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Sherman's Fourth Side of a Triangle

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Abstract. We give two simple ruler-and-compass constructions of the line which, like the sidelines of the triangle, is tangent to the incircle and cuts the circumcircle in a chord with midpoint on the nine-point circle.

1. Introduction

Consider the sides of a triangle as chords of its circumcircle. Each of these is tangent to the incircle and has its midpoint on the nine-point circle. Apart from these three chords, B. F. Sherman [3] has established the existence of a fourth one, which is also tangent to the incircle and bisected by the nine-point circle (see Figure 1). While Sherman called this the *fourth side* of the triangle, we refer to the line containing this fourth side as the Sherman line of the triangle. In this note we provide a simple euclidean construction of this Sherman line as a result of an analysis with barycentric coordinates.

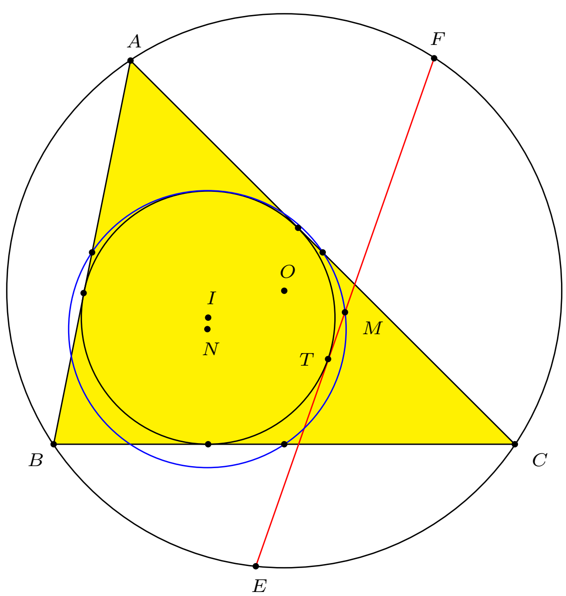

Figure 1. The fourth side of a triangle

2. Lines tangent to the incircle

Given a triangle ABC with sidelengths a, b, c, we say that the line with barycentric equation $px+qy+rz=0$ has line coordinates $[p,q,r]$. A line $px+qy+rz=0$

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is tangent to a conic $\mathscr{C}$ if and only if $[p:q:r]$ lies on the dual conic $\mathscr{C}^*$ (see, for example, [4, §10.6]).

Proposition 1. *If $\mathscr{C}$ is the inscribed conic tangent to the sidelines at the traces of the point $\left(\frac{1}{u}:\ \frac{1}{v}:\ \frac{1}{w}\right)$, its dual conic $\mathscr{C}^*$ is the circumconic*

$$\frac{u}{x}+\frac{v}{y}+\frac{w}{z}=0.$$

Proof. Since the barycentric equation of $\mathscr{C}$ is

$$u^2x^2+v^2y^2+w^2z^2-2vwyz-2wuzx-2uvxy=0,$$

the conic is represented by the matrix

$$M=\begin{pmatrix} u^2 & -uv & -uw\\ -uv & v^2 & -vw\\ -uw & -vw & w^2\end{pmatrix}.$$

This has adjoint matrix

$$M^*=8uvw\cdot\begin{pmatrix} 0 & w & v\\ w & 0 & u\\ v & u & 0\end{pmatrix}.$$

It follows that the dual conic $\mathscr{C}^*$ is the circumconic $uyz+vzx+wxy=0$. □

Applying this to the incircle, we have the following characterization of its tangent lines.

Proposition 2. *A line $px+qy+rz=0$ is tangent to the incircle if and only if*

$$\frac{b+c-a}{p}+\frac{c+a-b}{q}+\frac{a+b-c}{r}=0. \tag{1}$$

3. Lines bisected by the nine-point circle

Suppose a line $\mathscr{L}:\ px+qy+rz=0$ cuts out a chord EF of the circumcircle. The chord is bisected by the nine-point circle if and only if the pedal (orthogonal projection) P of the circumcenter O on $\mathscr{L}$ lies on the nine-point circle. We shall simply say that the line is bisected by the nine-point circle.

