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Improving Upon a Geometric Inequality of Third Order

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Abstract. We show that the best possible positive constant k in a certain geometric inequality of third order lies in the interval $[0.14119, 0.14364]$, which improves upon a previous known result where $k = 0$. We also consider a comparable question concerning a fourth order version of the inequality.

1. Introduction

Given a point P in the plane of triangle ABC, let R_1, R_2, and R_3 denote the respective distances AP, BP, and CP. Let a, b, and c be the lengths of the sides of triangle ABC, s the semi-perimeter, L the area, R the circumradius, and r the inradius.

Liu [4] conjectured the following geometric inequality which holds for all points P in the plane of an arbitrary triangle ABC:

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 24r^3. \tag{1}$$

This inequality was proven by Wu, Zhang and Chu in [5], where it was strengthened to

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 12Rr^2. \tag{2}$$

Observe that (1) and (2) both reduce to Euler's inequality $R \geq 2r$, see [1, p. 48, Th. 5.1], whenever P is taken to be the circumcenter of triangle ABC.

Note that (2) cannot be improved upon by a multiplicative factor since there is equality in the case when triangle ABC is equilateral with P its center. The following question involving an additional non-negative term on the right-hand side is raised by the authors at the end of [5]:

Problem. For a triangle ABC and an arbitrary point P, determine the best possible k such that the following inequality holds:

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 12[R + k(R - 2r)]r^2. \tag{3}$$

In this paper, we will prove the following result by a different method than that used in [5] to show (1) and (2).

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Theorem 1. *The best possible k such that inequality* (3) *holds lies in the interval* $[y, z]$*, where* $y \approx 0.14119$ *and* $z \approx 0.14364$*. In particular, we have*

$$(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}} \geq 12[R + \frac{7}{50}(R - 2r)]r^2.$$

2. Preliminary results

Lemma 2. [5, Eq. 3.1] *If* $j > 1$*, then*

$$(R_1R_2)^j + (R_2R_3)^j + (R_3R_1)^j \geq \frac{(abc)^j}{[a^{\frac{j}{j-1}} + b^{\frac{j}{j-1}} + c^{\frac{j}{j-1}}]^{j-1}}.$$

Lemma 3. *Suppose p is a fixed number with $0 < p \leq \frac{8}{27}$. Let $t := w(a, b, c) = ab + bc + ca$, where a, b and c are real numbers, and let M denote the maximum value of t subject to the constraints $a + b + c = 2$ and $abc = p$.*
(i) *M is achieved by some point (a, b, c), where two of a, b, c are the same and $a, b, c > 0$.*
(ii) *One may assume further that M is achieved by some point (a, b, c), where $a = b$ and $\frac{2}{3} \leq a < 1$.*
(iii) *If $v := \frac{M-1}{p}$, then v satisfies $p = g(v)$, where g is the function given by*

$$g(x) = \frac{-8x^2 + 36x - 27 - (9 - 8x)^{\frac{3}{2}}}{8x^3}. \tag{4}$$

Proof. (i) A standard argument using the method of Lagrange multipliers with two constraints shows that two of $\{a, b, c\}$ must be the same when t is maximized. Note that $a, b, c > 0$ when t is maximized, for if say $b, c < 0$, then $r = ab + bc + ca = 2(b + c) - (b + c)^2 + bc < 0$, and clearly t can achieve positive values for all choices of p (for example, choosing $a, b > 0$ and $c = \frac{2}{3}$). Note further that there is no minimum for t, for if c is negative, then

$$t = ab + bc + ca = ab + c(2 - c) < ab = \frac{p}{c},$$

so choosing c near zero implies t can assume arbitrarily large negative values.

(ii) By part (i) and symmetry, the equality $w(a, b, c) = M$ subject to the constraints is achieved by some point (a, b, c), where $a = b$ and $0 < a < 1$ (note that $c > 0$ implies $a < 1$). Then a is a positive root of $\alpha(x) = p$, where $\alpha(x) := 2x^2(1 - x)$. Note that the function α is increasing on $(0, \frac{2}{3})$, decreasing on $(\frac{2}{3}, 1)$, and has a maximum of $\frac{8}{27}$ at $x = \frac{2}{3}$, with $\alpha(0) = \alpha(1) = 0$. If $p = \frac{8}{27}$, then $a = b = c = \frac{2}{3}$, by the equality condition in the geometric-arithmetic mean inequality, so we will assume $p < \frac{8}{27}$. Then the equation $\alpha(x) = p$ has two roots in the interval $(0, 1)$, which we will denote by $r_1 < r_2$; note that $0 < r_1 < \frac{2}{3} < r_2 < 1$.

