under the same angle (this is the so-called *isoptic property*). Since it seems that the oldest reference to the point with such an isoptic property is to an unpublished work of G. T. Bennett given by H. F. Baker in his *Principles of Geometry*, volume 4 [1, p.17], in 1925, we propose to call the center of spiral similarities in the iterative process *Bennett's isoptic point*.

C. F. Parry and M. S. Longuet-Higgins [14] showed the existence of a point with property 7 using elementary geometry.

Mallison [12] defined W using property 3 and credited T. McHugh for observing that this implies property 5.

Several authors, including Wood [19] and De Majo [11], have looked at the properties of the isoptic point from the point of view of the unique rectangular hyperbola going through the vertices of the quadrilateral, and studied its properties related to cubics. For example, P.W. Wood [19] considered the diameters of the rectangular hyperbola that go through A, B, C, D. Denoting by $\bar{A}, \bar{B}$,$\bar{C}, \bar{D}$ the other endpoints of the diameters, he showed that the isogonal conjugates of these points in triangles BCD, CAD,ABD, ABC coincide. Starting from this, he proved properties 4 and 7 of the theorem. He also mentions the reversal of the iterative process using isogonal conjugation (also found in [19], [17], [5]). Another interesting property mentioned by Wood is that W is the Fregier point of the center of the rectangular hyperbola for the conic $ABCDO$, where O is the center of the rectangular hyperbola.

De Majo [11] uses the property that inversion in a point on the circle of similitude of two circles transforms the original circles into a pair of circles whose radii are inversely proportional to those of the original circles to show that that there is a common point of intersection of all 6 circles of similitude. He describes the iterative process and states property 1, as well as several other properties of W (including 8). Most statements are given without proofs.

Scimemi [17] describes a Möbius transformation that characterizes W: there exists a line going through W and a circle centered at W such that the product of the reflection in the line with the inversion in the circle maps each vertex of the first generation into a vertex of the second generation.

The question of proving that the third generation quadrilateral is similar to the original quadrilateral and finding the ratio of similarity was first formulated by J. Langr [8]. Independently, the result appeared in the form of a problem by V.V. Prasolov in [15, 16]. The expression for the ratio (under certain conditions) was obtained by J. Langr [8] , and the expression for the ratio (under certain conditions) was obtained by D. Bennett [2] (apparently, no relation to G. T. Bennett mentioned above), and J. King [7]. A paper by G. C. Shepard [18] found an expression for the ratio as well. (See [3] for a discussion of these works).

Properties 9 and 10 appear to be new.

For the convenience of the reader, we give a complete and self contained exposition of all the properties in the Theorem above, as well as proofs of several related statements.

In addition to investigating properties of W, we show that there is a unique point for which the feet of the perpendiculars to the sides lie on a straight line. In analogy with the case of a triangle, we call this line the *Simson line* of a quadrilateral and the point – the *Simson point.* The existence of such a point is stated in [6] where it is obtained as the intersection of the Miquel circles of the complete quadrilateral.

Finally, we introduce a version of isogonal conjugation for a quadrilateral and show that the isogonal conjugate of W is a parallelogram, and that of the Simson point is a degenerate quadrilateral whose vertices are at infinity, in analogy with the case of the points on the circumcircle of a triangle.

2. The iterative process

Let $A_1B_1C_1D_1$ be a quadrilateral. If $A_1B_1C_1D_1$ is cyclic, the center of the circumcircle can be found as the intersection of the four perpendicular bisectors to the sides of the quadrilateral.

Assume that $Q^{(1)} = A_1B_1C_1D_1$ is a noncyclic quadrilateral.[1] Is there a point that, in some sense, plays the role of the circumcenter? Let $Q^{(2)} = A_2B_2C_2D_2$ be the quadrilateral formed by the intersections of the perpendicular bisectors of the sides of $A_1B_1C_1D_1$. The vertices A_2, B_2, C_2, D_2 of the new quadrilateral are the circumcenters of the triangles $D_1A_1B_1$, $A_1B_1C_1$, $B_1C_1D_1$ and $C_1D_1A_1$ formed by vertices of the original quadrilateral taken three at a time.

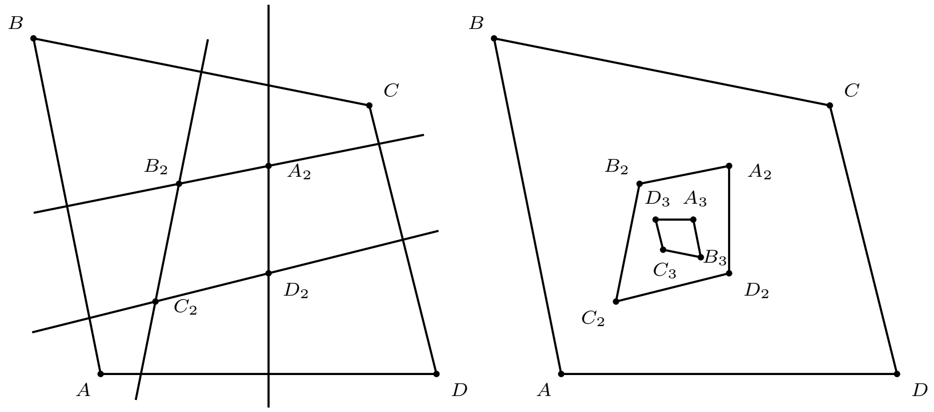

Figure 1. The perpendicular bisector construction and $Q^{(1)}, Q^{(2)}, Q^{(3)}$.

Iterating this process, *i.e.*, constructing the vertices of the next generation quadrilateral by intersecting the perpendicular bisectors to the sides of the current one, we obtain the successive generations, $Q^{(3)} = A_3B_3C_3D_3$, $Q^{(4)} = A_4B_4C_4D_4$ and so on, see Figure 1.

[1] Sometimes we drop the lower index 1 when denoting vertices of $Q^{(1)}$, so $ABCD$ and $A_1B_1C_1D_1$ are used interchangeably throughout the paper.

The first thing we note about the iterative process is that it can be reversed using isogonal conjugation. Recall that given a triangle ABC and a point P, the *isogonal conjugate* of P with respect to the triangle (denoted by $\mathrm{Iso}_{ABC}(P)$) is the point of intersection of the reflections of the lines AP, BP and CP in the bisectors of angles A, B and C respectively. One of the basic properties of isogonal conjugation is that the isogonal conjugate of P is the circumcenter of the triangle obtained by reflecting P in the sides of ABC (see, for example, [5] for more details). This property immediately implies

Theorem 2. *The original quadrilateral $A_1B_1C_1D_1$ can be reconstructed from the second generation quadrilateral $A_2B_2C_2D_2$ using isogonal conjugation:*

$$\begin{aligned} A_1 &= \mathit{Iso}_{D_2A_2B_2}(C_2), \\ B_1 &= \mathit{Iso}_{A_2B_2C_2}(D_2), \\ C_1 &= \mathit{Iso}_{B_2C_2D_2}(A_2), \\ D_1 &= \mathit{Iso}_{C_2D_2A_2}(B_2). \end{aligned}$$

The following theorem describes the basic properties of the iterative process.

Theorem 3. *Let $Q^{(1)}$ be a quadrilateral. Then*

(1) *$Q^{(2)}$ degenerates to a point if and only if $Q^{(1)}$ is cyclic.*
(2) *If $Q^{(1)}$ is not cyclic, the corresponding angles of the first and second generation quadrilaterals are supplementary:*

$$\angle A_1 + \angle A_2 = \angle B_1 + \angle B_2 = \angle C_1 + \angle C_2 = \angle D_1 + \angle D_2 = \pi.$$

(3) *If $Q^{(1)}$ is not cyclic, all odd generation quadrilaterals are similar to each other and all the even generation quadrilaterals are similar to each other:*

$$Q^{(1)} \sim Q^{(3)} \sim Q^{(5)} \sim \dots,$$
$$Q^{(2)} \sim Q^{(4)} \sim Q^{(6)} \sim \dots.$$

(4) *All odd generation quadrilaterals are related to each other via spiral similarities with respect to a common center.*
(5) *All even generation quadrilaterals are also related to each other via spiral similarities with respect to a common center.*
(6) *The angle of rotation for each spiral similarity is π (for a convex quadrilateral) or a 0 (for a concave quadrilateral). The ratio of similarity is*

$$r = \frac{1}{4}(\cot\alpha + \cot\gamma)\cdot(\cot\beta + \cot\delta), \tag{1}$$

where $\alpha = \angle A_1$, $\beta = \angle B_1$, $\gamma = \angle C_1$ and $\delta = \angle D_1$ are the angles of $Q^{(1)}$.
(7) *The center of spiral similarities is the same for both the odd and the even generations.*

Proof. The first and second statements follow immediately from the definition of the iterative process. To show that all odd generation quadrilaterals are similar to each other and all even generation quadrilaterals are similar to each other, it is enough to notice that both the corresponding sides and the corresponding diagonals of all odd (even) generation quadrilaterals are pairwise parallel.

