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Generalizing Orthocorrespondence

Manfred Evers

Abstract. In [3] B. Gibert investigates a transformation $P \mapsto P^{\perp}$ of the plane of a triangle ABC, which he calls *orthocorrespondence*. Important for the definition of this transformation is the tripolar line of $P^{\perp}$ with respect to ABC. This line can be interpreted as a polar-euclidean equivalent of the orthocenter H of the triangle ABC, the point P getting the role of the absolute pole of the polar-euclidean plane. We propose to substitute the center H by other triangle centers and will investigate the properties of such correspondences.

1. Foundations

1.1. *Introduction.* In [3] B. Gibert investigates the properties of orthocorrespondence, a mapping that every point P in the plane $\mathcal{E}$ of a triangle ABC assigns a point $P^{\perp}$, the tripole of the orthotransversal (line) $\mathcal{L}$ of P with respect to the triangle ABC. This orthotransversal $\mathcal{L}$ is described as follows: The perpendicular lines at P to AP, BP, CP intersect the lines BC, CA, AB respectively at points P_a, P_b, P_c which are collinear with the line $\mathcal{L}$.

We give an alternative description of the orthotransversal line $\mathcal{L}$, limiting ourselves to a point P which is neither an edge-point nor a point on the line at infinity. Let $A^*B^*C^*$ be the polar triangle of ABC with respect to a circle $\mathcal{S}$ with center P. Then $\mathcal{L}$ is the polar line with respect to $\mathcal{S}$ of the orthocenter H^* of $A^*B^*C^*$.

Because of this construction of the orthocorrespondent point $P^{\perp}$, we would like to call orthocorrespondence H^*-correspondence and generalize this by replacing H^* by some other point Q^* (especially by a center of the triangle $A^*B^*C^*$).

1.2. *Notations.* We always look on lines, conics, cubics, *etc.* as sets of points. Given a point R, a triangle Δ and a conic Γ, we write

- $R = (r_a : r_b : r_c)_\Delta$ if $(r_a : r_b : r_c)$ are homogeneous barycentric coordinates with respect to Δ,
- $\mathcal{L}_\Delta(R)$ for the tripolar line of R with respect to Δ,
- $\mathcal{C}_\Delta(R)$ for the circumconic and $\mathcal{J}_\Delta(R)$ for the inconic of Δ with perspector R,
- $\partial\Delta$ for the union of the three sidelines of Δ.

We suppose that the point R, $R = (r_a : r_b : r_c)_\Delta$, is not a point on $\partial\Delta$, so we have $r_a r_b r_c \neq 0$. In this case we say:

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$$\text{with respect to } \Delta,\ R \text{ is of type } \begin{cases} 0, & \text{if } \operatorname{sgn}(r_a) = \operatorname{sgn}(r_b) = \operatorname{sgn}(r_c), \\ a, & \text{if } \operatorname{sgn}(r_b) = \operatorname{sgn}(r_c) \neq \operatorname{sgn}(r_a), \\ b, & \text{if } \operatorname{sgn}(r_c) = \operatorname{sgn}(r_a) \neq \operatorname{sgn}(r_b), \\ c, & \text{if } \operatorname{sgn}(r_a) = \operatorname{sgn}(r_b) \neq \operatorname{sgn}(r_c). \end{cases}$$

In the plane of the original triangle ABC we use $\mathcal{L}_\infty$ for the line at infinity (instead of $\mathcal{L}_{ABC}(G)$ where G is the centroid of ABC), and we denote $\mathcal{E} - \mathcal{L}_\infty$ by $\mathcal{E}^-$. By d we denote the euclidean distance function. As usual, we do not define $\mathrm{d}(P, Q)$ for two points P and Q on the line $\mathcal{L}_\infty$, and we put $\mathrm{d}(P, Q) = \infty$ if exactly one of the points is infinite.

1.3. *Q^*-correspondent point and calculation of its coordinates.*
Let $P = (p_a : p_b : p_c)_{ABC}$ be a point in the plane of the triangle ABC, lying neither on a sideline of this triangle nor on $\mathcal{L}_\infty$. Let $A^*B^*C^*$ be the polar triangle of ABC with respect to a circle $\mathcal{S}$ with center P. For every point $Q^* = (q_a^* : q_b^* : q_c^*)_{A^*B^*C^*}$, we call the line $\mathcal{L}_\mathcal{S}(Q^*)$ the *Q^*-transversal of P* and its tripole with respect to ABC the *Q^*-correspondent of P*. The tripole we denote by $P\sharp Q^*$.

Remark. While the triangle $A^*B^*C^*$ and the point Q^* depend on the radius $r > 0$ of $\mathcal{S}$, the Q^*-transversal and the Q^*-correspondent of P do not.

Proposition. (1) *The Q^*-transversal of P has the equation*

$$(q_a^* p_b p_c)x + (q_b^* p_c p_a)y + (q_c^* p_a p_b)z = \Sigma_{cyclic}(q_a^* p_b p_c)x = 0.$$

(2) *If Q^* is not a vertex of the triangle $A^*B^*C^*$, then*

$$P\sharp Q^* = (p_a q_b^* q_c^* : p_b q_c^* q_a^* : p_c q_a^* q_b^*)_{ABC} = (p_a/q_a^* : \cdots : \cdots)_{ABC}.$$

Proof. (A) First, we calculate lengths a^*, b^*, c^* of the sides of $A^*B^*C^*$ for a finite point P not lying on any sideline of the triangle ABC. Let $(p_a, p_b, p_c) = (p_a, \cdots)$, $p_a + p_b + p_c = 1$, be the exact barycentric coordinates of P with respect to the triangle ABC and let a, b, c be the lengths of the sides and S be twice the area of ABC [1]. For a simpler calculation, we set the radius of the circle $\mathcal{S}$ to 1.

We then get $A^* = P + (B - C)^\perp / p'_a$ with $p'_a := p_a S = a \cdot \operatorname{sgn}(p_a) \cdot \mathrm{d}(P, BC)$. The difference of two points is interpreted as a vector of the two-dimensional vector space $\mathsf{V} = \mathbb{R}^2$ with euclidean norm $\|\cdots\|$, and $^\perp$ indicates a rotation of a vector by $+90°$: $(v_1, v_2)^\perp = (-v_2, v_1)$.

For a^* we get

$$\begin{aligned} (a^*)^2 =&\ \|B^* - C^*\|^2 = \|(C - A)/p'_b - (A - B)/p'_c\|^2 \\ =&\ (b/p'_b)^2 + 2S_A/(p'_b p'_c) + (c/p'_c)^2 \\ =&\ [(b/p_b)^2 + 2S_A/(p_b p_c) + (c/p_c)^2]/S^2. \end{aligned}$$

Note: We want to point out the following connection between the sidelengths a^*, b^*, c^* of the triangle $A^*B^*C^*$, the exact barycentric coordinates (p_a, p_b, p_c)

[1] We use Conway's triangle notation: $S = bc \sin A$, $S_A = (b^2 + c^2 - a^2)/2 = bc \cos A$, *etc.*

of the point P and the (exact) tripolar coordinates $(\mathrm{d}(P,A), \mathrm{d}(P,B), \mathrm{d}(P,C))$ of P (with respect to ABC):

$$\mathrm{d}(P,A) = \sqrt{(cp_b)^2 + 2S_A p_b p_c + (bp_c)^2} = Sa^*|p_b p_c|. \quad (*)$$

We also mention that the vertices A^*, B^*, C^* are finite points in the plane of triangle ABC as long as P is a finite point in this plane with $p_a p_b p_c \neq 0$.

(B) Calculation of the coordinates of the Q^*-correspondent $P\sharp Q^*$. Given a point Q^* with exact barycentric coordinates (q_a^*, q_b^*, q_c^*) with respect to $A^*B^*C^*$, we want to find an equation of the line $\mathcal{L}_\mathcal{S}(Q^*)$ as well as the coordinates of its tripole with respect to ABC. To achieve the results easily, we borrow a method from the theory of vector spaces which - in case of the two-dimensional vector space $\mathsf{V} = \mathbb{R}^2$ - considers an element of the dual space V^* of linear forms (often such a linear form is called a covector) as a one-dimensional affine subspace (a line) of V, see for example [2, Chapter I]. This method is not essential for the calculation of the polar line but will simplify it. We do not even have to know the coordinates of Q^* with respect to ABC, which are in fact

$$(p_a^2 p_b p_c S^2 + p_a p_b q_c^* S_B + p_a p_c q_b^* S_C - p_b p_c q_a^* a^2 : \cdots : \cdots).$$

Given a vector $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$, the dual vector is a 1-form $v^* = v_1 x + v_2 y$. To visualize this object, we identify v^* with the line $v_1 x + v_2 y = v_1^2 + v_2^2$, which is the polar line of $\vec{v}$ with respect to the unit circle $\{\vec{w} \in \mathbb{R}^2 | \; w_1^2 + w_2^2 = 1\}$. Within this interpretation, V^* is formed by all the lines of V that do not contain the zero vector, and additionally we have to include the line at infinity which represents $o^* = 0x + 0y$.