We will now show that the maximum value M is achieved when $a = b = r_2 > \frac{2}{3}$ by comparing it to the value of $w(a, b, c)$ when $a = b = r_1$. Let $\beta(x) := w(x, x, 2 - 2x) = 4x - 3x^2$. Note that $\beta(r_2) > \beta(r_1)$ iff $r_1 + r_2 < \frac{4}{3}$. To show the latter, first observe that $\alpha(x) > \alpha(\frac{4}{3} - x)$ for all $x \in (0, \frac{2}{3})$ since, for the function

$\gamma(x) := \alpha(x) - \alpha(\frac{4}{3} - x)$, we have $\gamma(\frac{2}{3}) = 0$ with $\gamma'(x) = -\frac{4}{3}(2-3x)^2 < 0$. Then $\alpha(\frac{4}{3} - r_1) < \alpha(r_1) = \alpha(r_2)$ implies $r_2 < \frac{4}{3} - r_1$, as desired, since $\alpha(x)$ is decreasing when $x > \frac{2}{3}$.

(iii) By part (ii), we have $v = \frac{\beta(a)-1}{p}$, where $\frac{2}{3} \leq a < 1$ satisfies $2a^2(1-a) = p$. Thus,

$$v = \frac{4a - 3a^2 - 1}{2a^2(1-a)} = \frac{3a-1}{2a^2}. \tag{5}$$

Note that $1 < v \leq \frac{9}{8}$ since $1 < \frac{3x-1}{2x^2} \leq \frac{9}{8}$ if $x \in [\frac{2}{3}, 1)$. Solving for a in terms of v in (5) gives

$$a = \frac{3 + (9-8v)^{\frac{1}{2}}}{4v}, \tag{6}$$

where we reject the other root since $a \geq \frac{2}{3}$. From (5) and (6), we may write

$$\begin{aligned} p = \ & 2a^2(1-a) = \frac{(1-a)(3a-1)}{v} = \frac{-3\left(\frac{3a-1}{2v}\right) + 4a - 1}{v} \\ = \ & \frac{-2v + 3 - (9-8v)a}{2v^2} = \frac{-8v^2 + 12v - (3 + (9-8v)^{\frac{1}{2}})(9-8v)}{8v^3}, \end{aligned}$$

which gives the requested relation. □

Lemma 4. *Let a, b, c be real numbers such that $a+b+c = 2$ with $0 < a, b, c < 1$. Then we have*

$$1 + abc < ab + bc + ca \leq 1 + \frac{9}{8}abc.$$

Proof. The proof of Lemma 3 shows the right inequality. The left one follows from expanding the obvious inequality $(1-a)(1-b)(1-c) > 0$, and noting $a + b + c = 2$. □

Lemma 5. *Let D consist of the set of ordered pairs (p, u) such that there exists a triangle of perimeter 2 having side lengths a, b, c with $p = abc$ and $u = \frac{ab+bc+ca-1}{abc}$. If $1 < u' \leq \frac{9}{8}$ is fixed, then $p = g(u')$ is the smallest p such that $(p, u') \in D$.*

Proof. Note first that $(p, u) \in D$ implies $0 < p \leq \frac{8}{27}$ and $1 < u \leq \frac{9}{8}$, the latter by Lemma 4. Given $p_o \in (0, \frac{8}{27}]$, let u_o denote the solution of the equation $g(u) = p_o$, where $u \in (1, \frac{9}{8}]$ and g is given by (4) above. Note that u_o is uniquely determined since $g(1) = 0$ and $g(\frac{9}{8}) = \frac{8}{27}$, with $g(x)$ increasing on $(1, \frac{9}{8}]$ as

$$g'(x) = \frac{81 - 8x(9-x) + (27-12x)(9-8x)^{\frac{1}{2}}}{8x^4} > 0.$$

Observe further that the proof of the third part of Lemma 3 can be modified slightly to show that points of the form $(g(u), u)$ always belong to D whenever $u \in (1, \frac{9}{8}]$. Thus, from the third part of Lemma 3, we see that u_o is the *largest* u such that $(p_o, u) \in D$.