Let $W_1 := A_1A_3 \cap B_1B_3$ be the center of spiral similarity taking $Q^{(1)}$ into $Q^{(3)}$. Similarly, let W_2 be the center of spiral similarity taking $Q^{(2)}$ into $Q^{(4)}$. Denote the midpoints of segments A_1B_1 and A_3B_3 by M_1 and M_3. (See fig. 2). To show that W_1 and W_2 coincide, notice that $B_1M_1A_2 \sim B_3M_3A_4$. Since the corresponding sides of these triangles are parallel, they are related by a spiral similarity. Since $B_1B_3 \cap M_1M_3 = W_1$ and $M_1M_3 \cap B_2B_4 = W_2$, it follows that $W_1 = W_2$. Let now W_3 be the center of spiral similarity that takes $Q^{(3)}$ into $Q^{(5)}$. By the same reasoning, $W_2 = W_3$, which implies that $W_1 = W_3$. Continuing by induction, we conclude that the center of spiral similarity for any pair of odd generation quadrilaterals coincides with that for any pair of even generation quadrilaterals. We denote this point by W. □

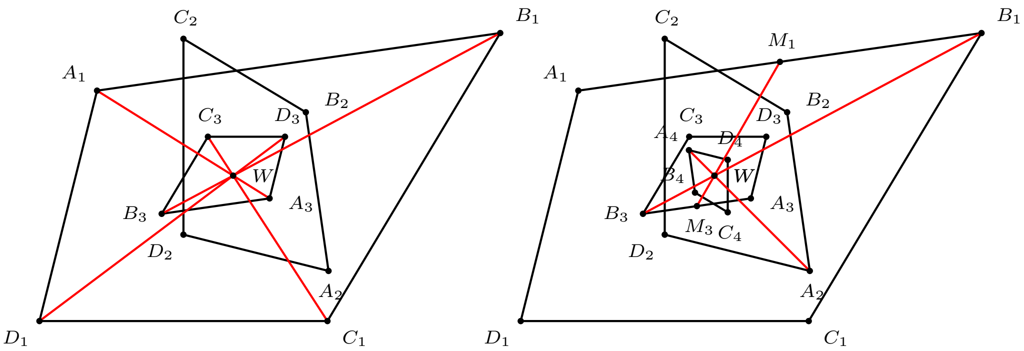

Figure 2. W as the center of spiral similarities.

From parts (2) and (3) of Theorem 3 we obtain the following corollary.

Corollary 4. *The even and odd generation quadrilaterals are similar to each other if and only if $Q^{(1)}$ is a trapezoid.*

The ratio of similarity $r = r(\alpha, \beta, \gamma, \delta)$ takes values in $(-\infty, 0] \cup [1, \infty)$ and characterizes the shape of $Q^{(1)}$ in the following way:

(1) $r \leq 0$ if and only if $Q^{(1)}$ is convex. Moreover, $r = 0$ if and only if $Q^{(1)}$ is cyclic.

(2) $r \geq 1$ if and only if $Q^{(1)}$ is concave. Moreover, $r = 1$ if and only if $Q^{(1)}$ is *orthocentric* (that is, each of the vertices is the orthocenter of the triangle formed by the remaining three vertices. Alternatively, an orthocentric quadrilateral is characterized by being a concave quadrilateral for which the two opposite acute angles are equal).

For convex quadrilaterals, r can be viewed as a measure of how noncyclic the original quadrilateral is. Recall that since the opposite angles of a cyclic quadrilateral add up to π, the difference

$$|(\alpha + \gamma) - \pi| = |(\beta + \delta) - \pi| \tag{2}$$

can be taken as the simplest measure of noncyclicity. This measure, however, treats two quadrilaterals with equal sums of opposite angles as equally noncyclic. The ratio r provides a refined measure of noncyclicity. For example, for a fixed sum of opposite angles, $\alpha + \gamma = C$, $\beta + \delta = 2\pi - C$, where $C \in (0, 2\pi)$, the convex quadrilateral with the smallest $|r|$ is the parallelogram with $\alpha = \gamma = \frac{C}{2}$, $\beta = \delta$.

Similarly, for concave quadrilaterals, r measures how different the quadrilateral is from being orthocentric.

Since the angles between diagonals are the same for all generations, it follows that the ratio is the same for all pairs of consecutive generations:

$$\frac{\text{Area}(Q^{(n)})}{\text{Area}(Q^{(n-1)})} = |r|.$$

Assuming the quadrilateral is noncyclic, there are the following three possibilities:

(1) When $|r| < 1$ (which can only happen for convex quadrilaterals), the quadrilaterals in the iterative process converge to W.
(2) When $|r| > 1$, the quadrilaterals in the inverse iterative process converge to W.
(3) When $|r| = 1$, all the quadrilaterals have the same area. The iterative process is periodic with period 4 for all quadrilaterals with $|r| = 1$, except for the following two special cases. If $Q^{(1)}$ is either a parallelogram with angle $\frac{\pi}{4}$ (so that $r = -1$) or forms an orthocentric system (so that $r = 1$), we have $Q^{(3)} = Q^{(1)}$, $Q^{(4)} = Q^{(2)}$, and the iterative process is periodic with period 2.

By setting $r = 0$ in formula (1), we obtain the familiar relations between the sides and diagonals of a cyclic quadrilateral $ABCD$:

$$AC \cdot BD = AB \cdot CD + BC \cdot AD, \qquad \text{(Ptolemy's theorem)} \tag{3}$$

$$\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{BA \cdot BC + DA \cdot DC.} \tag{4}$$

Since the vertices of the next generation depend only on the vertices of the previous one (but not on the way the vertices are connected), one can see that W and r for the (self-intersecting) quadrilaterals $ACBD$ and $ACDB$ coincide with those for $ABCD$. This observation allows us to prove the following

Corollary 5. *The angles between the sides and the diagonals of a quadrilateral satisfy the following identities:*

$$(\cot\alpha + \cot\gamma)\cdot(\cot\beta + \cot\delta) = (\cot\alpha_1 - \cot\beta_2)\cdot(\cot\delta_2 - \cot\gamma_1),$$
$$(\cot\alpha + \cot\gamma)\cdot(\cot\beta + \cot\delta) = (\cot\delta_1 - \cot\alpha_2)\cdot(\cot\beta_1 - \cot\gamma_2)$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$ are the directed angles formed between sides and diagonals of a quadrilateral (see Figure 3).

Proof. Since the (directed) angles of $ACBD$ are $-\alpha_1, \beta_2, \gamma_1, -\delta_2$ and the directed angles of $ACDB$ are $\alpha_2, \beta_1, -\gamma_2, -\delta_1$, the identities follow from formula (1) for the ratio of similarity. □

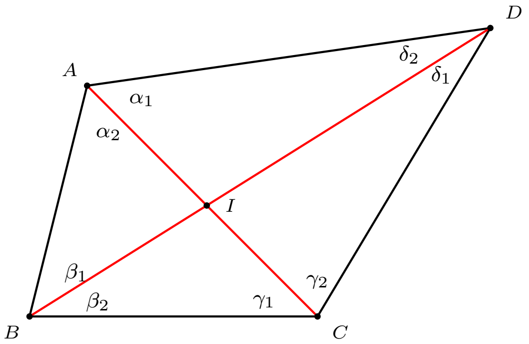

Figure 3. The angles between the sides and diagonals of a quadrilateral.

3. Properties of the center of spiral similarity

We will show that W, defined as the limit point of the iterated perpendicular bisectors construction in the case that $|r| < 1$ (or of its reverse in the case that $|r| > 1$), is the common center of all spiral similarities taking any of the triad circles into another triad circle in the iterative process.

First, we will prove that any of the triad circles of the first generation quadrilateral can be taken into another triad circle of the first generation by a spiral similarity centered at W (Theorem 9). This result allows us to view W as a generalization of the circumcenter for a noncyclic quadrilateral (Corollary 10 and Corollary 13), to prove its isoptic (Theorem 11), isodynamic (Corollary 14) and inversive (Theorem 15) properties, as well as to establish some other results. We then prove several statements that allow us to conclude (see Theorem 24) that W serves as the center of spiral similarities for any pair of triad circles of any two generations.

Several objects associated to a configuration of two circles on the plane will play a major role in establishing properties of W. We will start by recalling the definitions and basic constructions related to these objects.

3.1. *Preliminaries: circle of similitude, mid-circles and the radical axis of two circles.* Let o_1 and o_2 be two (intersecting[2]) circles on the plane with centers O_1 and O_2 and radii R_1 and R_2 respectively. Let A and B be the points of intersection of the two circles. There are several geometric objects associated to this configuration (see Figure 4):

(1) The *circle of similitude* $CS(o_1, o_2)$ is the set of points P on the plane such that the ratio of their distances to the centers of the circles is equal to the ratio of the radii of the circles:
$$\frac{PO_1}{PO_2} = \frac{R_1}{R_2}.$$
In other words, $CS(o_1, o_2)$ is the Apollonian circle determined by points O_1, O_2 and ratio R_1/R_2.

[2]Most of the constructions remain valid for non-intersecting circles. However, they sometimes have to be formulated in different terms. Since we will only deal with intersecting circles, we will restrict our attention to this case.

(2) The *radical axis* $RA(o_1, o_2)$ can be defined as the line through the points of intersection.
(3) The two *mid-circles* (sometimes also called the *circles of antisimilitude*) $MC_1(o_1, o_2)$ and $MC_2(o_1, o_2)$ are the circles that invert o_1 into o_2, and vice versa:

$$\mathrm{Inv}_{MC_i(o_1,o_2)}(o_1) = o_2, \qquad i = 1, 2.$$

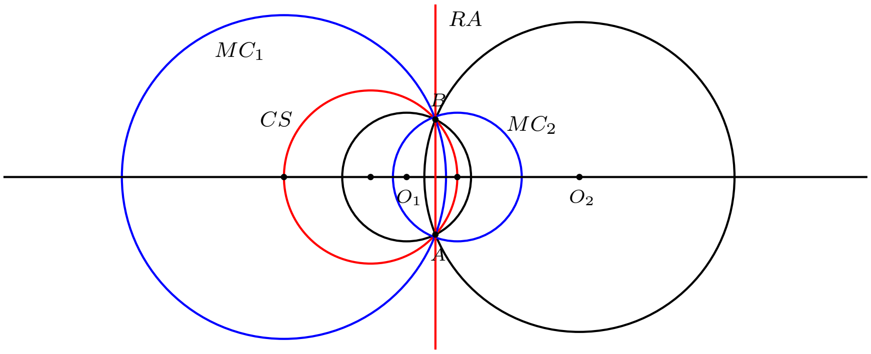

Figure 4. Circle of similitude, mid-circles and radical axis.

