Forum Geometricorum
Volume 12 (2012) 211–213.

FORUM GEOM
ISSN 1534-1178

Finding Integer-Sided Triangles With $P^2 = nA$

John F. Goehl, Jr.

Abstract. A surprising property of certain parameters leads to algorithms for finding integer-sided triangles with $P^2 = nA$, where P is the perimeter, A is the area, and n is an integer. Examples of triangles found for each of two values of n are given.

1. Introduction

MacLeod [1] considered the problem of finding integer-sided triangles with sides a, b, and c and $P^2 = nA$, where P is the perimeter, A is the area, and n is an integer. He showed that they could be found from solutions of the equation:

$$16(a+b+c)^3 = n^2(a+b-c)(a+c-b)(b+c-a). \tag{1}$$

It was shown that n must be an integer greater than or equal to 21. Define

$$2\alpha = a+b-c, \quad 2\beta = a+c-b, \qquad 2\gamma = b+c-a,$$

then

$$16(\alpha+\beta+\gamma)^3 = n^2\alpha\beta\gamma. \tag{2}$$

Note that the parameters α, β, and γ are the lengths of the segments into which the inscribed circle divides the sides.

2. Special case: n a prime number

Consider the special case when n is a prime number. Then $\alpha + \beta + \gamma = nw$ for some integer w. So equation (2) becomes $16nw^3 = \alpha\beta\gamma$. Then one of the parameters α, β, or γ must be divisible by n. Choose $\gamma = n\gamma'$ and so $16w^3 = \alpha\beta\gamma'$. Let $\alpha = 2^i\alpha_1$, $\beta = 2^j\beta_1$, and $\gamma' = 2^k\gamma_1$, where $i + j + k = 4$. Then $w^3 = \alpha_1\beta_1\gamma_1$. Note that it can be assumed that α_1, β_1, and γ_1 have no common factor since the sides of the corresponding triangle can be reduced by that factor to an equivalent triangle with the same P^2/A ratio. Hence $w = w'\alpha_0$ for some w' and a factor unique to α_1 so $\alpha_1 = \alpha_0^3$. Similarly, $\beta_1 = \beta_0^3$, $\gamma_1 = \gamma_0^3$, and $w = \alpha_0\beta_0\gamma_0$. Finally, the sides can be found from $\alpha = 2^i\alpha_0^3$, $\beta = 2^j\beta_0^3$, and $\gamma = 2^kn\gamma_0^3$.

Publication Date: July 6, 2012. Communicating Editor: Paul Yiu.

3. Algorithms

From equation (2), $16(\alpha+\beta+\gamma)^3 = n^2\alpha\beta\gamma = n^2 2^i\alpha_0^3 2^j\beta_0^3 2^k n\gamma_0^3$, or

$$2^i\alpha_0^3 + 2^j\beta_0^3 + 2^k n\gamma_0^3 = n\alpha_0\beta_0\gamma_0. \tag{3}$$

First note that

$$2^i\alpha_0^3 + 2^j\beta_0^3 = nv \tag{4}$$

for some v. Equation (4) is used to find allowed integer values of α_0, β_0, and v. Then allowed integer values of γ_0 are found from solutions of the cubic equation:

$$2^k\gamma_0^3 - \alpha_0\beta_0\gamma_0 + v = 0. \tag{5}$$

4. An example

Consider $n = 31$. Values for α_0 and β_0 up to 600 resulted in the integer solutions of equations (4) and (5) shown in Table 1. Solutions for which α_0 and β_0 have a common factor result in duplicate triangles and have been omitted. Entries for α_0, β_0, and v that result in duplicate triangles have also been omitted. In both tables that follow, the values for α, β, and γ and the values of the corresponding sides, $a = \alpha+\beta$, $b = \alpha+\gamma$, and $c = \beta+\gamma$ have been reduced by the common factor. The second solution in Table 1 is the triangle found by MacLeod.

i	4	3	3	3	3
j	0	1	1	0	0
k	0	0	0	1	1
α_0	2	1	5	17	29
β_0	3	3	13	18	35
v	5	2	174	1456	7677
γ_0	1	1	6	7	9
α	128	8	500	19652	195112
β	27	54	2197	2916	42875
γ	31	31	3348	10633	45198
a	155	62	2697	22568	237987
b	159	39	3848	30285	240310
c	58	85	5545	13549	88073

Table 1

5. General case: n a composite number

Consider a possible factorization of n: $n = n_1n_2n_3$. Similar arguments lead to $\alpha = 2^i n_1\alpha_0^3$, $\beta = 2^j n_2\beta_0^3$, and $\gamma = 2^k n_3\gamma_0^3$, where $i+j+k = 4$. All the MacLeod triangles are of this form.

6. General algorithm

With the above choices for α, β, and γ, equation (2) becomes

$$2^i n_1 \alpha_0^3 + 2^j n_2 \beta_0^3 + 2^k n_3 \gamma_0^3 = n_1 n_2 n_3 \alpha_0 \beta_0 \gamma_0. \tag{6}$$

First note that

$$2^i n_1 \alpha_0^3 + 2^j n_2 \beta_0^3 = n_3 v \tag{7}$$

for some v. Equation (7) is used to find allowed integer values of α_0, β_0, and v. Then allowed integer values of γ_0 are found from solutions of the cubic equation:

$$2^k \gamma_0^3 - n_1 n_2 \alpha_0 \beta_0 \gamma_0 + v = 0. \tag{8}$$

7. An example

Consider $n = 42$. Integer solutions of equations (7) and (8) are shown in Table 2. Note that the fourth entry in Table 2 is the triangle found by MacLeod.

i	0	2	2	0	0	0	0
j	0	2	2	2	2	2	2
k	4	0	0	2	2	2	2
n_1	1	1	1	2	2	2	2
n_2	1	1	1	3	3	3	3
n_3	42	42	42	7	7	7	7
α_0	11	43	227	1	4	92	109
β_0	19	47	487	1	1	53	121
v	195	17460	12114132	2	20	477700	3406970
γ_0	3	9	129	1	1	17	49
α	1331	159014	23394166	1	32	389344	1295029
β	6859	207646	231002606	6	3	446631	10629366
γ	18144	15309	45080469	14	7	34391	1647086
a	8190	366660	254396772	7	35	835975	11924395
b	19475	174323	68474635	15	39	423735	2942115
c	25003	222955	276083075	20	10	481022	12276452

Table 2

Reference

[1] A. J. MacLeod, On integer relations between the area and perimeter of Heron triangles, *Forum Geom.*, 9 (2009) 41–46.

John F. Goehl, Jr.: Department of Physical Sciences, Barry University, 11300 NE Second Avenue, Miami Shores, Florida 33161, USA

E-mail address: jgoehl@mail.barry.edu