So $u \leq u_o = g^{-1}(p_o)$ for all u such that $(p_o, u) \in D$, which implies $g(u) \leq p_o$ for all such u. Conversely, if $u' \in (1, \frac{9}{8}]$ is fixed and $(p, u') \in D$, then $g(u') \leq p$ for all such p. In particular, $p = g(u')$ is the smallest p such that $(p, u') \in D$. □

Lemma 6. *Let $f(u)$ be given by*

$$f(u)=\frac{[(3-6u)g(u)+2]^{-\frac{1}{2}}-3(u-1)^{\frac{1}{2}}}{3(u-1)^{\frac{1}{2}}-24(u-1)^{\frac{3}{2}}},$$

where $g(u)$ is given by

$$g(u)=\frac{-8u^2+36u-27-(9-8u)^{\frac{3}{2}}}{8u^3}.$$

If m is the minimum value of $f(u)$ on the interval $(1,\frac{9}{8}]$, then $m\approx 0.141194514$.

Proof. From the definitions, we have

$$\frac{d}{du}f(u)=\frac{\frac{6g(u)-(3-6u)\frac{d}{du}g(u)}{2((3-6u)g(u)+2)^{\frac{3}{2}}}-\frac{3}{2(u-1)^{\frac{1}{2}}}}{3(u-1)^{\frac{1}{2}}(9-8u)}-\frac{(25-24u)\left(\frac{1}{((3-6u)g(u)+2)^{\frac{1}{2}}}-3(u-1)^{\frac{1}{2}}\right)}{6(u-1)^{\frac{3}{2}}(9-8u)^2},$$

where

$$\frac{d}{du}g(u)=\frac{36-16u+12(9-8u)^{\frac{1}{2}}}{8u^3}+\frac{3(8u^2-36u+27+(9-8u)^{\frac{3}{2}})}{8u^4}.$$

The equation $\frac{d}{du}f(u)=0$ can be written as

$$\frac{(3-z)^{\frac{1}{2}}(3z^6-21z^5+40z^4-21z^3-3z^2+24z-18)+6(3-3z+3z^2-z^3)^{\frac{3}{2}}(1-z^2)^{\frac{3}{2}}}{(3-3z+3z^2-z^3)^{\frac{3}{2}}(1-z^2)^{\frac{3}{2}}z^4}=0,$$

where $u=(9-z^2)/8$. The last equation implies

$$\begin{aligned}&36z^{12}-324z^{11}+1197z^{10}-2421z^9+3111z^8-2877z^7+2014z^6\\&-702z^5-897z^4+1983z^3-2097z^2+1125z-180=0.\end{aligned}$$

With the aid of mathematical programming (such as Maple), one can show that the above polynomial equation has four real roots

$$z_1\approx -0.876333426, z_2\approx 0.257008823, z_3\approx 0.891710246, z_4\approx 2.374529908,$$

which implies

$$u_1\approx 1.029004966, u_2\approx 1.116743308, u_3\approx 1.025606605, u_4\approx 0.420200965.$$

Now $\frac{d}{du}f(u)|_{u=u_2}=0$, with $\frac{d}{du}f(u)|_{u=u_1}<0$ and $\frac{d}{du}f(u)|_{u=u_3}<0$. Thus, the equation $\frac{d}{du}f(u)=0$ has a unique real solution $u^*=u_2\approx 1.116743308$ on the interval $(1,\frac{9}{8})$.

Since $\lim_{u\to 1^+}f(u)=\infty$, $f(u^*)=0.141194514$, and $f(\frac{9}{8})=\lim_{u\to\frac{9}{8}^-}f(u)=\frac{1}{6}$, we see that the minimum value of $f(u)$ on the interval $(1,\frac{9}{8}]$ is approximately 0.141194514. □

Lemma 7. *Let $h(a)$ be given by*

$$h(a)=\frac{a^3(1-a)^3+2[a(1-a)(-a^2+4a-2)]^{\frac{3}{2}}-6a^2(1-a)^2(2a-1)^2}{6(1-a)^2(2a-1)^2(3a-2)^2}.$$

If n is the minimum value of $h(a)$ on the interval $(2-\sqrt{2},1)$, then $n\approx 0.143630168$.