Here are several important properties of these objects (see [6] and [4] for more details):

(1) $CS(o_1, o_2)$ is the locus of centers of spiral similarities taking o_1 into o_2. For any $E \in CS(o_1, o_2)$, there is a spiral similarity centered at E that takes o_1 into o_2. The ratio of similarity is R_2/R_1 and the angle of rotation is $\angle O_1EO_2$.
(2) Inversion with respect to $CS(o_1, o_2)$ takes centers of o_1 and o_2 into each other:

$$\mathrm{Inv}_{CS(o_1,o_2)}(O_1) = O_2.$$

(3) Inversion with respect to any of the mid-circles exchanges the circle of similitude and the radical axis:

$$\mathrm{Inv}_{MC_i(o_1,o_2)}(CS(o_1, o_2)) = RA(o_1, o_2), \qquad i = 1, 2.$$

(4) The radical axis is the locus of centers of all circles k that are orthogonal to both o_1 and o_2.
(5) For any $P \in CS(o_1, o_2)$, inversion in a circle centered at P takes the circle of similitude of the original circles into the radical axis of the images, and the radical axis of the original circles into the circle of similitude of the images:

$$CS(o_1, o_2)' = RA(o_1', o_2'),$$
$$RA(o_1, o_2)' = CS(o_1', o_2').$$

Here $'$ denotes the image of an object under the inversion in a circle centered at $P \in CS(o_1, o_2)$.

(6) Let K, L, M be points on the circles $o_1, o_2, CS(o_1, o_2)$ respectively. Then
$$\angle AMB = \angle AKB + \angle ALB, \tag{5}$$
where the angles are taken in the sense of directed angles.

(7) Let A_1B_1 be a chord of a circle k_1 and A_2B_2 be a chord of a circle k_2. Then A_1, B_1, A_2, B_2 are on a circle o if and only if $A_1B_1 \cap A_2B_2 \in RA(k_1, k_2)$.

It is also useful to recall the construction of the center of a spiral similarity given the images of two points. Suppose that A and B are transformed into A' and B' respectively. Let $P = AA' \cap BB'$. The center O of the spiral similarity can be found as the intersection $O = (ABP) \cap (A'B'P)$. (Here and henceforth (ABP) stands for the circle going through A, B, P). We will call point P in this construction the *joint point* associated to two given points A, B and their images A', B' under spiral similarity.

There is another spiral similarity associated to the same configuration of points. Let $P' = AB \cap A'B'$ be the joint point for the spiral similarity taking A and A' into B and B' respectively. A simple geometric argument shows that the center of this spiral similarity, determined as the intersection of the circles $(AA'P') \cap (BB'P')$, coincides with O. We will call such a pair of spiral similarities centered at the same point *associated spiral similarities*.

Let $H_{i,j}^W$ be the spiral similarity centered at W that takes o_i into o_j. The following Lemma will be useful when studying properties of the limit point of the iterative process (or of its inverse):

Lemma 6. *Let o_1 and o_2 be two circles centered at O_1 and O_2 respectively and intersecting at points A and B. Let $W, R, S \in CS(o_1, o_2)$ be points on the circle of similitude such that R and S are symmetric to each other with respect to the line of centers, O_1O_2. Then the joint points corresponding to taking $O_1 \to O_2$, $R \to R_{1,2} := H_{1,2}^W(R)$ by $H_{1,2}^W$ and taking $O_2 \to O_1$, $S \to S_{2,1} := H_{2,1}^W(S)$ by $H_{2,1}^W$ coincide. The common joint point lies on O_1O_2.*

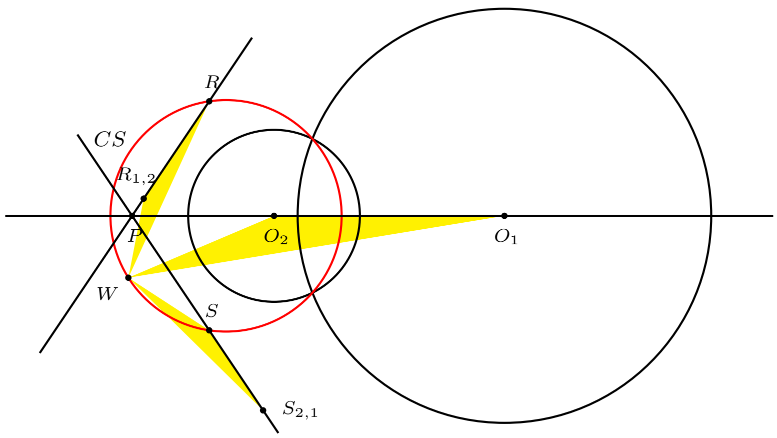

Figure 5. Lemma 6.

Proof. Perform inversion in the mid-circle. The image of $CS(o_1, o_2)$ is the radical axis $RA(o_1, o_2)$, *i.e.*, the line through A and B. The images of R and S lie on the line AB and are symmetric with respect to $I := AB \cap O_1O_2$. Similarly, the images of O_1 and O_2 are symmetric with respect to I and lie on the line of centers. By abuse of notation, we will denote the image of a point under inversion in the mid-circle by the same letter.

The lemma is equivalent to the statement that $P := (WO_1R) \cap O_1O_2$ lies on the circle (WO_2S). To show this, note that since P, R, O_1 and W lie on a circle, we have $|IP| \cdot |IO_1| = |IW| \cdot |IR|$. Since $|IO_2| = |IO_1|$ and $|IR| = |IS|$, it follows that $|IP| \cdot |IO_2| = |IW| \cdot |IS|$, which implies that W, P, O_2, S lie on a circle. After inverting back in the mid-circle, we obtain the result of the lemma. $\square$

Notice that the lemma is equivalent to the statement that

$$RR_{1,2} \cap SS_{2,1} = (WRO_1) \cap (WSO_2) \in O_1O_2.$$

3.2. *W as the center of spiral similarities for triad circles of $Q^{(1)}$.* Denote by o_1, o_2, o_3 and o_4 the triad circles $(D_1A_1B_1)$, $(A_1B_1C_1)$, $(B_1C_1D_1)$ and $(C_1D_1A_1)$ respectively.[3] For triad circles in other generations, we add an upper index indicating the generation. For example, $o_1^{(3)}$ denotes the first triad-circle in the 3rd generation quadrilateral, *i.e.*, circle $(D_3A_3B_3)$. Let T_1, T_2, T_3 and T_4 be the triad triangles $D_1A_1B_1$, $A_1B_1C_1$, $B_1C_1D_1$ and $C_1D_1A_1$ respectively.

Consider two of the triad circles of the first generation, o_i and o_j, $i \neq j \in \{1, 2, 3, 4\}$. The set of all possible centers of spiral similarity taking o_i into o_j is their circle of similitude $CS(o_i, o_j)$. If $Q^{(1)}$ is a nondegenerate quadrilateral, it can be shown that $CS(o_1, o_2)$ and $CS(o_1, o_4)$ intersect at two points and are not tangent to each other. Let W be the other point of intersection of $CS(o_1, o_2)$ and $CS(o_1, o_4)$.[4]

Let $H^W_{k,l}$ be the spiral similarity centered at W that takes o_k into o_l for any $k, l \in \{1, 2, 3, 4\}$.

Lemma 7. *Spiral similarities $H^W_{k,l}$ have the following properties:*

(1) $H^W_{1,2}(B_1) = A_1 \Longleftrightarrow H^W_{2,4}(A_1) = C_1$.
(2) $H^W_{1,2}(B_1) = A_1 \Longleftrightarrow H^W_{1,4}(B_1) = C_1$.

Proof. Assume that $H^W_{1,2}(B_1) = A_1$. Let $P_{1,2} := A_1B_1 \cap A_2B_2$ be the joint point of the spiral similarity (centered at W) taking B_1 into A_1 and A_2 into B_2. Since points $B_1, P_{1,2}, W, A_2$ lie on a circle (see Lemma 6), it follows that $\angle BWA_1 = \angle BP_{1,2}A_2 = \pi/2$. Thus, A_2B_1 is a diameter of $k_1 := (B_1P_{1,2}WA_2)$. Since o_1 is centered at A_2, the circles o_1 and k_1 are tangent at B_1. It is easy to see that the converse is also true: if o_1 and (B_1WA_2) are tangent at B_1, then $H^W_{1,2}(B_1) = A_1$.

[3] In short, the middle vertex defining the circle o_i is vertex number i (the first vertex being A_1, the second being B_1, the third being C_1 and the last being D_1).

[4] This will turn out to be the same point as the limit point of the iterative process defined in section 2, so the clash of notation is intentional.

Since $A_1, P_{1,2}, W, B_2$ lie on a circle, it follows that $\angle A_1WB_2 = \angle A_1P_{1,2}B_2 = \pi/2$. Since $B_1 \mapsto A_1$ and $A_2 \mapsto B_2$ under $H^W_{1,2}$, $\angle B_1WA_2 = \angle A_1WB_2 = \pi/2$. This implies that the circles $k_2 := (A_1P_{1,2}WB_2)$ and o_2 are tangent at A_1. It is easy to see that k_2 is tangent to o_2 if and only if k_1 is tangent to o_1.

Similarly to the above, let $P_{2,4} := A_1H^W_{2,4}(A_1) \cap B_2D_2$ be the joint point of the spiral similarity centered at W and taking o_2 into o_4. Then $P_{2,4} \in k_2$. Similarly to the argument above, k_2 is tangent to o_2 if and only if $k_4 := (C_1P_{2,4}WD_2)$ is tangent to o_4. This is equivalent to $H^W_{2,4}(A_1) = C_1$.