Obviously, the mapping Λ: Vx$\mathsf{V}^* \to \mathbb{R}, (\vec{v}, w^*) = ((v_1, v_2), w_1 x + w_2 y) \mapsto v_1 w_1 + v_2 w_2$ is a bilinear pairing. The mapping $\chi_P : \mathcal{E}^- \to \mathbb{R}^2, R \mapsto R - P$, is an affine chart with $\chi_P(P) = \vec{o}$ and $\chi_P(\mathcal{S}) = \{\vec{w} \in \mathbb{R}^2 | \; w_1^2 + w_2^2 = 1\}$. By means of this chart, we get a bilinear mapping

$$\Lambda_P : \mathcal{E}^- \text{x } \{\text{lines in } \mathcal{E} \text{ not passing through } P\} \to \mathbb{R}$$

with

$$\begin{cases} \Lambda_P(R, l) = 0, & \text{if } R = P \text{ or } l = \mathcal{L}_\infty \text{ or } l \parallel PR, \\ \Lambda_P(R, l) = 1/t, & \text{if } P + t(R - P) \text{ is a point on } l. \end{cases}$$

For every line l not passing through P, we get a linear form $\lambda = \Lambda_P(\cdots, l)$. Starting with a linear form λ, we find the corresponding line by $l = \{R \mid \lambda(R) = 1\}$.

Since we assume that P is not a point on any of the lines $\mathcal{L}_\infty, BC, CA, AB$, we have well defined 1-forms $\alpha := \Lambda_P(\cdots, BC), \beta := \Lambda_P(\cdots, CA), \gamma := \Lambda_P(\cdots, AB)$. For every point $R \in \mathcal{E} - P$, we can calculate the values $\alpha(R)$, $\beta(R)$, $\gamma(R)$ quite quickly once we know the values $\alpha(A)$, $\alpha(B)$, $\ldots$, $\gamma(C)$. But we already know that $\alpha(B) = \alpha(C) = 1$ and can easily calculate $\alpha(A) = 1 - 1/p_a$. Figure 1 gives an illustration of the mapping Λ_P.

Because A^*, B^*, C^* are the poles with respect to $\mathcal{S}$ of the lines $\alpha = 1, \beta = 1, \gamma = 1$, the point $Q^* = q_a^* A^* + q_b^* B^* + q_c^* C^*$ has a polar line $\mathcal{L}_\mathcal{S}(Q^*)$ with the equation $q_a^* \alpha + q_b^* \beta + q_c^* \gamma = 1$.

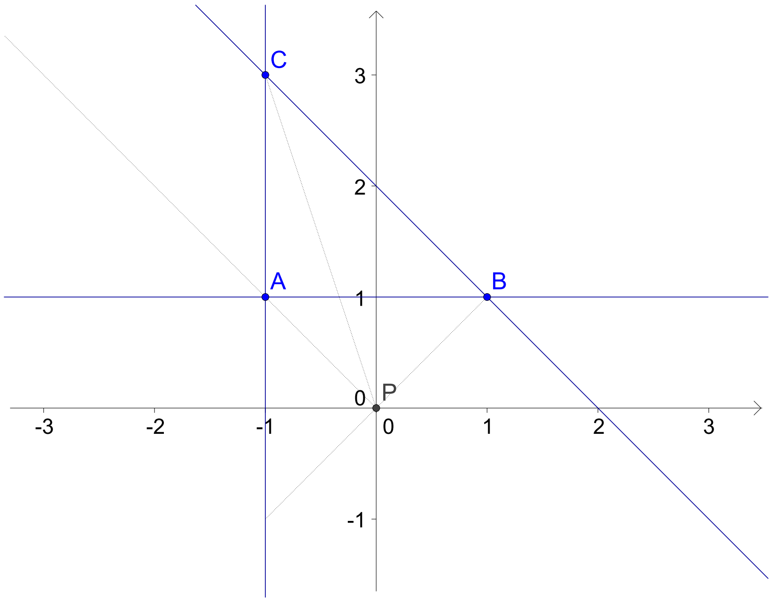

Figure 1. For the constellation shown here, we have $\Lambda_P(A, BC) = 0$, $\Lambda_P(B, CA) = -1$, $\Lambda_P(C, AB) = 1/3$.

We can now calculate the coordinates of the points of intersection of this Q^*-transversal with the sidelines of the triangle ABC. For example, the Q^*-transversal and the line BC intersect at $(0 : p_b q_c^* : -p_c q_b^*)_{ABC}$. Having calculated the three intersection points, the statements (1) and (2) of the proposition follow immediately. $\square$

We introduce the point $Q^{[P]} := (q_a^* : \cdots : \cdots)_{ABC}$, so we can write the point $P\sharp Q^* = P/Q^{[P]}$ as a barycentric quotient of two points.

1.4. *A first example.* For Q^* we choose the centroid $G^* = X_2{}^*$ of the triangle $A^*B^*C^*$.[2] For every finite point P not lying on any side line of the triangle ABC, we have the equations $G^{[P]} = G$ and $P\sharp G^* = P$. Of course, we like to extend the domain of the correspondence mapping to points on ∂ABC and on $\mathcal{L}_\infty$. For $Q^* = G^*$ we can get a continuous extension $\sharp G^* = \mathrm{id}_{\mathcal{E}}$.

Before investigating Q^*-correspondence for different triangle centers Q*, we contribute some

1.5. *Basic properties of Q^*-correspondence.*

1.5.1. If we take the cevian triangle of Q^* with respect to $A^*B^*C^*$ und construct its polar triangle with respect to $\mathcal{S}$ then we get the anticevian triangle of $P\sharp Q^*$ with respect to ABC, see Figure 2. The polar triangle of the anticevian triangle of Q^* with respect to $A^*B^*C^*$ is the cevian triangle of $P\sharp Q^*$ with respect to ABC.

[2]We adopt the notation X_n of [7] for triangle centers.

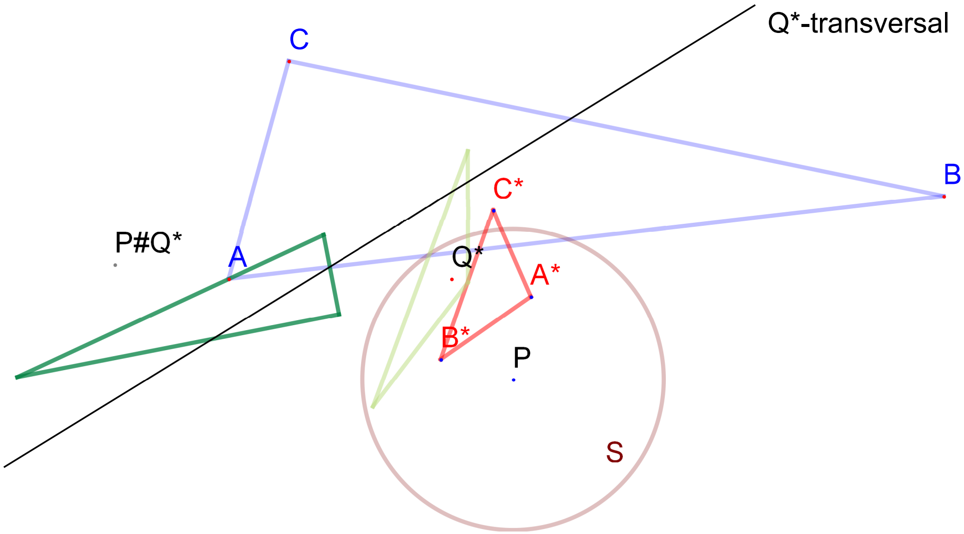

Figure 2. Besides the triangles ABC and $A^*B^*C^*$, the picture shows the cevian triangle of P with respect to $A^*B^*C^*$ (light green) and the anticevian triangle of $P\sharp Q^*$ with respect to ABC (green).