Proposition 3. *A line $px+qy+rz=0$ is bisected by the nine-point circle if and only if*

$$\frac{a^2(b^2+c^2-a^2)}{p}+\frac{b^2(c^2+a^2-b^2)}{q}+\frac{c^2(a^2+b^2-c^2)}{r}=0. \tag{2}$$

Proof. The pedal of O on the line $px+qy+rz=0$ is the point

$$\begin{aligned}P=&-b^2q^2-c^2r^2+(b^2+c^2-2a^2)qr+a^2rp+a^2pq\\ &:\ -c^2r^2-a^2p^2+(c^2+a^2-2b^2)rp+b^2pq+b^2qr\\ &:\ -a^2p^2-b^2q^2+(a^2+b^2-2c^2)pq+c^2qr+c^2rp.\end{aligned}$$

The superior of the pedal P is the point

$$Q = a^2p^2 - a^2qr + (b^2-c^2)rp - (b^2-c^2)pq$$
$$: b^2q^2 - b^2rp + (c^2-a^2)pq - (c^2-a^2)qr$$
$$: c^2r^2 - c^2pq + (a^2-b^2)qr - (a^2-b^2)rp.$$

The line $px+qy+rz=0$ is bisected by the nine-point circle if and only if Q lies on the circumcircle $a^2yz+b^2zx+c^2xy=0$. This condition is equivalent to

$$\begin{aligned}&a^2(b^2q^2-b^2rp+(c^2-a^2)pq-(c^2-a^2)qr)(b^2q^2-b^2rp+(c^2-a^2)pq-(c^2-a^2)qr)\\&+b^2(b^2q^2-b^2rp+(c^2-a^2)pq-(c^2-a^2)qr)(a^2p^2-a^2qr+(b^2-c^2)rp-(b^2-c^2)pq)\\&+c^2(a^2p^2-a^2qr+(b^2-c^2)rp-(b^2-c^2)pq)(b^2q^2-b^2rp+(c^2-a^2)pq-(c^2-a^2)qr)\\&=0.\end{aligned}$$

The quartic polynomial in p, q, r above factors as $-F\cdot G$, where

$$\begin{aligned}F=&\ a^2(b^2+c^2-a^2)qr+b^2(c^2+a^2-b^2)rp+c^2(a^2+b^2-c^2)pq,\\G=&\ a^2p^2+b^2q^2+c^2r^2-(b^2+c^2-a^2)qr-(c^2+a^2-b^2)rp-(a^2+b^2-c^2)pq.\end{aligned}$$

Now G can be rewritten as

$$G = S_A(q-r)^2 + S_B(r-p)^2 + S_C(p-q)^2.$$

As such, it is the square length of a vector of component p, q, r along the respective sidelines. Therefore, $G>0$, and we obtained $F=0$ as the condition for the line to be bisected by the nine-point circle. □

Corollary 4. *A line is bisected by the nine-point circle* (N) *if and only if it is tangent to the inscribed conic with center the nine-point center* N.

Proof. Let $px+qy+rz=0$ be a line bisected by the nine-point circle. By Proposition 3, it is tangent to the inscribed conic with perspector $\left(\frac{1}{u}:\frac{1}{v}:\frac{1}{w}\right)$, where

$$u:v:w = a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2).$$

The center of the inscribed conic is

$$\begin{aligned}&v+w : w+u : u+v\\=&\,b^2(c^2+a^2)-(b^2-c^2)^2 :\ b^2(c^2+a^2)^2-(c^2-a^2)^2 :\ c^2(a^2+b^2)-(a^2-b^2)^2.\end{aligned}$$

This is the center N of the nine-point circle. □

The inscribed conic with center N is called the MacBeath inconic. It is well known that this has foci O and H, the circumcenter and the orthocenter (see [4, §11.1.5]). The Sherman line is the *fourth* common tangent of the incircle and the inscribed conic with center N.

