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Alhazen' s Circular Billiard Problem

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Abstract. In this paper we give two simple geometric constructions of two versions of the famous Alhazen's circular billiard problem.

1. Introduction

The famous Alhazen problem [2, Problem 156] has to do with a circular billiard and there are two versions of the problem. The first case is to find at the edge of the circular billiard two points B, C such that a billiard ball moving from a given point A inside the circle of the billiard after reflection at B, C passes through the point A again (see Figure 1A). It is obvious that if O is the center of the circle and the points O, A, B, C are collinear then the problem is trivial.

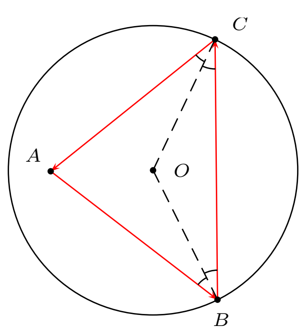

Figure 1A: The first case

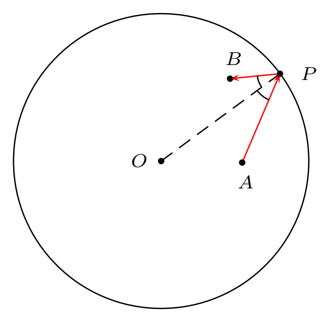

Figure 1B: The second case

The second case is, given two fixed points A and B inside the circle, to find a point P on the edge of the circular billiard such that the ball moving from A after one reflection at P will pass from B (see Figure 1B). It is obvious again that if the points A, B and O are on a diameter of the circle then the problem is trivial.

2. Alhazen' s problem 1

Given a point A inside a circle (O), to construct points B and C on the circle such that the reflection of AB at B passes through C and the reflection of BC at C passes through A.

Since the radii OB and OC are bisectors of angles B and C of triangle ABC, O is the incenter of ABC, which is isosceles with $AB = AC$ (see Figure 2). The

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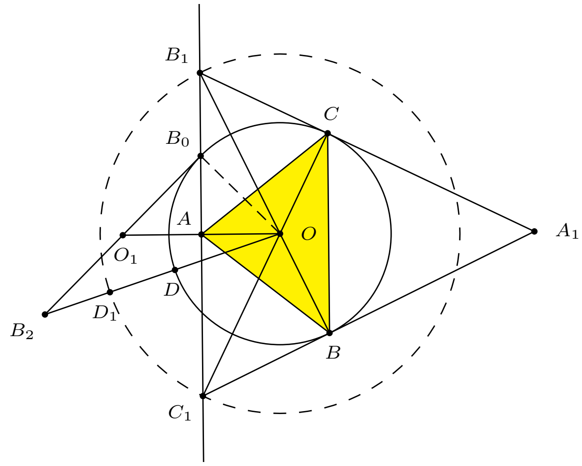

Figure 2.

points B and C are symmetric in OA. The tangents to the circle at B and C, together with the perpendicular to OA at A, bound the antipedal triangle $A_1B_1C_1$ of O (relative to ABC). Hence, O is the orthocenter of triangle $A_1B_1C_1$, and BB_1, CC_1 are altitudes of $A_1B_1C_1$ passing through O. Therefore, to construct the reflection points B and C, it is sufficient to construct B_1 and C_1.

Suppose the circle (O) has radius R and $OA = d$. If $OB_1 = x$, then from the similar right triangles B_1AO and B_1BC_1, we have

$$\frac{B_1A}{B_1O} = \frac{B_1B}{B_1C_1} \implies \frac{B_1A}{x} = \frac{x+R}{2B_1A}.$$

Since $B_1A^2 = x^2 - d^2$, this reduces to $x(x+R) = 2(x^2 - d^2)$, or

$$x^2 - Rx - 2d^2 = 0. \tag{1}$$

This has a unique positive solution x. This leads to the following construction.
(i) Let B_0 be an intersection of the given circle with the perpendicular to OA at A, O_1 the symmetric of O in A, and B_2 the symmetric of B_0 in O_1. Note that $O_1B_0 = OB_0 = R$.
(ii) Construct the segment OB_2 to intersect the given circle at D, and let D_1 be the midpoint of DB_2.
(iii) Construct the circle with center O to pass through D_1. The intersections of this circle with the line AB_0 are the points B_1 and C_1.

To validate this, let $OD_1 = y$. Then $OB_2 = 2y - R$. Applying Apollonius' theorem to the median OO_1 of triangle OB_0B_2, we have

$$(2y - R)^2 + R^2 = 2(2d)^2 + 2R^2.$$

This leads to

$$y^2 - Ry - 2d^2 = 0. \tag{2}$$

Comparison of (1) and (2) gives $y = x$.