1.5.2. The polar triangle of the pedal resp. antipedal triangle of Q^* with respect to $A^*B^*C^*$ is the antipedal resp. pedal triangle of $P\sharp Q^*$ with respect to ABC.

1.5.3. If Q^* is a point on B^*C^* different from B^* and C^* then $P\sharp Q^* = A$.

1.5.4. Suppose $Q^* = (q_a^* : q_b^* : q_c^*)_{A^*B^*C^*}$ is a point satisfying the equation $P\sharp Q^* = G = X_2$, then we have $Q^* = Q^{[P]} = P$.

In the following we denote the tripolar line of Q^* with respect to $A^*B^*C^*$ by q^*.

1.5.5. In 1.2 the point $P\sharp Q^*$ was defined as the tripole with respect to ABC of the line $\mathcal{L}_\mathcal{S}(Q^*)$. But we can get $P\sharp Q^*$ as the pole of q^* with respect to $\mathcal{S}$, as well.

1.5.6. The set $P\sharp q^* := \{P\sharp R^* \mid R^* \in q^*\}$ is the circumconic of ABC with perspector $P\sharp Q^*$, so we can write

$$P\sharp q^* = \mathcal{C}_{ABC}(P\sharp Q^*) = \mathcal{C}_{ABC}(P/Q^{[P]}).$$

Two examples:

- For $q^* = \mathcal{L}_{A^*B^*C^*}(G^*) = L_\infty$ we get $P\sharp q* = \mathcal{C}_{ABC}(P)$.
- If $q^* = \mathcal{L}_{A^*B^*C^*}(X_{648}^*)$ is the Euler line of $A^*B^*C^*$, we get $P\sharp q^* = \mathcal{C}_{ABC}(P\sharp X_{648}^*)$. For special cases, see 3.1 and 3.2.

1.5.7. The polar lines with respect to $\mathcal{S}$ of points on $\mathcal{C}_{A^*B^*C^*}(Q^*)$ agree with the tangent lines of $\mathcal{J}_{ABC}(P\sharp Q^*)$. In other words: The $\mathcal{S}$-dual of $\mathcal{C}_{A^*B^*C^*}(Q^*)$ is $\mathcal{J}_{ABC}(P\sharp Q^*)$.

Example: The $\mathcal{S}$-dual of the Steiner circumellipse $\mathcal{C}_{A^*B^*C^*}(G^*)$ is $\mathcal{J}_{ABC}(P)$. As special cases we get

- for $P = G$ the Steiner inellipse with center G ,
- for $P = Ge$ (Gergonne point) the incircle with center I (incenter),
- for $P = Na$ (Nagel point) the Mandart inellipse with center M (Mittenpunkt),
- for $P = K$ (symmedian point) the Brocard inellipse with center X_{39}.

1.5.8. The $\mathcal{S}$-dual of the inconic $\mathcal{J}_{A^*B^*C^*}(Q^*)$ of $A^*B^*C^*$ with perspector Q^* is $\mathcal{C}_{ABC}(P\sharp Q^*)$.

Examples:

- $\mathcal{J}_{A^*B^*C^*}(K^*)$ is the Brocard inellipse of $A^*B^*C^*$. Its $\mathcal{S}$-dual is $\mathcal{C}_{ABC}(P\sharp K^*)$, with $P\sharp K^* = (1/(p_a(p_b^2c^2 + 2p_bp_cS_A + p_c^2b^2)) : \cdots : \cdots)_{ABC}$. For the special case $P = O$, we get $P\sharp K^* = K$; the $\mathcal{S}$-dual of the Brocard inellipse of $A^*B^*C^*$ is the circumcircle of ABC.
- The $\mathcal{S}$-dual of the Steiner inellipse $\mathcal{J}_{A^*B^*C^*}(G^*)$ is $\mathcal{C}_{ABC}(P)$. As special cases we get
  - the circumellipse which is shown in Figure 5 for $P = I$,
  - the Steiner circumellipse for $P = G$,
  - the circumcircle for $P = K$,
  - the Kiepert hyperbola for $P = X_{523}$,
  - the Jerabek hyperbola for $P = X_{647}$.

1.6. *The I^*-correspondence (first part).* As mentioned above, we are mainly interested in the special case of Q^* being a triangle center of $A^*B^*C^*$. For further definitions we orient ourselves on the mapping $P \mapsto P\sharp I^*$ because I^* is the most important weak center of $A^*B^*C^*$, and it is a center for which the anticevians agree with extraversions: ${}_\tau I^* = I^*_\tau$, $\tau = 0, a, b, c$.

$(\mathrm{d}(P, A)\Delta|p_a| : \cdots : \cdots)$ are the homogeneous barycentric coordinates of I^* with respect to $A^*B^*C^*$ and of $I^{[P]}$ with respect to ABC. It can be easily seen that the mapping $\mathcal{E}^- - \partial ABC \to E, P \mapsto P/I^{[P]} = (\mathrm{sgn}(p_a)\mathrm{d}(P, B)\mathrm{d}(P, C) : \cdots : \cdots)_{ABC}$, cannot be extended to a continuous mapping with domain $\mathcal{E}^- - \{A, B, C\}$. But if we introduce the point

$$\begin{aligned} I^{[P,0]} &:= (a^P : b^P : c^P)_{ABC} \\ &:= (\mathrm{sgn}(p_a)a^* : \mathrm{sgn}(p_b)b^* : \mathrm{sgn}(p_c)c^*)_{ABC} \\ &= (p_a\mathrm{d}(P, A) : p_b\mathrm{d}(P, B) : p_c\mathrm{d}(P, C))_{ABC} \end{aligned}$$

and its anticevians $I^{[P,a]} := (-a^P : b^P : c^P)_{ABC}, \cdots$, all the mappings $\mathcal{E}^- - \{A, B, C\} \to E, P \mapsto P/I^{[P,\tau]} =: (P\sharp I^*)^\tau, \tau = 0, a, b, c$, are continuous. We get $(P\sharp I^*)^0 = (\mathrm{d}(P, B)\mathrm{d}(P, C) : \cdots : \cdots)_{ABC}$, which is a point of type 0, and the points $(P\sharp I^*)^\tau$, $\tau = a, b, c$, are the anticevians of $(P\sharp I^*)^0$.

We can see here that the same way the weak triangle center I^* comes in four versions (a main center I_0 and its three mates I_a, I_b, I_c), I^*-correspondence splits into four parts.

For $P \in \{A, B, C\}$ we have the equations $(P\sharp I^*)^\tau = P$, $\tau = 0, a, b, c$; the vertices are fixed points of all four I^*-correspondences.

Let us suppose now that the point P is a point on $\mathcal{L}_\infty$. Since we have $\lim_{R\to P}(a^R : b^R : c^R) = \lim_{R\to P}(p_a \mathrm{d}(R,A) : p_b \mathrm{d}(R,B) : p_c \mathrm{d}(R;C))) = (p_a : p_b : p_c)$, we put $(a^P : b^P : c^P) := (p_a : p_b : p_c)$ and define $I^{[P,0]} := (a^P : b^P : c^P)_{ABC}, \cdots$
We get $(P\sharp I^*)^\tau := P/I^{[P,\tau]} = {}_\tau G,\ \tau = 0, a, b, c$.
Conclusion: All four mappings $\mathcal{E}^- - \{A, B, C\} \to \mathcal{E}, P \mapsto (P\sharp I^*)^\tau,\ \tau = 0, a, b, c$, can be extended to continuous mappings $\mathcal{E} \to \mathcal{E}$.

1.6.1. *Special cases.*

- $P = I = X_1 : (I\sharp I^*)^0 = X_{174}$ (Yff-center of congruence).
- $P = G = X_2 : (G\sharp I^*)^0 = (1/\sqrt{2b^2 + 2c^2 - a^2} : \cdots : \cdots)_{ABC} = \sqrt{X_{598}}$.
- $P = O = X_3$, and suppose O is of type τ: $(O\sharp I^*)^\tau = {}_\tau G$.
- $P = H = X_4$, and suppose H is of type τ: $(H\sharp I^*)^\tau = {}_\tau X_{52}$.
- $(L_\infty \sharp I^*)^\tau = {}_\tau G$. (More accurately, we should write: $(L_\infty \sharp I^*)^\tau = \{{}_\tau G\}$.)