N. Dergiades has kindly suggested the following alternative proof of Corollary 4. The orthogonal projection of a focus on a tangent of a conic lies on the auxiliary

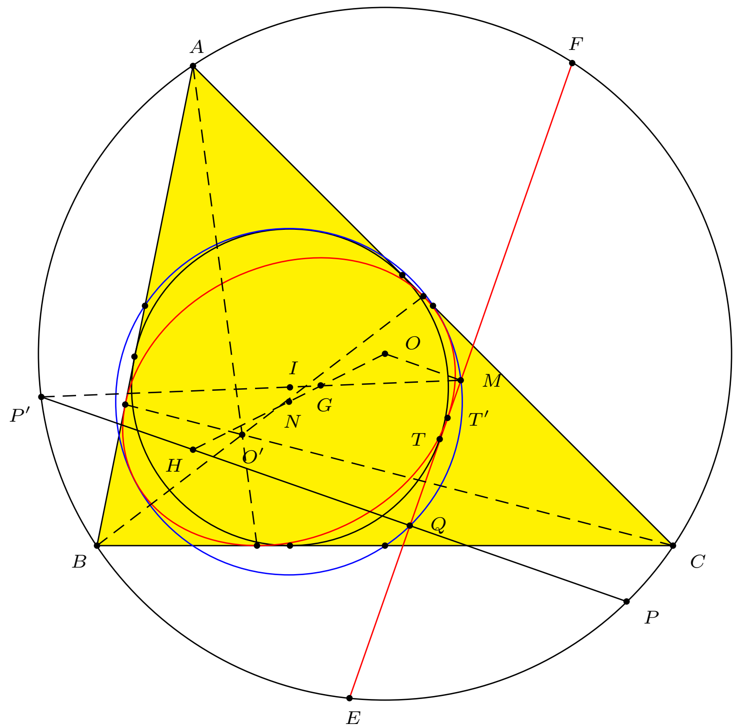

Figure 2. The fourth side of a triangle as a common tangent

circle. Since the MacBeath inconic has the nine-point circle as auxiliary circle ([1, Problem 130]), and the orthogonal projection of the focus O on the Sherman line lies on the nine-point circle, the Sherman line must be tangent to the MacBeath inconic.

4. Construction of the Sherman line

The Sherman line, being tangent to the incircle and bisected by the nine-point circle, has its line coordinates $[p : q : r]$ satisfying both (1) and (2). Regarding $px + qy + rz = 0$ as the trilinear polar of the point $S = \left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$, we have a simple characterization of S leading to an easy ruler-and-compass construction of the Sherman line.

Proposition 5. *The Sherman line is the trilinear polar of the intersection of*
(i) *the trilinear polar of the Gergonne point,*
(ii) *the isotomic line of the trilinear polar of the circumcenter* (see Figure 2).

Proof. The point S is the intersection of the two lines with equations

$$\begin{aligned} (b+c-a)x &+ (c+a-b)y &+ (a+b-c)z &= 0, \quad (3)\\ a^2(b^2+c^2-a^2)x &+ b^2(c^2+a^2-b^2)y &+ c^2(a^2+b^2-c^2)z &= 0. \quad (4)\end{aligned}$$

These two lines can be easily constructed as follows.
(3) is the trilinear polar of the Gergonne point $\left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right)$.