The second statement follows since $H^W_{1,4}(B_1) = H^W_{2,4} \circ H^W_{1,2}(B_1) = H^W_{2,4}(A_1) = C_1$. (Here and below the compositions of transformations are read right to left). □

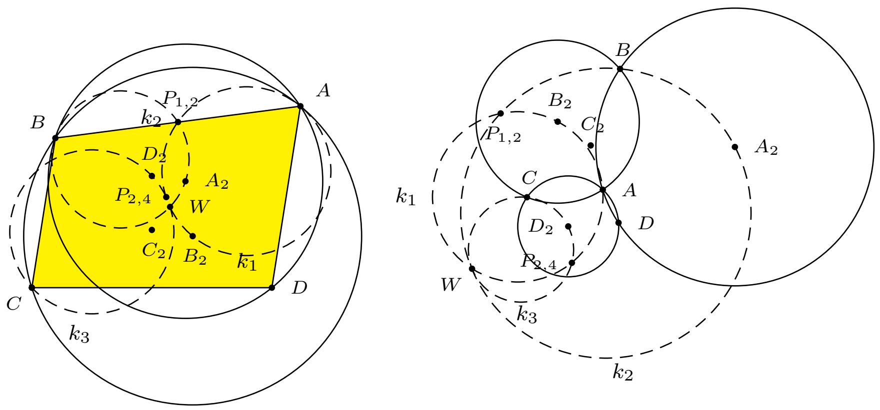

Figure 6. Proofs of Lemma 7 and Lemma 8.

Notice that circles o_1 and o_4 have two common vertices, A_1 and D_1. The next Lemma shows that $H^W_{1,4}$ takes B_1 (the third vertex on o_1) to C_1 (the third vertex on o_4). This property is very important for showing that any triad circle from the first generation can be transformed into another triad circle from the first generation by a spiral similarity centered at W. Similar properties hold for $H^W_{1,2}$ and $H^W_{2,4}$. Namely, we have

Lemma 8. $H^W_{1,4}(B_1) = C_1$, $H^W_{1,2}(D_1) = C_1$, $H^W_{4,2}(D_1) = B_1$.

Proof. Lemma 7 shows that $H^W_{1,2}(B_1) = A_1$ implies $H^W_{1,4}(B_1) = C_1$. Assume that $H^W_{1,2}(B_1) \neq A_1$. To find the image of B_1 under $H^W_{1,4}$, represent the latter as the composition $H^W_{2,4} \circ H^W_{1,2}$. First, $H^W_{1,2}(B_1) = P_{1,2}B_1 \cap (P_{1,2}B_2W)$, where $P_{1,2}$ is as in Lemma 7, see Figure 6. For brevity, let $B_{1,2} := H^W_{1,2}(B_1)$. (The indices refer to the fact that $B_{1,2}$ is the image of B under spiral similarity taking o_1 into o_2).

Now we construct $H^W_{1,4}(B_1) = H^W_{2,4}(B_{1,2})$. By Lemma 6, $H^W_{1,4}(B_1) = P_{2,4}B_{1,2} \cap (WP_{2,4}D_2)$, where $P_{2,4}$ is as in Lemma 7. Applying Lemma 6 to the circle $(WP_{2,4}D_2)$, we conclude that it passes through C_1. Since by assumption $H^W_{1,2}(B_1) \neq$

A_1, it follows that $H^W_{2,4} \circ H^W_{1,2}(B_1) = C_1$. Thus, $H^W_{1,4}(B_1) = C_1$. The other statements in the Lemma can be shown in a similar way. $\square$

The last Lemma allows us to show that W lies on all of the circles of similitude $CS(o_i, o_j)$.

Theorem 9. $W \in CS(o_i, o_j)$ *for all* $i, j \in \{1, 2, 3, 4\}$.

Proof. By definition, $W \in CS(o_1, o_2) \cap CS(o_1, o_4) \cap CS(o_2, o_4)$. We will show that $W \in CS(o_3, o_i)$ for any $i \in \{1, 2, 4\}$.

Recall that $B_1 \in CS(o_1, o_2) \cap CS(o_2, o_3)$. Let $\widetilde{W}$ be the second point in the intersection $CS(o_1, o_2) \cap CS(o_2, o_3)$, so that $CS(o_1, o_2) \cap CS(o_2, o_3) = \{B_1, \widetilde{W}\}$. By Lemma 8, $H^{\widetilde{W}}_{1,2}(D_1) = C_1$. Since $H^{\widetilde{W}}_{1,2}(A_2) = B_2$, it follows that $H^{\widetilde{W}}_{1,2} = H^W_{1,2}$, which implies that $\widetilde{W} = W$. Therefore, W is the common point for all the circles of similitude $CS(o_i, o_j)$, $i, j \in \{1, 2, 3, 4\}$. $\square$

3.3. *Properties of W.* The angle property (5) of the circle of similitude implies

Corollary 10. *The angles subtended by the quadrilateral's sides at* W *are as follows (see Figure 7):*

$$\begin{aligned}
\angle AWB &= \angle ACB + \angle ADB, \\
\angle BWC &= \angle BAC + \angle BDC, \\
\angle CWD &= \angle CAD + \angle CBD, \\
\angle DWA &= \angle DBA + \angle DCA.
\end{aligned}$$

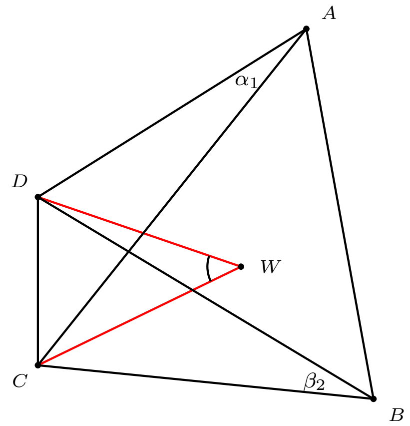

Figure 7. $\angle CWD = \angle CAD + \angle CBD$.

This allows us to view W as a replacement of the circumcenter in a certain sense: the angle relations above are generalizations of the relation $\angle AOB = \angle ACB + \angle ADB$ between the angles in a cyclic quadrilateral $ABCD$ with circumcenter O. (Of course, in this special case, $\angle ACB = \angle ADB$).

Since $W \in CS(o_i, o_j)$ for all i, j, W can be used as the center of spiral similarity taking any of the triad circles into another triad circle. This implies the following

Theorem 11. *(Isoptic property) All the triad circles o_i subtend equal angles at W.*

In particular, W is inside of all of the triad circles in the case of a convex quadrilateral and outside of all of the triad circles in the case of a concave quadrilateral. (This was pointed out by Scimemi in [17]). If W is inside of a triad circle, the isoptic angle equals to $\angle TOT'$, where T and T' are the points on the circle so that TT' goes through W and $TT' \perp OW$. (See Figure 8, where $\angle T_1A_2W$ and $\angle T_4B_2W$ are halves of the isoptic angle in o_1 and o_4 respectively). If W is outside of a triad circle centered at O and WT is the tangent line to the circle, so that T is point of tangency, $\angle OTW$ is half of the isoptic angle. Inverting in a triad circle of the second generation, we get that the triad circles are viewed at equal angles from the vertices opposite to their centers (see Figure 8).

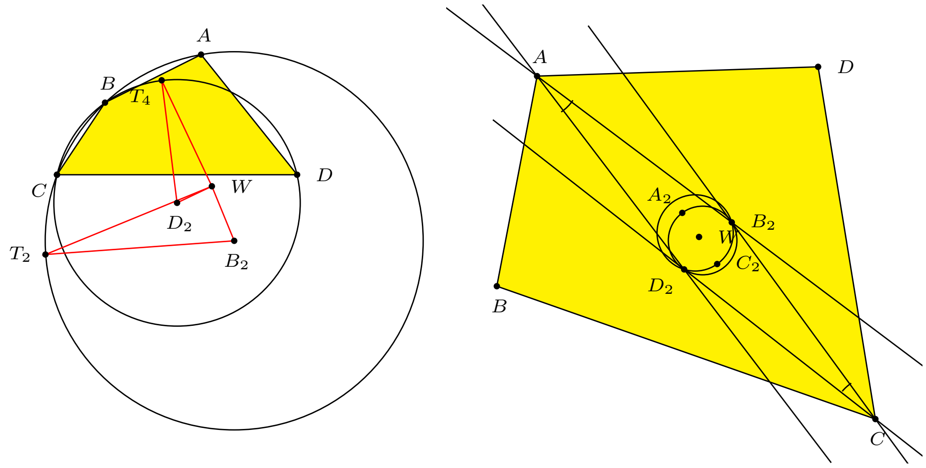

Figure 8. The isoptic angles before and after inversion.

Recall that the *power of a point* P with respect to a circle o centered at O with radius R is the square of the length of the tangent from P to the circle, that is,

$$h = |PO|^2 - R^2.$$

The isoptic property implies the following

Corollary 12. *The powers of W with respect to triad circles are proportional to the squares of the radii of the triad circles.*

This property of the isoptic point was shown by Neville in [13] using tetracyclic coordinates and the Darboux-Frobenius identity.

Let a, b, c, d be sides of the quadrilateral. For any $x \in \{a, b, c, d\}$, let F_x be the foot of the perpendicular bisector of side x on the opposite side. (E.g., F_a is the intersection of the perpendicular bisector to the side AB and the side CD). The following corollary follows from Lemma 8 and expresses W as the point of intersection of several circles going through the vertices of the first and second generation quadrilaterals, as well as the intersections of the perpendicular bisectors of the original quadrilateral with the opposite sides (see Figure 9).

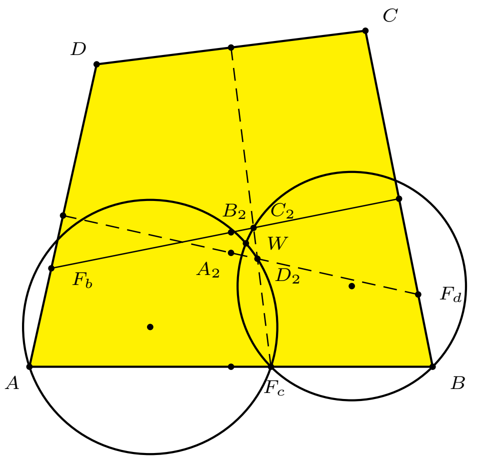

Figure 9. W as the intersection of circles $(A_1F_cD_2)$ and $(B_1F_cC_2)$ in (6).