1.7. *The Definition of Q^*-correspondence for other centers of $A^*B^*C^*$.*
Let $Q^* = (q_a^* : q_b^* : q_c^*)_{A^*B^*C^*}$ be any triangle center of $A^*B^*C^*$ and let f^* be a barycentric center function, homogeneous in its arguments, with
$Q^* = ((f^*(a^* : b^* : c^*) : f^*(b^* : c^* : a^*) : f^*(c^* : a^* : b^*))_{A^*B^*C^*}$.
We take the definition of $(a^P : b^P : c^P)$ from the last subsection, introduce the points

$$\begin{cases} Q^{[P,0]} := (f^*(a^P : b^P : c^P) : f^*(b^P : c^P : a^P) : f^*(c^P : a^P : b^P))_{ABC}, \\ Q^{[P,a]} := (f^*(-a^P : b^P : c^P) : f^*(b^P : c^P : -a^P) : f^*(c^P : -a^P : b^P))_{ABC}, \\ \textit{etc}. \end{cases}$$

and put $(P\sharp Q^*)^\tau := P/Q^{[P,\tau]},\ \tau = 0, a, b, c$.

The Q^*-correspondent $(P\sharp Q^*)^\tau$ of P is well defined if and only if at least one of the three coordinates in the definition is not zero. We denote the set of points P where all the points $(P\sharp Q^*)^\tau,\ \tau = 0, a, b, c$, are defined by $\mathrm{dom}(Q^*)$.

The mappings $(\cdots\sharp Q^*)^\tau : \mathrm{dom}(Q^*) \to \mathcal{E},\ \tau = 0, a, b, c$, are continuous.

If Q^* is a strong center of $A^*B^*C^*$ then for every point P in $\mathrm{dom}(Q^*)$ the set $\{(P\sharp Q^*)^\tau \mid \tau = 0, a, b, c\}$ consists of only one point, $P\sharp Q^*$.

Examples.

1.7.1. Taking $P = H$, we have $(a^P : b^P : c^P) = (a : b : c)$. So we get $I^{[H,0]} = I,\ G^{[H,0]} = G^{[H]} = G,\ O^{[H,0]} = O^{[H]} = O, \cdots$ (see also 3.2.)

1.7.2. Let P be a point on $\mathcal{L}_\infty$. We get $P\sharp G^* = P,\ P\sharp O^* = P\sharp H^* = G$. The points G^*, O^*, H^* are points on the Euler line of the (degenerate) triangle $A^*B^*C^*$. If Q^* is any point on this line, $P\sharp Q^*$ is a point on the circumconic of ABC through G and P. The perspector of this conic is $P\sharp(X_{648})^*$.

Two special cases:

- Taking $P = X_{30}$ (Euler infinity point), we get $X_{648}^{[P]} = X_{648}$ and $P\sharp X_{648}^* = X_{30}/X_{648}$.

- If P is one of the two infinite points of the Kiepert hyperbola $\mathcal{C}_{ABC}(X_{523})$, we get $P\sharp X^*_{648} = X_{523}$.

1.7.3. If we take $Q^* = K^* = X_6{}^*$, we get

- $K^{[P]} = (p_a{}^2(c^2 p_b{}^2 + 2S_A p_b p_c + b^2 p_c{}^2) : \cdots : \cdots)_{ABC}$.
- $P\sharp K^* = (p_b p_c \, \mathrm{d}^2(P,B)\, \mathrm{d}^2(P,C) : \cdots : \cdots)_{ABC}$
  $= (1/(p_a(c^2 p_b{}^2 + 2S_A p_b p_c + b^2 p_c{}^2)) : \cdots : \cdots)_{ABC}$.
- $\mathrm{dom}(K^*) = \mathcal{E} - \{A, B, C\}$.

Special cases:

- $K^{[I]} = M = X_9;\ I\sharp K^* = Ge = X_7$.
- $K^{[G]} = X_{599};\ G\sharp K^* = X_{598}$.
- $K^{[O]} = X_{577};\ O\sharp K^* = X_{264} = G/O$.
- $K^{[H]} = K;\ H\sharp K^* = X_{264}$.
- $K^{[K]} = X_{574};\ K\sharp K^* = X_{598}$.
- If P is not a point on a sideline of ABC then we have $\lim_{t\to 0}(A + tP)/K^{[A+tP]} = A$.
- If P is a point on a sideline of ABC but not a triangle vertex then $P\sharp K^*$ is the vertex opposite this sideline. For a point P on AB, different from A, we therefore get $\lim_{t\to 0}(A + tP)/K^{[A+tP]} = C$. This shows that K^*-correspondence $\sharp K^* : \mathrm{dom}(K^*) \to \mathcal{E}, P \mapsto P\sharp K^*$, does not have any extension that is continuous in A, B, C.
- $\mathcal{L}_\infty \sharp K^* = \mathcal{C}_{ABC}(G)$ (Steiner circumellipse).

If instead of P we take its isogonal conjugate K/P, we get $K^{[K/P]} = (a^2(c^2 p_b{}^2 + 2S_A p_b p_c + b^2 p_c^2) : \cdots : \cdots)_{ABC}$ and $(K/P)\sharp K^* = P\sharp K^*$.

1.7.4. We take $Q^* = Ge^* = X_7^*$ and get

$$\begin{aligned}
(P\sharp Ge^*)^0 &= (p_a(-a^P + b^P + c^P) : p_b(a^P - b^P + c^P) : p_c(a^P + b^P - c^P)),\\
(P\sharp Ge^*)^a &= (p_a(a^P + b^P + c^P) : p_b(-a^P - b^P + c^P) : p_c(-a^P + b^P - c^P)),\\
&\vdots
\end{aligned}$$

A careful analysis shows that $\mathrm{dom}(Ge^*) = \mathcal{E}$.

Special cases:

- The vertices A, B, C are fixed points of all four Ge^*-correspondences.
- If $P = (0 : t : 1 - t)_{ABC}$ is a point on BC and $t(1-t) > 0$ then
  $$(P\sharp Ge^*)^\tau = \begin{cases} (\ 2t(1-t)a : tg(t) : (1-t)g(t))_{ABC} \text{ for } \tau = 0,\\ (-2t(1-t)a : tg(t) : (1-t)g(t))_{ABC} \text{ for } \tau = a,\\ (0 : -t : 1-t)_{ABC} \text{ for } \tau = b, c,\end{cases}$$
  where the polynomial function g is defined by
  $$g(t) := \sqrt{-t(1-t)a^2 + (1-t)b^2 + tc^2}.$$

- If $P = (0 : t : 1-t)_{ABC}$ is a point on BC and $t(1-t) < 0$ then $$(P\sharp Ge^*)^\tau = \begin{cases} (0 : -t : 1-t)_{ABC} \text{ for } \tau = 0, a \\ (\ 2|t|(1-t)a : tg(t) : (1-t)g(t))_{ABC} \text{ for } \tau = b, \\ (-2|t|(1-t)a : tg(t) : (1-t)g(t))_{ABC} \text{ for } \tau = c. \end{cases}$$
- For a point $P = (p_a : p_b : p_c)_{ABC}$ on $\mathcal{L}_\infty$ we get $(P\sharp Ge^*)^0 = (p_a^2 : p_b^2 : p_c^2)_{ABC}$ (this is a point on the Steiner inellipse of ABC), $(P\sharp Ge^*)^a = (0 : 1 : 1)_{ABC}$ *etc.*

1.8. *Fixed points of Q^*-correspondence.*

(A) Fixed points on a sideline of ABC. For different centers Q^* the situation can be quite different: For $Q^* = H^*$ (see [7]), $Q^* = I^*$ (see 1.6), $Q^* = Ge^*$ (see 1.7.3), the vertices of ABC are the only edgepoints which are fixed points of the correspondence mapping. (In case of the weak center $Q^* = I^*$, the vertices are fixed points for all four correspondences $(\cdots\sharp I^*)^\tau$, $\tau = 0, a, b, c$.) The correspondence of $Q^* = (X_{110})^* = (a^2/(b^2-c^2) : \cdots : \cdots)_{ABC}$ has exactly six fixed points on the sidelines, the vertices of ABC and the vertices of the orthic triangle. For some centers, as for $Q^* = (X_{76})^* = G^*/K^*$, every point on a sideline of ABC is a fixed point. In contrast, K^*-correspondence has no proper fixed point on a sideline of ABC (see 1.7.2).