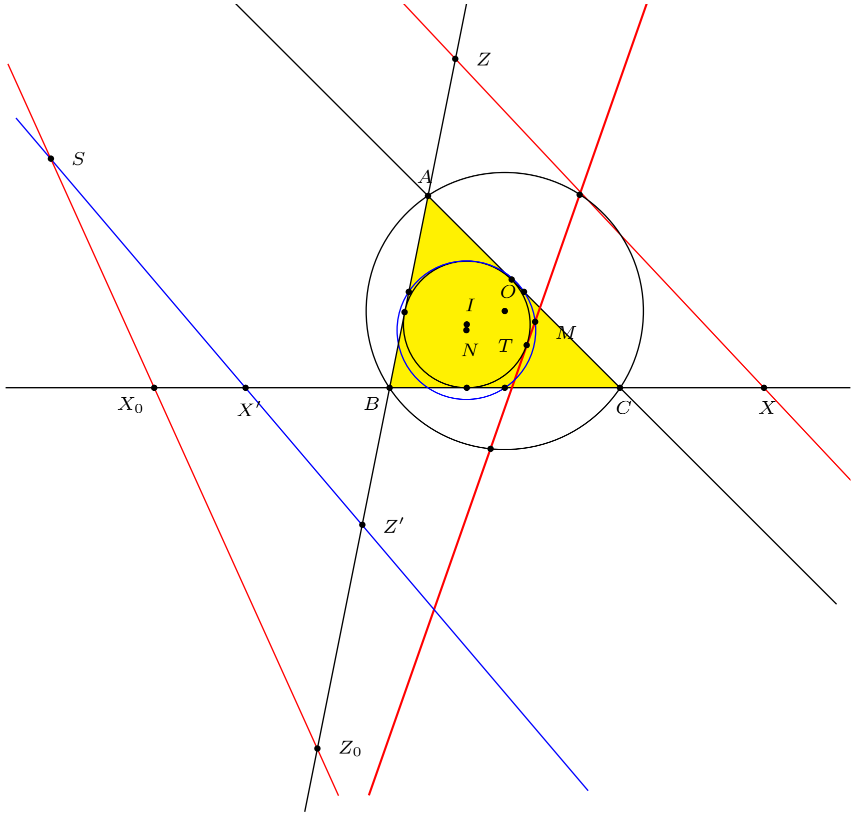

Figure 3. Construction of the tripole of the Sherman line

(4) is the trilinear polar of the isotomic conjugate of the circumcenter. It can also be constructed as follows. If the trilinear polar of the circumcenter O intersects the sidelines at X, Y, Z respectively, and if X', Y', Z' are points on the respective sidelines such that

$$BX' = XC, \qquad CY' = YA, \qquad AZ' = ZB,$$

then (4) is the line containing X', Y', Z'. This is called the isotomic line of the line containing X, Y, Z. □

5. Coordinates

For completeness, we record the barycentric coordinates of various points associated with the Sherman line configuration.

5.1. *Points on the Sherman line.* The Sherman line is the trilinear polar of

$$S = (f(a,b,c) : \ f(b,c,a) : \ f(c,a,b)),$$

where

$$f(a,b,c) := (b-c)(a^2(b+c) - 2abc - (b+c)(b-c)^2).$$

The point of tangency with the incircle is

$$T = ((b+c-a)f(a,b,c)^2 : \ (c+a-b)f(b,c,a)^2 : \ (a+b-c)f(c,a,b)^2).$$

This is the triangle center X_{3326} in [2]. The point of tangency with the MacBeath inconic is the point

$$T' = (a^2 S_A \cdot f(a,b,c)^2 : \ b^2 S_B \cdot f(b,c,a)^2 : \ c^2 S_C \cdot (c,a,b)^2).$$

See [5].

The pedal of O on the Sherman line is the point

$$M = ((b+c-a)(b-c)S_A f(a,b,c) \cdot g(a,b,c) : \cdots : \cdots),$$

where

$$g(a,b,c) = -2a^4 + a^3(b+c) + a^2(b-c)^2 - a(b+c)(b-c)^2 + (b^2-c^2)^2.$$

The triangle centers S, T', and M do not appear in Kimberling's *Encyclopedia of Triangle Centers* [2]. However, the superior of M is the point

$$P' = \left(\frac{1}{S_A \cdot f(a,b,c)} : \ \cdots : \ \cdots\right)$$

on the circumcircle, and the line HP' is perpendicular to the Sherman line (see Figure 2). P' is the triangle center X_{1309}.