Corollary 13. *W is a common point of the following eight circles:*

$$\begin{array}{cccc} (A_1F_bB_2), & (A_1F_cD_2), & (B_1F_cC_2), & (B_1F_dA_2), \\ (C_1F_dD_2), & (C_1F_aB_2), & (D_1F_aA_2), & (D_1F_bC_2). \end{array} \tag{6}$$

Remark. This property can be viewed as the generalization of the following property of the circumcenter of a triangle:

Given a triangle ABC with sides a, b, c opposite to vertices A, B, C, let F_{kl} denote the feet of the perpendicular bisector to side k on the side l (or its extension), where $k, l \in \{a, b, c\}$. Then the circumcenter is the common point of three circles going through vertices and feet of the perpendicular bisectors in the following way [5]:

$$O = (ABF_{ab}F_{ba}) \cap (BCF_{bc}F_{cb}) \cap (CAF_{ca}F_{ac}), \tag{7}$$

see Figure 10.

The similarity between (7) and (6) supports the analogy of the isoptic point with the circumcenter.

The last corollary provides a quick way of constructing W. First, construct two vertices (e.g., A_2 and D_2) of the second generation by intersecting the perpendicular bisectors. Let F_d be the intersection of the lines A_2D_2 and B_1C_1. Then W is obtained as the second point of intersection of the two circles $(B_1F_bA_2)$ and $(C_1F_bD_2)$.

[5]Note also that this statement is related to Miquel's theorem as follows. Take any three points P, Q, R on the three circles in (7), so that A, B, C are points on the sides PQ, QR, PQ of PQR. Then the statement becomes Miquel's theorem for PQR and points A, B, C on its sides, with the extra condition that the point of intersection of the circles (PAC), (QAB), (RBC) is the circumcenter of ABC.

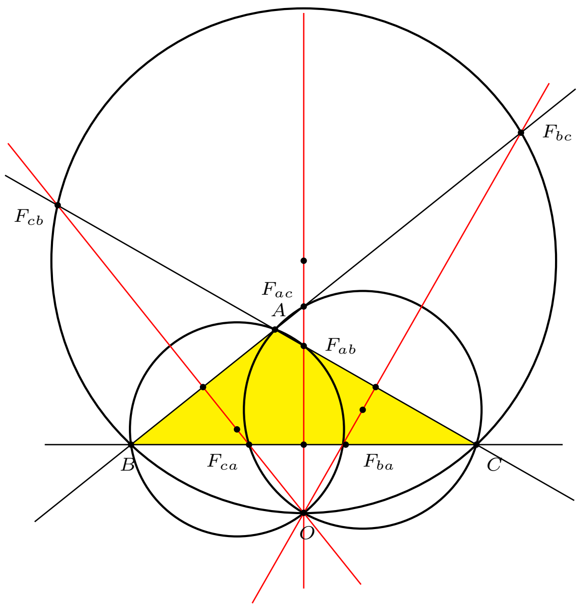

Figure 10. Circumcenter as intersection of circles in (7).

Recall the definition of isodynamic points of a triangle. Let $A_1A_2A_3$ be a triangle with sides a_1, a_2, a_3 opposite to the vertices A_1, A_2, A_3. For each $i, j \in \{1, 2, 3\}$, where $i \neq j$, consider the circle o_{ij} centered at A_i and going through A_j. The circle of similitude $CS(o_{ij}, o_{kj})$ of two distinct circles o_{ij} and o_{kj} is the Apollonian circle with respect to points A_i, A_k with ratio $r_{ik} = \frac{a_k}{a_i}$. It is easy to see that the three Apollonian circles intersect in two points, S and S', which are called the *isodynamic points* of the triangle.

Here are some properties of isodynamic points (see, e.g., [6], [4] for more details):

(1) The distances from S (and S') to the vertices are inversely proportional to the opposite side lengths:

$$|SA_1| : |SA_2| : |SA_3| = \frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3}. \tag{8}$$

Equivalently,

$$|SA_i| : |SA_j| = \sin\alpha_j : \sin\alpha_i, \qquad i \neq j \in \{1, 2, 3\},$$

where α_i is the angle $\angle A_i$ in the triangle. The isodynamic points can be characterized as the points having this distance property. Note that since the radii of the circles used to define the circles of similitude are the sides, the last property means that distances from isodynamic points to the vertices are inversely proportional to the radii of the circles.

(2) The pedal triangle of a point on the plane of $A_1A_2A_3$ is equilateral if and only if the point is one of the isodynamic points.

(3) The triangle whose vertices are obtained by inversion of A_1, A_2, A_3 with respect to a circle centered at a point P is equilateral if and only if P is one of the isodynamic points of $A_1A_2A_3$.

It turns out that W has properties (Corollary 14, Theorem 30, Theorem 27) similar to properties 1–3 of S.

Corollary 14. *(Isodynamic property of W) The distances from W to the vertices of the quadrilateral are inversely proportional to the radii of the triad-circles going through the remaining three vertices:*

$$|WA_1| : |WB_1| : |WC_1| : |WD_1| = \frac{1}{R_3} : \frac{1}{R_4} : \frac{1}{R_1} : \frac{1}{R_2},$$

where R_i is the radius of the triad-circle o_i. Equivalently, the ratios of the distances from W to the vertices are as follows:

$$\begin{aligned} |WA_1| : |WB_1| &= |A_1C_1|\sin\gamma : |B_1D_1|\sin\delta, \\ |WA_1| : |WC_1| &= \sin\gamma : \sin\alpha, \\ |WB_1| : |WD_1| &= \sin\delta : \sin\beta. \end{aligned}$$

From analysis of similar triangles in the iterative process, it is easy to see that the limit point of the process satisfies the above distance relations. Therefore, W (defined at the beginning of this section as the second point of intersection of $CS(o_1, o_2)$ and $CS(o_1, o_4)$) is the limit point of the iterative process.

One more property expresses W as the image of a vertex of the first generation under the inversion in a triad circle of the second generation. Namely, we have the following

Theorem 15 (Inversive property of W)**.**

$$W = Inv_{o_1^{(2)}}(A_1) = Inv_{o_2^{(2)}}(B_1) = Inv_{o_3^{(2)}}(C_1) = Inv_{o_4^{(2)}}(D_1). \tag{9}$$

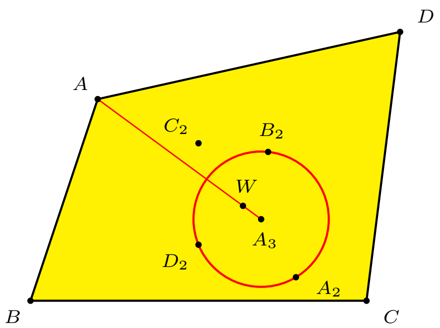

Figure 11. Inversive property of W

Proof. To prove the first equality, perform inversion in a circle centered at A_1. The image of a point under the inversion will be denoted by the same letter with a prime. The images of the circles of similitude $CS(o_1, o_2)$, $CS(o_4, o_1)$ and $CS(o_2, o_4)$ are the perpendicular bisectors of the segments $A'_2B'_2$, $D'_2A'_2$ and $B'_2D'_2$ respectively. By Theorem 9, these perpendicular bisectors intersect in W'. Since W' is the circumcenter of $D'_2A'_2B'_2$, it follows that $\text{Inv}_{o_1^{(2)\prime}}(W') = A'_1$. Inverting back in the same circle centered at A_1, we obtain $\text{Inv}_{o_1^{(2)}}(W) = A_1$. The rest of the statements follow analogously. □

The fact that the inversions of each of the vertices in triad circles defined by the remaining three vertices coincide in one point was proved by Parry and Longuet-Higgins in [14].

Notice that the statement of Theorem 15 can be rephrased in a way that does not refer to the original quadrilateral, so that we can obtain a property of circumcenters of four triangles taking a special configuration on the plane. Recall that an inversion takes a pair of points which are inverses of each other with respect to a (different) circle into a pair of points which are inverses of each other with respect to the image of the circle, that is if $S = \text{Inv}_k(T)$, then $S' = \text{Inv}_{k'}(T')$, where $'$ denotes the image of a point (or a circle) under inversion in a given circle. Using this and property 2 of circles of similitude, we obtain the corollary below. In the statement, A, B, C, P, X, Y, Z, O play the role of $A'_2, B'_2, D'_2, A_1, B'_1, C'_1, D'_1, W'_1$ in Theorem 15.

Corollary 16. *Let P be a point on the plane of ABC. Let points O, X, Y and Z be the circumcenters of ABC, APB, BPC and CPA respectively.* Then

$$\mathit{Inv}_{(ZOX)}(A) = \mathit{Inv}_{(XOY)}(B) = \mathit{Inv}_{(YOZ)}(C) = \mathit{Inv}_{(XYZ)}(P). \tag{10}$$

Furthermore,

$$Iso_{ZOX}(A) = Y, \quad Iso_{XOY}(B) = Z, \quad Iso_{YOZ}(C) = X.$$

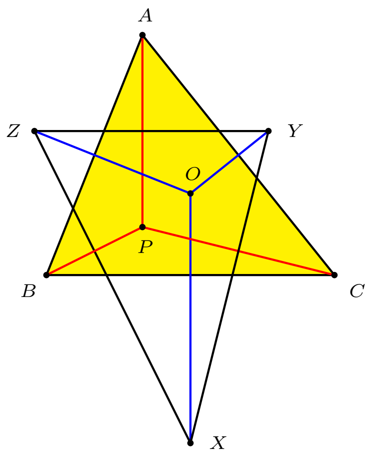

Figure 12. Corollary 16.

Combining the description of the reverse iterative process (Theorem 2) and the inversive property of W (Theorem 15), we obtain one more direct way of constructing W without having to refer to the iterative process:

Theorem 17. *Let A, B, C, D be four points in general position. Then*

$$W = Inv_{o_3} \circ Iso_{T_3}(A_1) = Inv_{o_4} \circ Iso_{T_4}(B_1) = Inv_{o_1} \circ Iso_{T_1}(C_1) = Inv_{o_2} \circ Iso_{T_2}(D_1),$$

where o_i is the ith triad circle, and T_i is the ith triad triangle.