(B) Fixed points not lying on a sideline of ABC. If we assume P is a finite point not lying on any side line of the triangle ABC, the equation $P\sharp Q^* = P$ is true if and only if $Q^* = G^*$ or $A^*B^*C^*$ is equilateral. $A^*B^*C^*$ is equilateral if and only if P is one of the two Fermat points X_{13}, X_{14}.

Suppose that F is a Fermat point and that Q^* is a weak center of $A^*B^*C^*$. If F is of type 0 then $(F\sharp Q^*)^0 = F$, $(F\sharp Q^*)^a$ is a point on the line AF, *etc.* If $P = F$ is of type a then $((F\sharp Q^*)^a = F$ and $(F\sharp Q^*)^0$ is a point on the line AF, $(F\sharp Q^*)^b$ is a point on the line BF, *etc.* We give a proof of the last statement: If $P = F$ is of type a then ${}_aQ^*$ is identical with the center G^* of the equilateral triangle $A^*B^*C^*$ and the points ${}_0Q^*$, ${}_bQ^*$, ${}_cQ^*$ lie on the lines G^*A^*, G^*B^*, G^*C^*, respectively. The polar line of ${}_0Q^*$ with respect to $\mathcal{S}$ passes through the pole of G^*A^* which is the point $(0 : -p_b : p_c)_{ABC}$. Therefore, $(F\sharp Q^*)^0$ is a point on the line through A and $(0 : p_b : p_c)_{ABC}$. But this line also goes through $P = F$. The same way follows that $(F\sharp Q^*)^b$, $(F\sharp Q^*)^c$ are points on BF resp. CF.

1.9. *Points P with an isosceles triangle $A^*B^*C^*$.* We assume that $A^*B^*C^*$ is an isosceles triangle with $b^* = c^*$. The last equation leads to the following condition for the exact coordinates (p_a, p_b, p_c) of the point P:

$$p_b^2((p_b-1)c^2 + p_c(b^2-c^2)) = p_c^2((p_c-1)b^2 + p_b(c^2-a^2)).$$

The locus of points P satisfying the last equation is (after completion) a cubic which passes through the points A, B, C, A being a dubble point. We denote this algebraic curve (a strophoide) by $\mathcal{K}(A; B, C)$. Since A is a dubble point of this curve, one can find a rational parametrisation for it. $\mathcal{K}(A; B, C)$ also passes through the vertex H_A of the orthic triangle $H_AH_BH_C$, the two Fermat points and the infinite point $(-2:1:1)_{ABC}$ on the triangle median AG (see Figure 3).

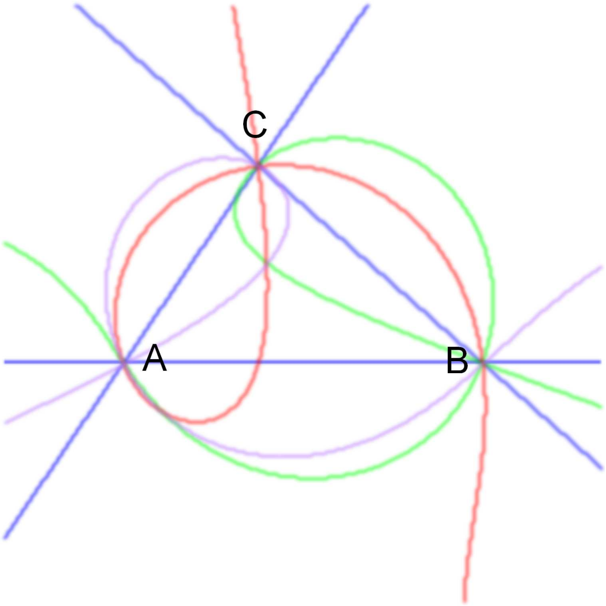

Figure 3. Here are shown the cubics $\mathcal{K}(A;B,C)$, $\mathcal{K}(B;C,A)$, $\mathcal{K}(C;A,B)$. See 1.9 for a definition of these curves.

1.10. *The image of the circumcircle of ABC under Q^*-correspondence.* If P is a point on this circle but not a triangle vertex then ABC and $A^*B^*C^*$ are similar triangles: $a^*:b^*:c^* = a:b:c$. Therefore, if Q^* is a center of $A^*B^*C^*$ with a center function f^*, we get $Q^{[P]} = (f^*(a,b,c):\cdots:\cdots)_{ABC}$ and $P\sharp Q^*$ is a point on the circumconic $\mathcal{C}_{ABC}(K/Q^{[P]})$.

Examples.

- $\mathcal{C}_{ABC}(K)\sharp G^* := \{P/G^{[P]} \mid P \in \mathcal{C}_{ABC}(K)\} = \mathcal{C}_{ABC}(K/G) = \mathcal{C}_{ABC}(K)$.
- $\mathcal{C}_{ABC}(K)\sharp I_\tau^* = \mathcal{C}_{ABC}(K/I_\tau) = \mathcal{C}_{ABC}(I_\tau)$ for $\tau = 0, a, b, c$ (see Figure 4.)
- $\mathcal{C}_{ABC}(K)\sharp O^* = \mathcal{C}_{ABC}(K/O) = \mathcal{C}_{ABC}(H)$.
- $\mathcal{C}_{ABC}(K)\sharp H^* = \mathcal{C}_{ABC}(K/H) = \mathcal{C}_{ABC}(O)$, see [7].
- If we put $P\sharp K^* = P$ for $P = A, B, C$ (see 1.7.3) then $\mathcal{C}_{ABC}(K)\sharp K^* = \mathcal{C}_{ABC}(G)$.

We also look at the isotomic conjugates of these circumconics:

- $\{G^{[P]}/P \mid P \in \mathcal{C}_{ABC}(K)\} = \mathcal{L}_{ABC}(K)$.
- $\{I_\tau^{[P]}/P \mid P \in \mathcal{C}_{ABC}(K)\} = \mathcal{L}_{ABC}(I_\tau), \tau = 0, a, b, c$.
- $\{O^{[P]}/P \mid P \in \mathcal{C}_{ABC}(K)\} = \mathcal{L}_{ABC}(H)$.
- $\{H^{[P]}/P \mid P \in \mathcal{C}_{ABC}(K)\} = \mathcal{L}_{ABC}(O)$.
- $\{K^{[P]}/P \mid P \in \mathcal{C}_{ABC}(K)\} = \mathcal{L}_\infty$.

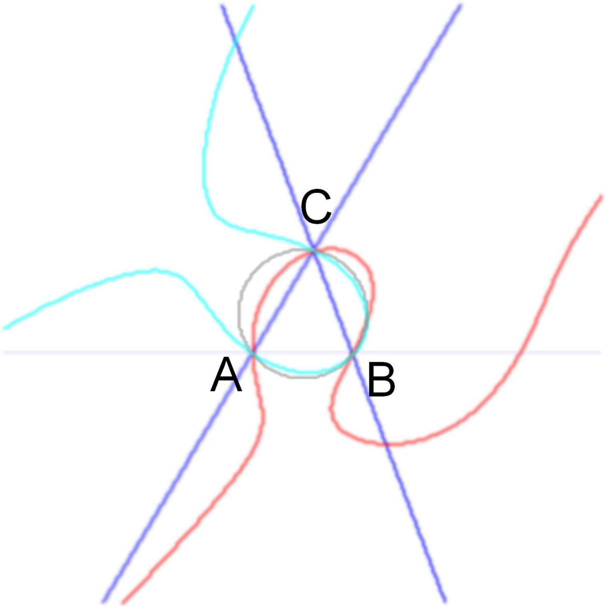

Figure 4. This shows the circumcircle (grey) and the cubics $\mathcal{V}(K^*, O)$ (cyan) and $\mathcal{V}(K^*, H)$ (red) for the triangle ABC, see 1.11.1.