5.2. *A second construction of the Sherman line.* It is known that the MacBeath inconic is the envelope of the perpendicular bisector of HP as P traverses the circumcircle ([4, §11.1.5]). Therefore, the reflection of H in the Sherman line, like those in the three sidelines of ABC, is a point on the circumcircle. This reflection is the point

$$P = \left(\frac{a^2}{2a^4 - 2a^3(b+c) - a^2(b^2-4bc+c^2) + 2a(b+c)(b-c)^2 - (b^2-c^2)^2} : \cdots : \cdots\right),$$

According to [2], P is the triangle center X_{953}, the isogonal conjugate of the infinite point

$$X_{952} = (2a^4 - 2a^3(b+c) - a^2(b^2-4bc+c^2) + 2a(b+c)(b-c)^2 - (b^2-c^2)^2 : \cdots : \cdots).$$

This is the infinite point of the line joining the incenter to the nine-point center, namely,

$$\sum_{\text{cyclic}} (b-c)(b+c-a)(a^2-b^2+bc-c^2)x = 0.$$

This observation leads to a very easy (second) construction of the Sherman line:
(i) Construct lines through A, B, C parallel to the line IN.
(ii) Construct the reflections of the lines in (i) in the respective angle bisectors of the triangle.
(iii) The three lines in (ii) intersect at a point P on the circumcircle.
(iv) The perpendicular bisector of HP is the Sherman line.
See Figure 4. For a simpler construction, it is sufficient to construct one line in (i) and the corresponding reflection in (ii).

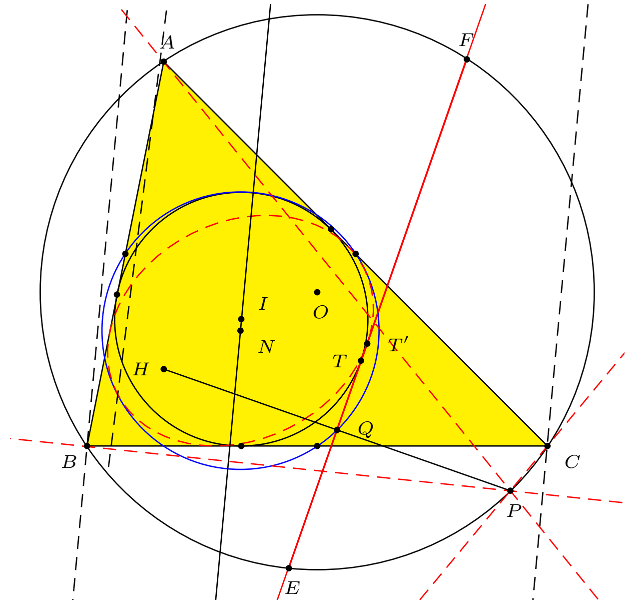

Figure 4. Construction of the Sherman line

5.3. *Pedal of orthocenter on the Sherman line.* The midpoint of the segment HP is the point

$$Q = ((b+c-2a)(b-c)f(a,b,c) : (c+a-2b)(b-c)f(b,c,a) : (a+b-2c)(b-c)f(c,a,b))$$

on the nine-point circle. This is the triangle center X_{3259} in [2] (see Figure 2).

5.4. *Distances.* Finally, we record the length of the fourth side EF of the triangle:

$$EF^2 = \frac{16r(4R^2+5Rr+r^2-s^2)(4R^3-(2r^2+s^2)R+r(s^2-r^2))}{(4R^2+4Rr+3r^2-s^2)^2},$$

where R, r, and s are the circumradius, inradius, and semiperimeter of the given triangle. The distance from O to the Sherman line is

$$OM = \frac{(R-2r)(2R+r-s)(2R+r+s)}{4R^2+4Rr+3r^2-s^2}.$$

References

[1] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[3] B. F. Sherman, The fourth side of a triangle, *Math. Mag.*, 66 (1993) 333–337.

[4] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

[5] P. Yiu, Hyacinthos message 14416, November 7, 2006.

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