Proof. Using mathematical programming such as Maple, one can show that the equation $\frac{d}{da}h(a) = 0$ has a unique real solution $a^* \approx 0.741049808$ on the interval $2-\sqrt{2} < a < 1$. Since $h(2-\sqrt{2}) = 2.178511254$, $h(\frac{2}{3}) = \lim_{a\to\frac{2}{3}} h(a) = \frac{1}{4}$, $h(a^*) = 0.143630168$, and $\lim_{a\to 1^-} h(a) = \infty$, we see that the minimum of $h(a)$ on the interval $2-\sqrt{2} < a < 1$ is approximately 0.143630168. $\square$

3. Proof of the main result

3.1. *The lower bound.* We first treat the lower bound in Theorem 1. By Lemma 2 with $j = \frac{3}{2}$, we may consider the inequality

$$\frac{(abc)^{\frac{3}{2}}}{(a^3+b^3+c^3)^{\frac{1}{2}}} \geq 12[R + k(R-2r)]r^2,$$

which can be rewritten as

$$\frac{(abc)^{\frac{3}{2}}}{(a^3+b^3+c^3)^{\frac{1}{2}}} \geq \frac{3(1+k)abcL}{s^2} - \frac{24kL^3}{s^3}, \tag{7}$$

using the facts $abc = 4Rrs$ and $L = rs$, see [3, Section 1.4]. By homogeneity, we may take $s = 1$ in (7). Recalling $L = \sqrt{s(s-a)(s-b)(s-c)}$ (see [2, p. 12, 1.53]), we wish to find the best possible k such that the inequality

$$\frac{(abc)^{\frac{3}{2}}}{(a^3+b^3+c^3)^{\frac{1}{2}}} \geq 3(1+k)abc[(1-a)(1-b)(1-c)]^{\frac{1}{2}} - 24k[(1-a)(1-b)(1-c)]^{\frac{3}{2}} \tag{8}$$

holds for all a, b, c satisfying $a+b+c = 2$ with $0 < a, b, c < 1$.

Let $p = abc$ and $t = ab+bc+ca$. From the algebraic identity,

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= (a+b+c)((a+b+c)^2 - 3(ab+bc+ca)), \end{aligned}$$

and $a+b+c = 2$, we get

$$a^3+b^3+c^3 = 3p + 2(2^2-3t) = 3p - 6t + 8.$$

Furthermore, we have

$$(1-a)(1-b)(1-c) = 1-(a+b+c)+(ab+bc+ca)-abc = t-p-1.$$

Thus, (8) may be rewritten in terms of p and t as

$$\frac{p^{\frac{3}{2}}}{(3p-6t+8)^{\frac{1}{2}}} \geq 3(1+k)p(t-p-1)^{\frac{1}{2}} - 24k(t-p-1)^{\frac{3}{2}}. \tag{9}$$

Dividing both sides of (9) by $p^{\frac{3}{2}}$, and letting $u = \frac{t-1}{p}$, we obtain the following inequality in p and u over the domain D defined above in Lemma 5:

$$\frac{1}{(3p-6pu+2)^{\frac{1}{2}}} \geq 3(1+k)(u-1)^{\frac{1}{2}} - 24k(u-1)^{\frac{3}{2}}. \tag{10}$$

Next consider the function $h(p,u,k)$ defined by

$$h(p,u,k) = \frac{1}{(3p-6pu+2)^{\frac{1}{2}}} - 3(1+k)(u-1)^{\frac{1}{2}} + 24k(u-1)^{\frac{3}{2}}.$$

Since for each given $u \in (1, \frac{9}{8}]$, we have

$$\frac{d}{dp}h(p,u,k) = \frac{6u-3}{2(3p-6pu+2)^{\frac{3}{2}}} > 0$$

for all $p \in (0, \frac{8}{27})$, we may consider for each u, the *smallest* p such that $(p,u) \in D$ when determining the best possible constant k. That is, we may replace p with $g(u)$ when determining the best possible k in (10), by Lemma 5, where $u \in (1, \frac{9}{8}]$ and g is given by (4).