1.11. *The preimage under Q^*-correspondence / Q^*-associates.* The mapping $\sharp G^* : \mathcal{E} \to \mathcal{E}$ is bijective. But in general, Q^*-correspondence is neither injective nor surjective. Gibert proved (see [3]) that for $Q^* = H^*$ there are up to two points having the same correspondent[3]. Points having the same correspondent he calls *associates*. We shall take this terminus here. As we could see in 1.7.3, a point P is a K^*-associate of its isogonal conjugate. There are centers Q^* with more than two Q^*-associates, $Q^* = O^*$ for example (see in 2.3.4). Q^*-correspondence doesn't have to be surjective, either. For example, for $Q^* = K^*$ there is no point $P\sharp Q^*$ on a sideline of the triangle ABC except for the vertices of this triangle.

We now describe a way of constructing the preimage of a point $R = (r_a : r_b : r_c)_{ABC}$ under Q^*-correspondence. We want to determine all points P with $P\sharp Q^* = R$ and omit all the special cases $(P\sharp Q^*)^\tau$, $\tau = 0, a, b, c$. (These can be easily adapted.)

We start with a point P and choose a point Q^* which is a triangle center of $A^*B^*C^*$ with barycentric center function $f^*(a^*, b^*, c^*)$. The Q^*-transversal of P, $\mathcal{L}_\mathcal{S}(Q^*)$, is the set of points $(x : y : z)_{ABC}$ satisfying the equation

$$\Sigma_{cyclic} p_b p_c f^*(a^*, b^*, c^*)\, x = 0.$$

Given a point T, we denote the set of points P with T a point on $\mathcal{L}_\mathcal{S}(Q^*)$ by $\mathcal{V}(Q^*, T)$. If T is not an edgepoint, the set dom$(Q^*) \cap \mathcal{V}(Q^*, T)$ is the preimage of the circumconic $\mathcal{C}_{ABC}(T)$. If $T = (0 : t : 1-t)_{ABC}$, $t(1-t) \neq 0$, is a

[3]Gibert proved in fact that - using proper multiplicity - there are exactly two real or two complex points having the same correspondent.

point on BC but not a vertex then $\mathrm{dom}(Q^*) \cap \mathcal{V}(Q^*, T)$ is the preimage of the line through the points A and $(0 : t : t-1)_{ABC}$. Finally, if T is a triangle vertex then $\mathrm{dom}(Q^*) \cap \mathcal{V}(Q^*, T)$ is the preimage of this vertex.

Now we can present the preimage of a point R which is not a vertex of ABC: It is the set $\mathcal{V}(Q^*, T_1) \cap \mathcal{V}(Q^*, T_2) \cap \mathrm{dom}(Q^*)$ for any two different points T_1, T_2 on $\mathcal{L}_{ABC}(R)$.

1.11.1. Example. We want to determine the preimage of the point X_{648}, the tripole of the Euler line, under K^*-correspondence. So we choose two different points on $\mathcal{L}_{ABC}(X_{648})$, G and O for instance. For every point T, the set $\mathcal{V}(K^*, T)$ is a cubic curve. For $T = G$, this cubic is the union of the line at infinity and the circumcircle of ABC. We now look at $\mathcal{V}(K^*, O)$. There is exactly one infinite point, let us say P_1, on this curve, so this point is mapped to X_{648} by K^*-correspondence. In general, $\mathcal{V}(K^*, O)$ and the circumcircle have four common points. Three of them are the points A, B, C; the fourth common point is a the finite point, P_2, which is mapped to X_{648} by K^*-correspondence. For an isosceles but not equilateral triangle ABC, the point X_{648} agrees with one of the edges A, B, C, and so does the point P_2. See Figure 4 for a picture. For more examples, see 2.1.4 and 2.3.4.

1.12. *Pivotal curves.* In [3] Gibert introduces algebraic curves consisting of all points P for which the line through P and its orthocorrespondent $P\sharp H^*$ passes through a given point R. Such a curve Gibert calls *orthopivotal*, the point R being the *orthopivot*. We transfer Gibert's concept to other correspondences. Given a point $R = (r_a : r_b : r_c)_{ABC}$, the set of points P such that the points $R, P, P\sharp Q^*$ are collinear is

$$\{P = (p_a : p_b : p_c)_{ABC} \in \mathrm{dom}(Q^*) \mid \Sigma_{cyclic} r_a q_a^*(q_b^* - q_c^*) p_b p_c \ = \ 0 \}.$$

We call this set Q^**-pivotal set with pivot point* R. For a triangle center Q^* the coordinates q_a^*, q_b^*, q_c^* depend on P, of course.

For a strong center Q^*, the Q^*-pivotal set is an open set (with respect to the Zariski topology) of an algebraic curve which we denote by $\mathcal{P}(Q^*, R)$. For most strong centers, these curves are of high degree (> 4) and rather complicated. Thus, we do not go into an analysis of these. But for all of the curves $\mathcal{P}(Q^*, R)$, one can state that if R is not an edgepoint, they pass through the vertices A, B, C, the two Fermat points and the point R. Gibert gives a detailed description of the orthopivotal curves $\mathcal{P}(H^*, R)$. These are cubics. The question arises: What are the other pivotal curves of degree 3? The answer is: There aren't any! ***Proof:*** If $\mathcal{P}(Q^*, R)$ has degree 3 then the correspondent center Q^* must have a (homogeneous and bisymmetric) barycentric centerfunction $f^*(a^*, b^*, c^*) = 1/(ma^{*2} + n(b^{*2} + c^{*2}))$ with two different real numbers n, m. (For $i < 100$ there are just three such centers X_i, namely, X_4, X_{76} and X_{83}.) For all of these centers Q^* one gets $\mathcal{P}(Q^*, R)$ = $\mathcal{P}(H^*, R)$ because the points $P, P\sharp Q^*, P\sharp H^*$ are always collinear, as one can verify by simple calculation.

For a weak center Q^*, the set of points P so that for some $\tau \in \{0, a, b, c\}$ the three points $P, (P\sharp Q^*)^\tau$ and R are collinear is an open set of an algebraic curve which we denote by $\mathcal{P}(Q^*, R)$. In 3.1 we present a picture of $\mathcal{P}(I^*, R)$.

2. Q^*-correspondence for "classical" triangle centers Q^*.

2.1. *I^*-correspondence (second part).*

2.1.1. Geometric construction of the image and preimage points. For each point $P \in \mathcal{E}^- - \{A, B, C\}$ we define six points $P_A, P_A^{'}, P_B, P_B^{'}, P_C, P_C^{'}$ by: P_A is the intersection of BC with the internal bisector of the angle $\angle BPC$, and $P_A^{'}$ is the intersection of BC with the external bisector of this angle. Similarly we define the points $P_B, P_B^{'}, P_C, P_C^{'}$.

P. Yiu [10] shows the following properties of these six points: The triangle $P_AP_BP_C$ is the cevian triangle of some point that lies inside the triangle and that we call ${}_0R$. The tripolar line $\mathcal{L}_{ABC}({}_0R)$ of ${}_0R$ intersects the side lines BC, CA, AB in $P_A^{'}, P_B^{'}, P_B^{'}$, respectivly. The points $P_A^{'}, P_B, P_C$ are collinear with the line $\mathcal{L}_{ABC}({}_aR)$, the points $P_B{}', P_C, P_A$ collinear with the line $\mathcal{L}_{ABC}({}_bR)$ and the points $P_C^{'}, P_A, P_B$ collinear with the line $\mathcal{L}_{ABC}({}_cR)$. Further more, Yiu shows: The circles with diameters $P_AP_A^{'}, P_BP_B^{'}, P_CP_C^{'}$ - they are called *generalized Apollonian circles* [9], [10] [4] - have their centers on the line $\mathcal{L}_{ABC}(R^2),\ R^2 = ({r_a}^2 : {r_b}^2 : {r_c}^2)_{ABC}$, and they are in the same pencil of circles through the point P and its image P' under the reflection in the circumcircle of ABC. (If P is a point on the circumcircle then all three circles are mutually tangent to each other and P is the point of tangency.)

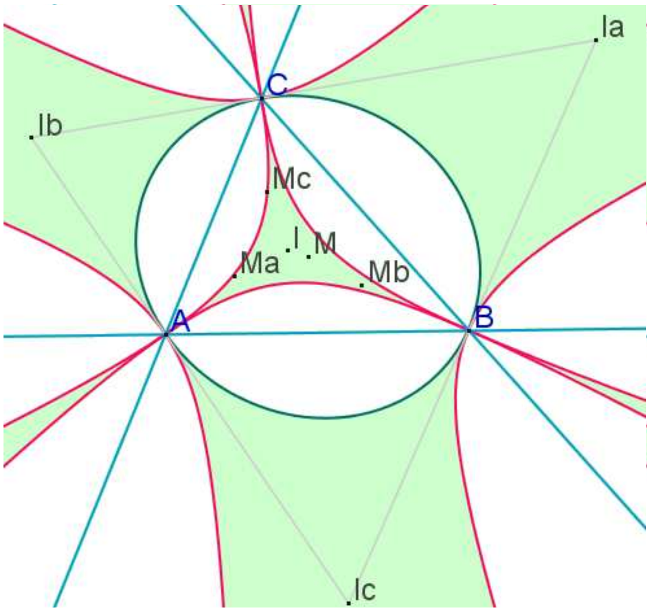

Figure 5.