This property suggests a surprising relation between inversion and isogonal conjugation.

Taking into account that the circumcenter and the orthocenter of a triangle are isogonal conjugates of each other, we obtain the following

Corollary 18. *W is the point at infinity if and only if the vertices of the quadrilateral form an orthocentric system.*

3.4. *W as the center of similarity for any pair of triad circles.* To show that W is the center of spiral similarity for any pair of triad circles (of possibly different generations), we first need to prove Lemmas 19—21 below.

The following lemma shows that given three points on a circle — two fixed and one variable — the locus of the joint points of the spiral similarities taking one fixed point into the other applied to the variable point is a line.

Lemma 19. *Let $M, N \in o$ and $W \notin o$. For every point $L \in o$, define*

$$J := (MWL) \cap NL.$$

The locus of points J is a straight line going through W.

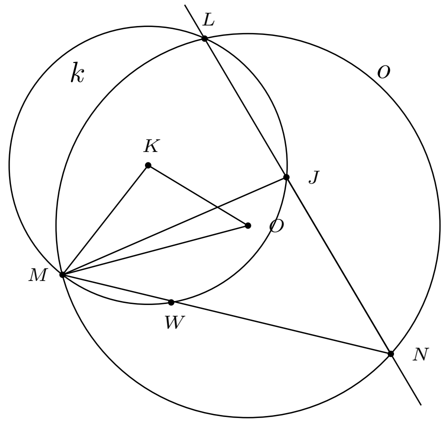

Figure 13. Lemma 19.

Proof. For each point $L \in o$, let K be the center of the circle $k := (MWL)$. The locus of centers of the circles k is the perpendicular bisector of the segment MW. Since $M \in o \cap k$, there is a spiral similarity centered at M with joint point L that takes k into o. This spiral similarity takes $K \mapsto O$ and $J \mapsto N$, where O is the center of o. Thus, $MOK \simeq MNJ$. Since M, O, K are fixed and the locus of K is a line (the perpendicular bisector), the locus of points J is also a line.

To show that the line goes through W, let $L = NW \cap o$. Then $J = W$. $\square$

In the setup of the lemma above, let $H^W_{L,N}$ be the spiral similarity centered at W that takes L into N. Let M' be the image of M under this spiral similarity. Then J is the joint point for the spiral similarity taking $L \mapsto N$ and $M \mapsto M'$.

The following two results are used for proving that W lies on the circle of similitude of o_3 and $o_1^{(2)}$.

Lemma 20. *Let AC, ZX be two distinct chords of a circle o, and W be the center of spiral similarity taking ZX into AC. Let $H^W_{B,C}$ be the spiral similarity centered at W that takes a point $B \in o$ into C. Then $H^W_{B,C}(Z) \in o$.*

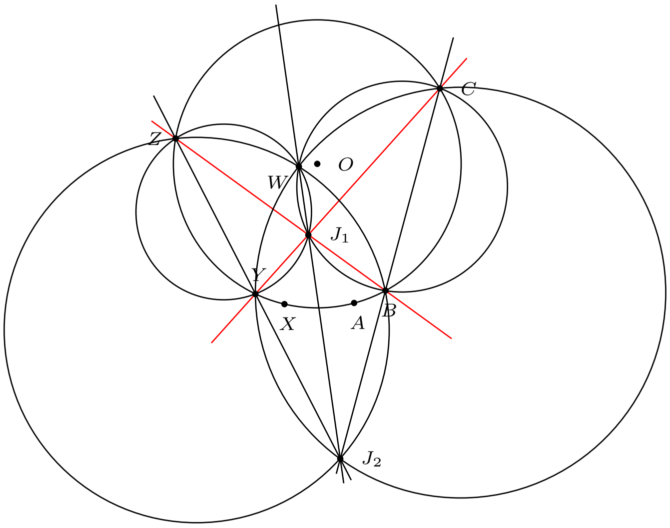

Figure 14. Lemma 20.

Proof. Let l be the locus of the joint points corresponding to $M = Z$, $N = C$ in Lemma 19. Let J_1 be the joint point corresponding to $L = B$. Then $J_1, W \in l$.

Let J_2 be the joint point corresponding to $M = C$, $N = Z$ and $L = B$ in Lemma 19.

Let $Y = J_2C \cap J_1Z$. By properties of spiral similarity, $Y = H^W_{B,C}(Z)$.

Notice that by definition of J_1, points J_1, B, C are on a line. Similarly, by definition of J_2, points J_2, B, Z are on a line as well. By definition of Y, points Y, J_2, C are on a line, as are points Z, Y, J_1. The intersections of these four lines form a complete quadrilateral. By Miquel's theorem, the circumcircles of the triangles BJ_1Z, BJ_2C, J_2YZ, CJ_1Y have a common point, the Miquel point for the complete quadrilateral. By definitions of J_1 and J_2 , $(BJ_2C) \cap (BZJ_1) = \{B, W\}$. Thus, the Miquel point is either B or W. It is easy to see that B can not be the Miquel point (if $B \neq C, Z$). Thus, W is the Miquel point of the complete quadrilateral. This implies that (YCJ_1), (YZJ_2) both go through W.

Consider the circles $k_1 = (ZWJ_2Y)$ and $k_2 = (CWJ_2B)$. Then $RA(k_1, k_2) = l$. Since $ZY \cap BC = J_1 \in l = RA(k_1, k_2)$, by property 7 in section 3.1, points Z, Y, B, C are on a circle. Thus, $Y \in o$. $\square$

Remark. Notice that in the proof of the Lemma above there are three spiral similarities centered at W that take each of the sides of XYZ into the corresponding side of CBA. We will call such a construction a *cross-spiral* and say that the two triangles are obtained from each other via a cross-spiral. [6]

Lemma 21. *Let PQ be a chord on a circle o centered at O. If $W \notin (POQ)$, there is a spiral similarity centered at W that takes PQ into another chord of the circle o.*

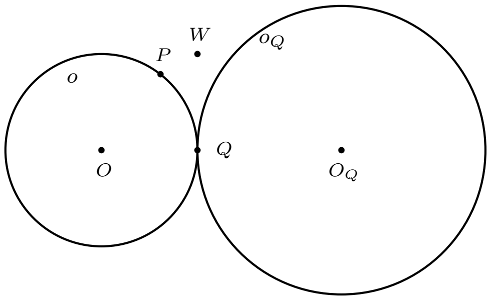

Figure 15. Proof of Lemma 21.

Proof. Let $H^W_{P,P'}$ be the spiral similarity centered at W that takes P into another point P' on circle o. As P' traces out o, the images $H^W_{P,P'}(Q)$ of Q trace out another circle, o_Q. To see this, consider the associated spiral similarity and notice that $H^W_{P,Q}(P') = Q'$. Since P' traces out o, $H^W_{P,Q}(o) = o_Q$. Since $Q = H^W_{P,P}(Q) \in o_Q$, it follows that $Q \in o \cap o_Q$.

Suppose that o and o_Q are tangent at Q. From $H_{P,Q}(o) = o_Q$ it follows that the joint point is Q, and therefore the quadrilateral $PQWO$ must be cyclic. Since $W \notin (POQ)$, this can not be the case. Thus, the intersection $o \cap o_Q$ contains two points, Q and Q'. This implies that there is a unique chord, $P'Q'$, of o to which PQ can be taken by a spiral similarity centered at W. $\square$

Theorem 22. $W \in CS(o_3, o_1^{(2)})$.

Proof. We've shown previously that W is on all six circles of similitude of $A_1B_1C_1D_1$. Since W has the property that

$$H^W_{C_1,B_2} : C_1 \mapsto B_2, D_1 \mapsto A_2,$$
$$H^W_{B_1,A_2} : B_1 \mapsto A_2, C_1 \mapsto D_2,$$

it follows that
$$H^W_{B_2,C_1} H^W_{B_1,A_2}(B_1) = H^W_{B_2,C_1}(A_2) = D_1.$$

[6]Clearly, the sides of any triangle can be taken into the sides of any other triangle by three spiral similarities. The special property of the cross-spiral is that the centers of all three spiral similarities are at the same point.

Since the spiral similarities centered at W commute, it follows that

$$H^W_{B_2,C_1}H^W_{B_1,A_2}(B_2) = H^W_{B_1,A_2}H^W_{B_2,C_1}(B_2) = H^W_{B_1A_2}(C_1) = D_2.$$

This means that there is a spiral similarity centered at W that takes B_1D_1 into B_2D_2. Therefore, $B_1C_1D_1$ and $D_2A_2B_2$ are related by a cross-spiral centered at W.

We now show that there is a cross-spiral that takes $D_2A_2B_2$ into another triangle, XYZ, with vertices on the same circle, $o_1^{(2)} = (D_2A_2B_2)$. This will imply that there is a spiral similarity centered at W that takes $B_1C_1D_1$ into XYZ. This, in turn, implies that W is a center of spiral similarity taking o_3 into $o_1^{(2)}$.

Assume that $W \in (B_2A_3D_2)$. Since inversion in $(D_2A_2B_2)$ takes W into A_1 and $(B_2A_3D_2)$ into B_2D_2, it follows that $A_1 \in B_2D_2$. This can not be the case for a nondegenerate quadrilateral. Thus, $W \in (B_2A_3D_2)$.

By Lemma 21, there is a spiral similarity centered at W that takes the chord B_2D_2 into another chord, XZ, of the circle $(D_2A_2B_2)$. Thus, there is a spiral similarity taking B_2D_2 into XZ and centered at W.

By Lemma 20, there is a point $Y \in o_1^{(2)}$ such that XYZ and $B_2A_2D_2$ are related by a cross-spiral centered at W. (See also the remark after Lemma 20).