We rewrite (10) when $p = g(u)$ as $f(u) \geq k$, where

$$f(u) = \frac{[(3-6u)g(u)+2]^{-\frac{1}{2}} - 3(u-1)^{\frac{1}{2}}}{3(u-1)^{\frac{1}{2}} - 24(u-1)^{\frac{3}{2}}}.$$

Therefore, we seek the minimum value m of $f(u)$ over the interval $(1, \frac{9}{8}]$, and choosing $k = m$ will yield the largest k for which inequality (10), and hence (7), holds. By Lemma 6, we have $m \approx 0.14119$. By Lemma 2, we see that inequality (3) holds with $k = m$ and thus the best possible k in that inequality is at least m, which establishes our lower bound.

3.2. *The upper bound.* We now treat the upper bound given in Theorem 1. For this, we consider the original inequality (3), rewritten as

$$\frac{(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}}}{p^{\frac{3}{2}}} \geq 3(1+k)(u-1)^{\frac{1}{2}} - 24k(u-1)^{\frac{3}{2}}, \quad (11)$$

where we have divided through both sides by $p^{\frac{3}{2}}$, and u and p are as before with $a+b+c=2$. Equivalently, we consider the inequality

$$\frac{\frac{(R_1R_2)^{\frac{3}{2}} + (R_2R_3)^{\frac{3}{2}} + (R_3R_1)^{\frac{3}{2}}}{p^{\frac{3}{2}}} - 3(u-1)^{\frac{1}{2}}}{3(u-1)^{\frac{1}{2}} - 24(u-1)^{\frac{3}{2}}} \geq k, \quad (12)$$

and seek to find a triangle ABC of perimeter 2 and a point P in its plane such that the left-hand side is small. We take ABC to be an acute isosceles triangle and the point P to be the orthocenter of triangle ABC. Note that the sides of triangle ABC are a, a, and $2-2a$ for some a, where $2-\sqrt{2} < a < 1$. After several straightforward calculations, we see that (12) in this case may be rewritten in terms of a as $h(a) \geq k$, where

$$h(a) = \frac{a^3(1-a)^3 + 2[a(1-a)(-a^2+4a-2)]^{\frac{3}{2}} - 6a^2(1-a)^2(2a-1)^2}{6(1-a)^2(2a-1)^2(3a-2)^2}.$$

By Lemma 7, we see that the minimum value of $h(a)$ on the interval $(2-\sqrt{2}, 1)$ is approximately 0.14364, which gives our upper bound for k.

4. Fourth order inequalities

Liu [4] conjectured the following geometric inequality of fourth order,

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq 8(R^2 + 2r^2)r^2, \tag{13}$$

which was proven in [5], where it was strengthened to

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq 8(R + r)Rr^2. \tag{14}$$

Note that, since $R \geq 2r$, both (13) and (14) imply the inequality

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq 48r^4, \tag{15}$$

which is the $k = 2$ case of Theorem 4.4 in [5]. Here, we apply the prior reasoning and sharpen inequality (15), obtaining a new lower bound for the sum which is incomparable to the bounds given in (13) and (14). We also provide an alternate proof of inequality (14), though it does not appear that we are able to sharpen it using the present method.

4.1. *Sharpened form of* (15). We prove the following strengthened version of inequality (15).

Theorem 8. *For any triangle ABC and point P in its plane, we have*

$$(R_1R_2)^2 + (R_2R_3)^2 + (R_3R_1)^2 \geq 6(7R - 6r)r^3. \tag{16}$$

Proof. By Lemma 2 when $j = 2$, it suffices to show

$$\frac{(abc)^2}{a^2 + b^2 + c^2} \geq 6(7R - 6r)r^3 \tag{17}$$

for all triangles ABC with sides a, b, and c such that $a + b + c = 2$. Note that $4Rr = abc$, $r^2 = L^2 = (1 - a)(1 - b)(1 - c) = ab + bc + ca - abc - 1$, and $a^2 + b^2 + c^2 = 4 - 2(ab + bc + ca)$, since $a + b + c = 2$. Letting $p = abc$ and $t = ab + bc + ca$, we see that inequality (17) may thus be reexpressed as