[4]The original Apollonian circles we get for $P = I$.

Since ${}_0R = (\mathrm{d}(P,B)\mathrm{d}(P,C) : \cdots : \cdots)_{ABC}$, this point agrees with $(P\sharp I)^0$, and ${}_\tau R$ agrees with $(P\sharp I)^\tau$ for $\tau = a,b,c$. P' is the I^*-associate of P. A routine calculation gives its coordinates: $P' = (p_a^2a^2b^2c^2 + p_ap_ba^2c^2(a^2-c^2) + p_ap_ca^2b^2(a^2-b^2) + p_bp_ca^4\cdot(a^2-b^2-c^2) : \cdots : \cdots)_{ABC}$.

Question: Given a point R, what is the number n_R of (real) points P with $R = (P\sharp I^*)^\tau$ for some $\tau \in \{0,a,b,c\}$? In [10] Yiu gives the following answer: The number n_R is 2, 1, or 0 according as the line $\mathcal{L}_{ABC}(R^2)$ intersects the circumcircle of ABC in 0, 1, or 2 points. Additionally, one could ask for a partition of $\mathcal{E}$ illustrating the domains of points R with $n_R = 0$ resp. 1 resp. 2. The set of points R with $n_R = 1$ is the union of circumconics $\mathcal{C}_{ABC}(I_\tau)$, $\tau = 0,a,b,c$. The set of points R with $n_R = 2$ is the union of the open green domains shown in Figure 5. We also can get a partition of the plane by lines showing the domains of points $R^{-1} = G/R$ with $n_R = 0,1,2$ (see Figure 6).

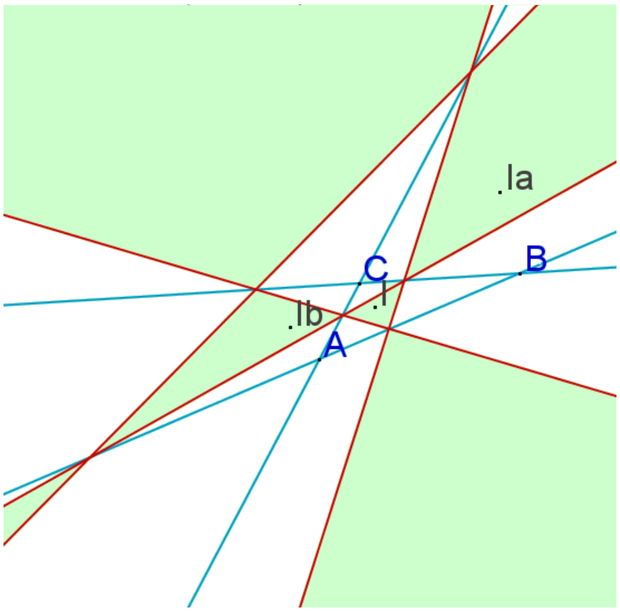

Figure 6.

The set of points R^{-1} with $n_R = 1$ is the union of lines $\mathcal{L}_{ABC}(I_\tau)$. The set of points R^{-1} with $n_R = 2$ is the union of the green areas. This way we can link Yiu's [10] and Weaver's [9] work to a problem that was put and solved by Bottema in [1]: Given a triplet (r_a, r_b, r_c) of real numbers, what is the number of points P satisfying $(r_a : r_b : r_c) = (\mathrm{d}(P,A) : \mathrm{d}(P,B) : \mathrm{d}(P,C))$? Identifying $(r_a : r_b : r_c)$ with the point $R = (r_a : r_b : r_c)_{ABC}$, Bottema's answer can be formulated as follows: The number of points depends on $\mathrm{d}(R,BC), \mathrm{d}(R,CA)$ and $\mathrm{d}(R,AB)$ being the sidelengths of a triangle (two points), a degenerate triangle (one point) or not a triangle (zero points).

Given a point $R = (r_a, r_b, r_c)_{ABC}$ of type τ, the points P and P' with $(P\sharp I^*)^\tau = (P'\sharp I^*)^\tau = R$ have coordinates

$$((b^2+c^2+({r_b}^2-{r_c}^2))\sqrt{a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2}$$
$$\pm [(c^2+a^2-b^2)\sqrt{c^4-2c^2({r_a}^2+{r_b}^2)+({r_a}^2-{r_b}^2)^2}$$
$$+(a^2+b^2-c^2)\sqrt{b^4-2b^2({r_c}^2+{r_a}^2)+({r_c}^2-{r_a}^2)^2}]$$
$$:\cdots:\cdots)_{ABC}.$$

We get real values for points R with $n_R \geq 1$.

2.1.2. There is a direct connection between I^*-correspondence and orthocorrespondence: The I^*-correspondent $P\sharp I^*$ agrees with the orthocorrespondent of P for the cevian triangle of $P\sharp I^*$. This is a consequence of the well known fact that the orthocenter H^* of the triangle $A^*B^*C^*$ is the incenter of its orthic triangle which we denote by Δ^*. Since the tripolar of any point with respect to a given triangle agrees with the tripolar of this point with respect to its cevian triangle, we have $\mathcal{L}_{A^*B^*C^*}(H^*) = \mathcal{L}_{\Delta^*}(H^*)$. The polar triangles of $A^*B^*C^*$ and Δ^* with respect to $\mathcal{S}$ are ABC and the cevian triangle of $P\sharp I^*$, respectivly.
Consequences: (1) P' is the orthoassociate of P with respect to the cevian triangle of $P\sharp I^*$.
(2) The circumcircle of ABC is identical with the polar circle of the cevian triangle of $P\sharp I^*$.
(3) The orthocorrespondent $P\sharp H^*$ of P with respect to ABC agrees with the I^*-correspondent of P for the anticevian triangle of $P\sharp H^*$.
(4) The polar circle of ABC is identical with the circumcircle of the anticevian triangle of $P\sharp H^*$.

2.1.3. The image of the sidelines. $\bigcup_{\tau=0,a,b,c}(AB\sharp I^*)^\tau$ is an analytic curve which is shown in Figure 7.

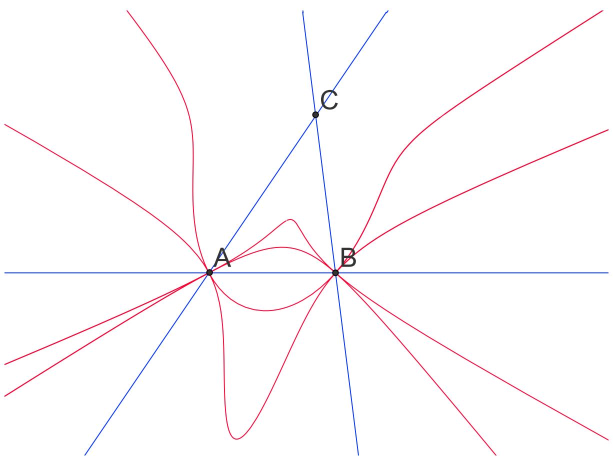

Figure 7. The red curve is the image of the sideline AB under the mappings $(I^*)^\tau$, $\tau = 0, a, b, c$.

2.1.4. *The preimage of $\mathcal{L}_\infty$ under I^*-correspondence.* A point P has the image point $(P\sharp I^*)^a$ on the line of infinity if and only if $1/\mathrm{d}(P,A) = 1/\mathrm{d}(P,B) + 1/\mathrm{d}(P,C)$. The set of points P satisfying the last equation is an analytic curve (an oval) $\mathcal{O}_a$ which is invariant under inversion with respect to the circumcircle $\mathcal{C}_{ABC}(K)$. The union of the three ovals $\mathcal{O}_\tau,\ \tau = a,b,c$, is the algebraic curve $\{P \mid \Sigma_{cyclic}\, \mathrm{d}^2(P,B)\mathrm{d}^2(P,C)(\mathrm{d}^2(P,B)\mathrm{d}^2(P,C) - 2\mathrm{d}^4(P,A)) = 0\}$ (see Figure 8).