By composing the two cross-spirals, we conclude that $XYZ \sim D_1C_1B_1$. Since $(XYZ) = o_1^{(2)}$ and $(D_1C_1B_1) = o_3$, it follows that $W \in CS(o_1^{(2)}, o_3)$. □

Corollary 23. *$W \in CS(o_i^{(1)}, o_j^{(k)})$ for any i, j, k.*

Proof. Since there is a spiral similarity centered at W that takes A_1B_1 into C_2D_2, Theorem 22 implies that $W \in CS(o_1, o_4^{(2)})$. Since $W \in CS(o_1, o_2)$, it follows that $W \in CS(o_4^{(2)}, o_2)$. Since W is on two circles of similitude for the second generation, it follows that it is on all four. Furthermore, we can apply Theorem 22 to the triad circles of the second and third generation to show that W is also on all four circles of similitude of the third generation.

Finally, a simple induction argument shows that $W \in CS(o_j^{(1)}, o_i^{(k)})$. Assuming $W \in CS(o_j^{(1)}, o_i^{(k-1)})$, Theorem 22 implies that $W \in CS(o_i^{(k-1)}, o_i^{(k)})$. Thus, $W \in CS(o_j^{(1)}, o_i^{(k)})$. □

Using this, we can show that W lies on all the circles of similitude:

Theorem 24. *$W \in CS(o_i^{(k)}, o_j^{(l)})$ for all $i, j \in \{1, 2, 3, 4\}$ and any k, l.*

Recall that the *complete quadrangle* is the configuration of 6 lines going through all possible pairs of 4 given vertices.

Theorem 25. *(Inversion in a circle centered at W) Consider the complete quadrangle determined by a nondegenerate quadrilateral. Inversion in W transforms*

- 6 *lines of the complete quadrilateral into the* 6 *circles of similitude of the triad circles of the image quadrilateral;*
- 6 *circles of similitude of the triad circles into the* 6 *lines of the image quadrangle.*

Proof. Observe that the 6 lines of the quadrangle are the radical axes of the triad circles taken in pairs. Since W belongs to all the circles of similitude of triad circles, by property 5 in section 3.1, inversion in a circle centered in W takes radical axes into the circles of similitude. This implies the statement. □

4. Pedal properties

4.1. *Pedal of W with respect to the original quadrilateral.* Since W has a distance property similar to that of the isodynamic points of a triangle (see Corollary 14), it is interesting to investigate whether the analogy between these two points extends to pedal properties. In this section we show that the pedal quadrilateral of W with respect to $A_1B_1C_1D_1$ (and, more generally, with respect to any $Q^{(n)}$) is a nondegenerate parallelogram. Moreover, W is the unique point whose pedal has such a property. These statements rely on the fact that W lies on the intersection of two circles of similitude, $CS(o_1, o_3)$ and $CS(o_2, o_4)$.

First, consider the pedal of a point that lies on one of these circles of similitude.

Lemma 26. *Let $P_aP_bP_cP_d$ be the pedal quadrilateral of P with respect to $ABCD_1$. Then*

- *$P_aP_bP_cP_d$ is a trapezoid with $P_aP_d||P_bP_c$ if and only if $P \in CS(o_2, o_4)$;*
- *$P_aP_bP_cP_d$ is a trapezoid with $P_aP_b||P_cP_d$ if and only if $P \in CS(o_1, o_3)$.*

Proof. Assume that $P \in CS(o_2, o_4)$. Let $K = AC \cap P_aP_d$ and $L = AC \cap P_bP_c$. We will show that $\angle AKP_d + \angle CLP_c = \pi$, which implies $P_aP_d||P_bP_c$.

Let $\theta = \angle APP_a$. Since AP_aPP_d is cyclic, $\angle AP_dP_a = \theta$. Then

$$\angle AKP_d = \pi - \alpha_1 - \theta. \tag{11}$$

On the other hand, $\angle CLP_c = \pi - \gamma_2 - \angle LP_cC$. Since PP_bCP_c is cyclic, it follows that $\angle LP_cC = \angle P_bPC$.

We now find the latter angle. Since $P \in CS(o_2, o_4)$, by property (5) of the circle of similitude (see §3.1), it follows that $\angle APC = \pi + \delta + \beta$. Since P_aPP_bB is cyclic, $\angle P_aPP_b = \pi - \beta$. Therefore, $\angle P_bPC = \delta - \theta$. This implies that

$$\angle CLP_c = \pi - \gamma_2 - \delta + \theta. \tag{12}$$

Adding (11) and (12), we obtain $\angle AKP_d + \angle CLP_c = \pi$.

Reasoning backwards, it is easy to see that $P_aP_d||P_bP_c$ implies that $P \in CS(o_2, o_4)$. □

Let S be the second point of intersection of $CS(o_1, o_3)$ and $CS(o_2, o_4)$, so that $CS(o_1, o_3) \cap CS(o_2, o_4) = \{W, S\}$. The Lemma above implies that the pedal quadrilateral of a point is a parallelogram if and only if this point is either W or S.

Theorem 27. *The pedal quadrilateral of W is a parallelogram whose angles equal to those of the Varignon parallelogram.*

Proof. Since $W \in CS(o_1, o_2) \cap CS(o_3, o_4)$, property (5) of the circle of similitude implies that

$$\begin{aligned} \angle AWB &= \angle ACB + \angle ADB = \gamma_1 + \delta_2, \\ \angle CWD &= \angle CAD + \angle CBD = \alpha_1 + \beta_2, \end{aligned}$$

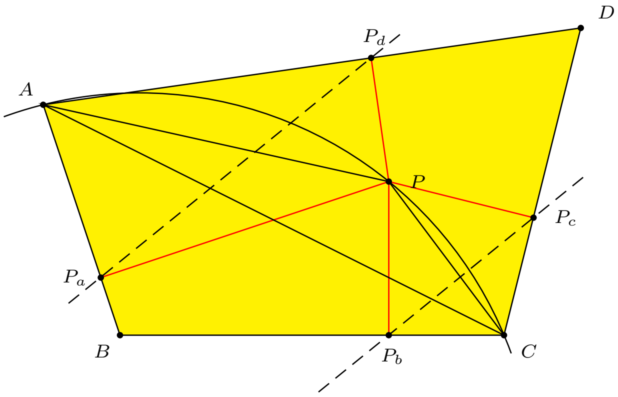

Figure 16. The pedal quadrilateral of a point on $CS(o_2, o_4)$ has two parallel sides.

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are the angles between the quadrilateral's sides and diagonals, as before (see Figure 3). Let $\angle AWW_a = x$ and $\angle W_cWC = y$. Since the quadrilaterals W_aWW_dA and W_cWW_bC are cyclic, $\angle W_aW_dA = x$ and $\angle W_cW_bC = y$. Therefore,

$$\angle W_aW_bB = \angle AWB - \angle AWW_a = \gamma_1 + \delta_2 - x,$$
$$\angle W_cW_dD = \angle CWD - \angle W_cWC = \alpha_1 + \beta_2 - y.$$

Finding supplements and adding, we obtain

$$\begin{aligned}\angle W_aW_dW_c + \angle W_aW_bW_c &= (\pi - x - \alpha_1 - \beta_2 + y) + (\pi - y - \gamma_1 - \delta_2 + x)\\ &= 2\pi - \alpha_1 - \beta_2 - \gamma_1 - \delta_2\\ &= 2\pi - (2\pi - 2\angle AIC) = 2\angle AIC,\end{aligned}$$

where $\angle AIC$ is the angle formed by the intersection of the diagonals. Thus, $W_aW_bW_cW_d$ is a parallelogram with the same angles as those of the Varignon parallelogram $M_aM_BM_bM_c$, where M_x is the midpoint of side x, for any $x \in \{a, b, c, d\}$. $\square$

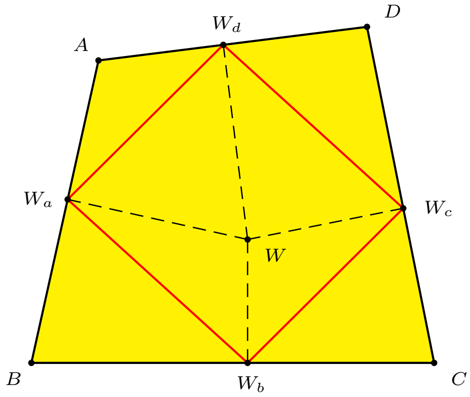

Figure 17. The pedal parallelogram of W.

It is interesting to note the following

Corollary 28. *The pedal of W with respect to the self-intersecting quadrilateral $ACBD$ (whose sides are the two diagonals and two opposite sides of the original quadrilateral) is also a parallelogram.*

The Theorem above also implies that the pedal of W is nondegenerate. (We will see later that the pedal of S degenerates to four points lying on a straight line). While examples show that the pedal of W and the Varignon parallelogram have different ratios of sides (and, therefore, are not similar in general), it is easy to see that they coincide in the case of a cyclic quadrilateral:

Corollary 29. *The Varignon parallelogram $M_aM_bM_cM_d$ is a pedal parallelogram of a point if and only if the quadrilateral is cyclic and the point is the circumcenter. In this case, $M_aM_bM_cM_d = W_aW_bW_cW_d$.*

Theorem 30. *The pedal quadrilateral of a point with respect to quadrilateral $ABCD$ is a nondegenerate parallelogram if and only if this point is W.*

Proof. By Lemma 26, if $P \in CS(o_1, o_3) \cap CS(o_2, o_4)$, then both pairs of opposite sides of the pedal quadrilateral $P_aP_bP_cP_d$ are parallel.