$$\frac{p^2}{4 - 2t} \geq \frac{21}{2}p(t - p - 1) - 36(t - p - 1)^2.$$

Dividing through both sides of the last inequality by p^2, letting $u = \frac{t-1}{p}$, and rearranging, we see that it is equivalent to

$$w(p, u) := \frac{1}{1 - pu} - 21(u - 1) + 72(u - 1)^2 \geq 0. \tag{18}$$

Since for each $u \in (1, \frac{9}{8}]$, we have

$$\frac{d}{dp}w(p, u) = \frac{u}{(1 - pu)^2} > 0$$

for all $p \in (0, \frac{8}{27})$, it suffices to prove (18) in the case when $p = g(u)$, by Lemma 5, where g is given by (4). Rearranging inequality (18) when $p = g(u)$, and cancelling a factor of $9 - 8u$, we show equivalently that $\ell(u) \geq 0$, where

$$\ell(u) = (72u^2 - 165u + 93)(9 - 8u)^{\frac{1}{2}} - 144u^3 + 492u^2 - 619u + 279.$$

To do so, first observe that

$$\ell'(u) = \frac{-1440u^2 + 3276u - 1857}{(9-8u)^{\frac{1}{2}}} - 432u^2 + 984u - 619,$$

whence $\ell'(1) = -88 < 0$ and $\lim_{u\to\frac{9}{8}^-} \ell'(u) = \infty$. Since

$$\ell''(u) = \frac{17280u^2 - 39024u + 22056}{(9-8u)^{\frac{3}{2}}} + (984 - 864u) > 0, \qquad 1 < u < \frac{9}{8},$$

being the sum of two positive terms, it follows that the equation $\ell'(u) = 0$ has a unique real solution u^* on the interval $(1, \frac{9}{8})$. By any numerical method, we have $u^* \approx 1.123717946$. It follows that $\ell(u^*) \approx 0.205071273$ is the minimum value of the function ℓ on the interval $(1, \frac{9}{8}]$. In particular, we have $\ell(u) \geq 0$ if $1 < u \leq \frac{9}{8}$, which establishes (18) and completes the proof of (16). $\square$

Remark: Note that right-hand side of (16) is at least as large as the right-hand side of (14) when $R \leq \frac{9}{4}r$ and is smaller when $R > \frac{9}{4}r$.

4.2. *An alternate proof of* (14). Here, we provide an alternative proof for (14) to the one given in [5]. By the $j = 2$ case of Lemma 2, it is enough to show

$$\frac{(abc)^2}{a^2+b^2+c^2} \geq 8(R+r)Rr^2 \tag{19}$$

for a triangle ABC with side lengths a, b and c, where we may assume $a+b+c = 2$. Upon dividing through both sides of inequality (19) by $(abc)^2$, we see that it may be rewritten in terms of $p = abc$ and $u = \frac{ab+bc+ca-1}{abc}$ as

$$\frac{1}{2(1-pu)} \geq \frac{1}{2} + 2(u-1). \tag{20}$$

It suffices to show (20) in the case when $p = g(u)$, where g is given by (4), by Lemma 5, since the difference of the two sides is an increasing function of p for each u. To show the inequality

$$(4u-3)(1-ug(u)) \leq 1, \qquad 1 < u \leq \frac{9}{8},$$

i.e.,

$$\frac{(4u-3)(16u^2 - 36u + 27 + (9-8u)^{\frac{3}{2}})}{8u^2} \leq 1,$$

we first rewrite it as

$$-64u^3 + 200u^2 - 216u + 81 \geq (4u-3)(9-8u)^{\frac{3}{2}}.$$

Cancelling factors of $9 - 8u$ from both sides of the last inequality then gives

$$8u^2 - 16u + 9 \geq (4u-3)(9-8u)^{\frac{1}{2}}. \tag{21}$$

Finally, to show that (21) holds for $1 < u \leq \frac{9}{8}$, note that for the function

$$v(u) := \frac{8u^2 - 16u + 9}{4u-3} - (9-8u)^{\frac{1}{2}},$$

we have $v(1) = 0$ with

$$v'(u) = 2 - \frac{6}{(4u-3)^2} + \frac{4}{(9-8u)^{\frac{1}{2}}} > 0, \qquad 1 < u < \frac{9}{8}.$$

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