3. Alhazen's problem 2

Given two points A and B inside a circle (O), to construct a point P on the circle such that the reflection of AP at P passes through B.

It is well known that P cannot be constructed with ruler and compass only; see, for example, [3]. The analysis below leads to a simple construction with conics.

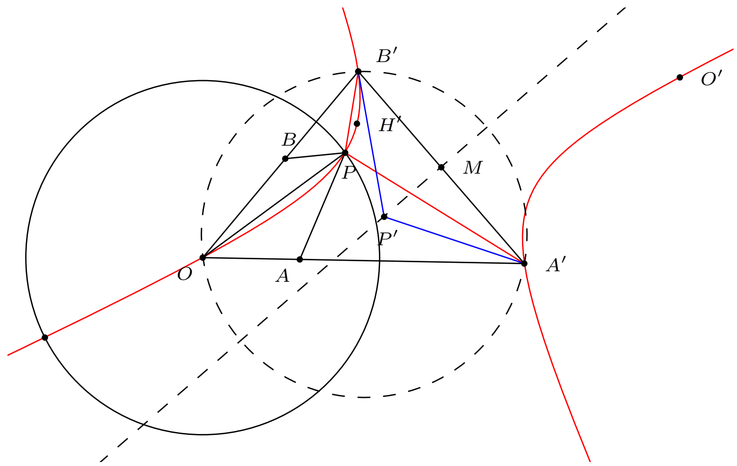

Figure 3

Let A' and B' be the inverses of A and B in the circle (O). Since $OA \cdot OA' = OP^2$, the triangles $PA'O$ and APO are similar, and $\angle PA'O = \angle APO$. Similarly, $\angle PB'O = \angle BPO$. Since $\angle APO = \angle BPO$, we have $\angle PA'O = \angle PB'O$. Consider the reflections of PA' and PB' respectively in the bisectors of angles A' and B' of triangle $OA'B'$. These reflection lines intersect at the isogonal conjugate P' of P (in triangle $OA'B'$). Note that $\angle P'A'B' = \angle PA'O = \angle PB'O = \angle P'B'A'$. Therefore, P' is a point on the perpendicular bisector of $A'B'$ (which contains the circumcenter center of $O'A'B'$). It follows that P lies on the isogonal conjugate of the perpendicular bisector of $A'B'$. This is a rectangular circum-hyperbola of triangle $OA'B'$, whose center is the midpoint of $A'B'$. It also contains the orthocenter of the triangle. This leads to the following construction of the point P.

(i) Construct the orthocenter H' of triangle $OA'B'$ and complete the parallelogram $OA'O'B'$.

(ii) The point P can be constructed as an intersection of the given circle (O) with the conic (rectangular hyperbola) containing O, A', B', H' and O'.

We conclude with two special cases when P can be constructed easily with ruler and compass.

3.1. *Special case: A and B on a diameter*. If the points A, B, O are collinear, then the triangle $OA'B'$ degenerates into a line. Let O_1 be the harmonic conjugate of O relative to AB; see Figure 4. The point P lies on the circle with diameter OO_1 ([1]).

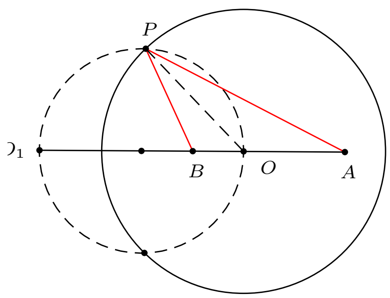

Figure 4

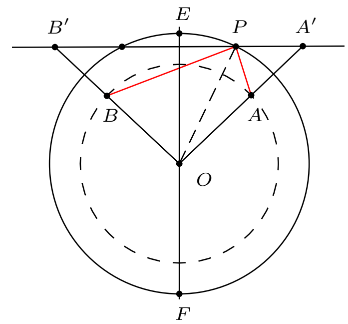

Figure 5

3.2. *Special case:* $OA = OB$. If $OA = OB = d$, then $OA'B'$ is isosceles and the rectangular circum-hyperbola degenerates into a pair of perpendicular lines, the perpendicular bisector of AB and the line $A'B'$. The first line gives the endpoints E and F of the diameter perpendicular to AB. The second line $A'B'$ intersects the circle (O) at two real points (solution to Alhazen's problem) if and only if $\angle AOB < 2\arccos\frac{d}{R}$ (see Figure 5).

References

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