Assume that the pedal quadrilateral $P_aP_bP_cP_d$ of P is a nondegenerate parallelogram. Since P_dAP_aP is a cyclic quadrilateral,

$$\begin{aligned} |P_aP_d| &= \frac{|PA|}{2\sin\alpha}, \\ |P_bP_c| &= \frac{|PC|}{2\sin\gamma}. \end{aligned}$$

The assumption $|P_aP_d| = |P_bP_c$— implies that $|PA| : |PC| = \sin\gamma : \sin\alpha$. Similarly, $|P_aP_b| = |P_cP_d|$ implies $|PB| : |PD| = \sin\delta : \sin\beta$, so that P must be on the Apollonian circle with respect to A, C with ratio $\sin\gamma : \sin\alpha$ and on the Apollonian circle with respect to B, D with ratio $\sin\delta : \sin\beta$. These Apollonian circles are easily shown to be $CS(o_1^{(0)}, o_3^{(0)})$ and $CS(o_2^{(0)}, o_4^{(0)})$, the circles of similitude of the previous generation quadrilateral. One of the intersections of these two circles of similitude is W. Let Y be the other point of intersection. Computing the ratios of distances from Y to the vertices, one can show that the pedal of Y is an isosceles trapezoid. That is, instead of two pairs of equal opposite sides, it has one pair of equal opposite sides and two equal diagonals. This, in particular, means that Y does not lie on $CS(o_1, o_3) \cap CS(o_2, o_4)$. It follows that W is the only point for which the pedal is a nondegenerate parallelogram. □

Remark. Note that another interesting pedal property of a quadrilateral was proved by Lawlor in [9, 10]. For each vertex, consider its pedal triangle with respect to the triangle formed by the remaining vertices. The four resulting pedal triangles are directly similar to each other. Moreover, the center of similarity is the so-called *nine-circle point*, denoted by H in Scimemi's paper [17].

4.2. *Simson line of a quadrilateral.* Recall that for any point on the circumcircle of a triangle, the feet of the perpendiculars dropped from the point to the triangle's sides lie on a line, called the *Simson line* corresponding to the point (see Figure

18). Remarkably, in the case of a quadrilateral, Lemma 26 and Theorem 30 imply that there exists a unique point for which the feet of the perpendiculars dropped to the sides are on a line (see Theorem 31 below).

In the case of a noncyclic quadrilateral, this point turns out to be the second point of intersection of $CS(o_1, o_3)$ and $CS(o_2, o_4)$, which we denote by S. For a cyclic quadrilateral $ABCD$ with circumcenter O, even though all triad circles coincide, one can view the circles (BOD) and (AOC) as the replacements of $CS(o_1, o_3)$ and $CS(o_2, o_4)$ respectively. The second point of intersection of these two circles, $S \in (BOD) \cap (AOC)$, $S \neq W$ also has the property that the feet of the perpendiculars to the sides lie on a line. Similarly to the noncyclic case (see Lemma 26), one can start by showing that the pedal quadrilateral of a point is a trapezoid if and only if the point lies on one of the two circles, (BOD) or (AOC).

In analogy with the case of a triangle, we will call the line $S_aS_bS_cS_d$ the *Simson line* and S the *Simson point of a quadrilateral*, see Fig. 18.

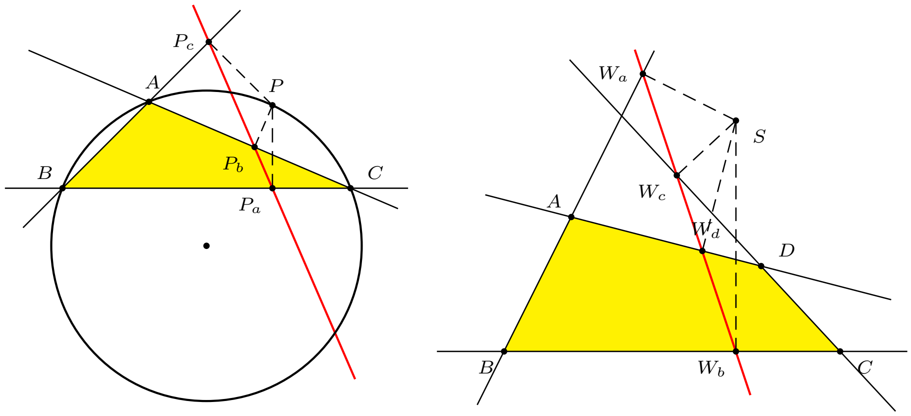

Figure 18. A Simson line for a triangle and the Simson line of a quadrilateral.

Theorem 31. *(The Simson line of a quadrilateral) The feet of the perpendiculars dropped to the sides from a point on the plane of a quadrilateral lie on a straight line if and only if this point is the Simson point.*

Unlike in the case of a triangle, where every point on the circumcircle produces a Simson line, the Simson line of a quadrilateral is unique. When the original quadrilateral is a trapezoid, the Simson point is the point of intersection of the two nonparallel sides. In particular, when the original quadrilateral is a parallelogram, the Simson point is point at infinity. The existence of this point is also mentioned in [6].

Recall that all circles of similitude intersect at W. The remaining $\binom{6}{2} = 15$ intersections of pairs of circles of similitude are the Simson points with respect to the $\binom{6}{4} = 15$ quadrilaterals obtained by choosing 4 out of the lines forming the complete quadrangle. Thus for each of the 15 quadrilaterals associated to a complete quadrangle there is a Simson point lying on a pair of circles of similitude.

4.3. *Isogonal conjugation with respect to a quadrilateral.* Recall that the isogonal conjugate of the first isodynamic point of a triangle is the Fermat point, *i.e.*, the point minimizing the sum of the distances to vertices of the triangle. Continuing to explore the analogy of W with the isodynamic point, we will now define isogonal conjugation with respect to a quadrilateral and study the properties of W and S with respect to this operation.

Let P be a point on the plane of $ABCD$. Let l_A, l_B, l_C, l_D be the reflections of the lines AP, BP, CP, DP in the bisectors of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ respectively.

Definition. Let $P_A = l_A \cap l_B$, $P_B = l_B \cap l_C$, $P_C = l_C \cap l_D$, $P_D = l_D \cap l_A$. The quadrilateral $P_A P_B P_C P_D$ will be called the *isogonal conjugate of* P with respect to $ABCD$ and denoted by $Iso_{ABCD}(P)$.

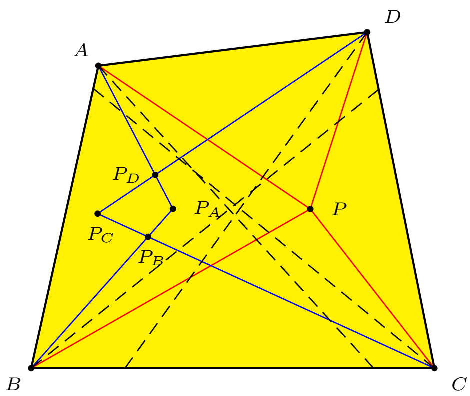

Figure 19. Isogonal conjugation with respect to a quadrilateral

The following Lemma relates the isogonal conjugate and pedal quadrilaterals of a given point:

Lemma 32. *The sides of the isogonal conjugate quadrilateral and the pedal quadrilateral of a given point are perpendicular to each other.*

Proof. Let b_A be the bisector of the $\angle DAB$. Let $I = l_A \cap P_a P_d$ and $J = b_A \cap P_a P_d$. Since $AP_a PP_d$ is cyclic, it follows that $\angle P_d AP = \angle P_d P_a P$. Since $PP_a \perp P_a A$, it follows that $AI \perp P_a P_d$. Therefore, $P_A P_D \perp P_a P_d$. The same proof works for the other sides, of course. □

The Lemma immediately implies the following properties of the isogonal conjugates of W and S:

Theorem 33. *The isogonal conjugate of W is a parallelogram. The isogonal conjugate of S is the degenerate quadrilateral whose four vertices coincide at infinity.*

The latter statement can be viewed as an analog of the following property of isogonal conjugation with respect to a triangle: the isogonal conjugate of any point on the circumcircle is the point at infinity.

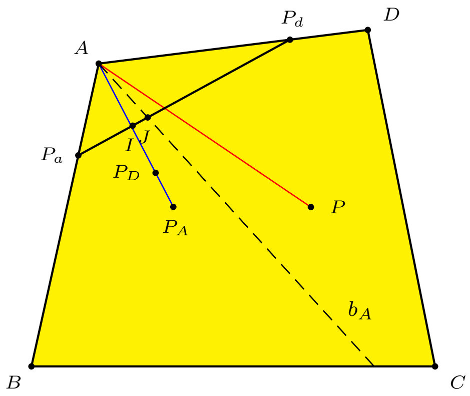

Figure 20. Lemma 32.

4.4. *Reconstruction of the quadrilateral.* The paper by Scimemi [17] has an extensive discussion of how one can reconstruct the quadrilateral from its central points. Here we just want to point out the following 3 simple constructions:

(1) Given W and its pedal parallelogram $W_aW_bW_cW_d$ with respect to $A_1B_1C_1D_1$, one can reconstruct $A_1B_1C_1D_1$ by drawing lines through W_a, W_b, W_c, W_d perpendicular to WW_a, WW_b, WW_c, WW_d respectively. The construction is actually simpler than reconstructing $A_1B_1C_1D_1$ from midpoints of sides *i.e.*, vertices of the Varignon parallelogram and the point of intersection of diagonals.

(2) Similarly, one can reconstruct the quadrilateral from the Simson point S and the four pedal points of S on the Simson line.

(3) Given three vertices A_1, B_1, C_1 and W, one can reconstruct D_1. Here is one way to do this. The given points determine the circles $o_2 = (A_1B_1C_1)$, $CS(o_2, o_1) = (A_1WB_1)$ and $CS(o_2, o_3) = (B_1WC_1)$. Given o_2 and $CS(o_2, o_1)$, we construct the center of o_1 as $A_2 = \text{Inv}_{CS(o_2,o_1)}(B_2)$ (see property 2 in the Preliminaries of Section 3). Similarly, $C_2 = \text{Inv}_{CS(o_2,o_3)}(B_2)$. Then D_1 is the second point of intersection of o_1 (the circle centered at A_2 and going through A_1, B_1) and o_3 (the circle centered at C_2 and going through B_1, C_1). Alternatively, one can use the property that $D_1 = \text{Iso}_{T_2} \circ \text{Inv}_{o_2}(W)$.

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