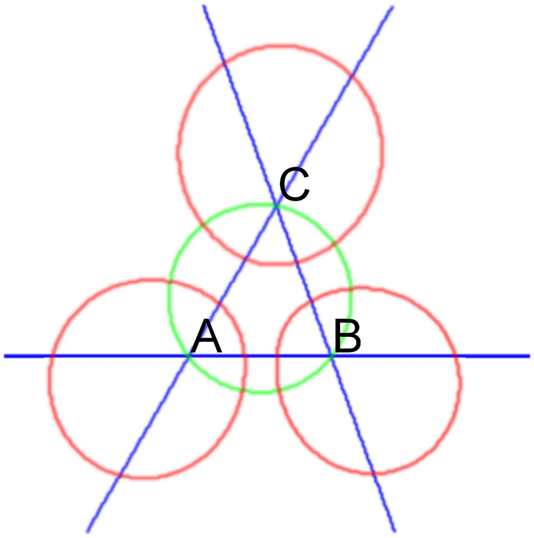

Figure 8. The set of points P with $(P\sharp Q^*)^\tau$ a point on $\mathcal{L}_\infty$, $\tau = a,b,c$, is an algebraic curve which is the union of the three (red) ovals.

2.1.5. *The $\mathcal{S}$-duals of the incircle and the excircles of the triangle $A^*B^*C^*$.* Because of the strong connection between the incenter and the incircle and the excenters and their correspondent excircles, we take a brief look at the incircle and the excircles of $A^*B^*C^*$, $J_{A^*B^*C^*}(Ge^*_\tau)$, $\tau = 0,a,b,c$, and their $\mathcal{S}$-duals, $\mathcal{C}_{ABC}((P\sharp Ge^*)^\tau)$, $\tau = 0,a,b,c$. The point P is a focus of each of these circumconics, and the lines $\mathcal{L}_{ABC}(P\sharp I^*)^\tau)$, $\tau = a,b,c$, are the corresponding directrices. Figure 9 shows the situation for $P = O$.

2.1.6. *I^*-pivotal curves.* We take the notation $\mathcal{P}(Q^*,R)$ from 1.11. For the weak center $Q^* = I^*$, this set is an algebraic curve, given by the equation

$$\Sigma_{cyclic}\, (d_a^2 d_b^2 (xr_b - yr_a)^4 - 2d_a^2 d_b d_c (xr_b - yr_a)^2 (xr_c - zr_a)^2) = 0,$$

with $d_a := c^2y^2 + 2yzS_A + b^2z^2$, *etc*. For a picture, see Figure 10.

2.2. *G^*-correspondence.* In 1.4 we already saw that $P\sharp G^* = P$ for every point P in the triangle plane.

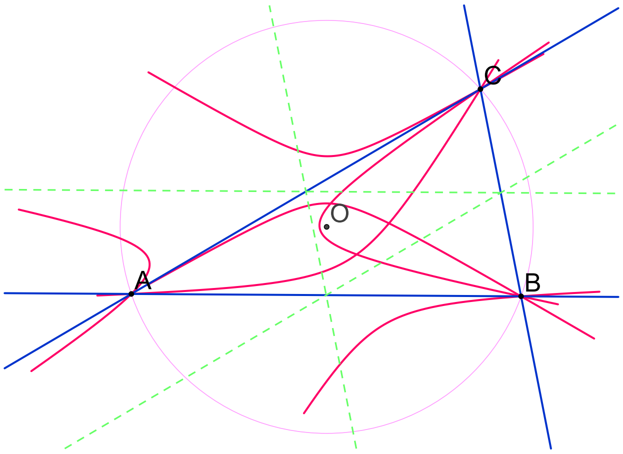

Figure 9. This shows the (pink) circumcircle $\mathcal{C}_{ABC}((O\sharp Ge_\tau^*)^0) = \mathcal{C}_{ABC}(K)$ and the (red) circumconics $C_{ABC}((O\sharp Ge^*)^\tau)$, $\tau = a, b, c$. The three (green) lines $\mathcal{L}_{ABC}((O\sharp I^*)^\tau)$, $\tau = a, b, c$, are the sidelines of the medial triangle.

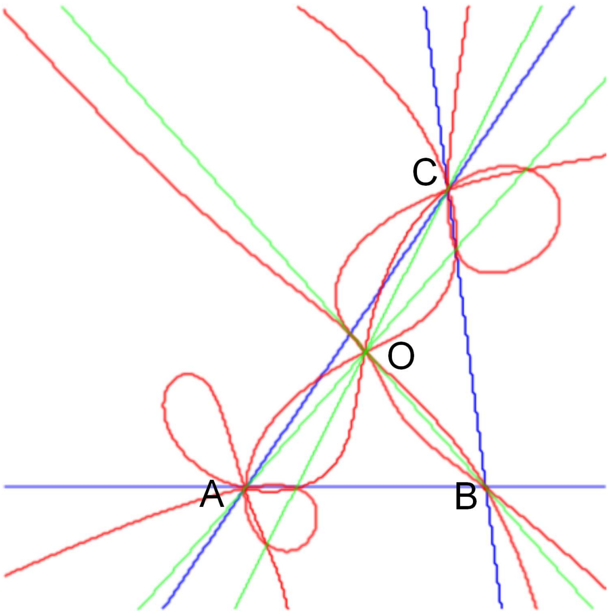

Figure 10. Besides the (red) algebraic curve $\mathcal{P}(I^*, O)$, the picture shows the lines AO, BO, CO (green). Without any proof, we state that all (ten) singular points of $\mathcal{P}(I^*, O)$ lie on these lines. Six singular points are points on ∂ABC. And for each $\tau = 0, a, b, c$, one is of type τ.

2.3. *O^*-correspondence.*

2.3.1. *Calculation of dom(O^*).* We have

$$O^{[P]} = ((p_b p_c(b^2 p_c^2 + 2p_b p_c S_A + c^2 p_b^2)(-p_a^2 S_A + p_a p_b S_B + p_a p_c S_C + p_b p_c a^2) \\ : \cdots : \cdots)_{ABC}.$$

First, we look at the sets $\{(p_a : p_b : p_c)_{ABC} \mid b^2 p_c^2 + 2p_b p_c S_A + c^2 p_b^2 = 0\}$ and $\{(p_a : p_b : p_c)_{ABC} \mid -p_a^2 S_A + p_a p_b S_B + p_a p_c S_C + p_b p_c a^2 = 0\}$. The first set contains one real point, the vertex A. The second set is the circle with diameter BC. From this it follows that the first coordinate of $P\sharp O^*$ is zero if and only if P is a point of the line BC or a point on one of the circles with diameter AB resp. AC. This implies: dom$(O^*) = \mathcal{E} - \{A, B, C, H_A, H_B, H_C\}$, where H_A, H_B, H_C are the vertices of the orthic triangle of ABC.

2.3.2. *Special images.* As special cases for $O^{[P]}$ and $P\sharp O^*$ we get

- for $P = I : O^{[I]} = I$ and $I\sharp O^* = G$,
- for $P = G : O^{[G]} = ((a^2 - 2b^2 - 2c^2)(5\Delta a^2 - b^2 - c^2) : \cdots : \cdots)_{ABC}$ $= X_{1384}/X_{1383}$ and $G\sharp O^* = X_{1383}/X_{1384}$,
- for $P = O : O^{[O]} = X_{1147}$ and $O\sharp O^* = O/X_{1147}$,
- for $P = H : O^{[H]} = O$ and $H\sharp O^* = X_{2052}$.

2.3.3. *The image of the sidelines.* If $P = (0 : t : 1-t)$ is a point on BC, different from B, C and H_A, then $P\sharp O^* = (t(t-1)(a^2(2t-1) - b^2 + c^2) : -2t(a^2 t(t-1) + b^2(1-t) + c^2 t) : 2(1-t)(a^2 t(t-1) + b^2(1-t) + c^2 t))_{ABC}$. The infinite point on BC is mapped to the point G. The image set $BC\sharp O^*$ can be extended to a connected analytic curve. This curve we denote by $\mathcal{A}(BC, O^*)$. See Figure 11 for a picture.

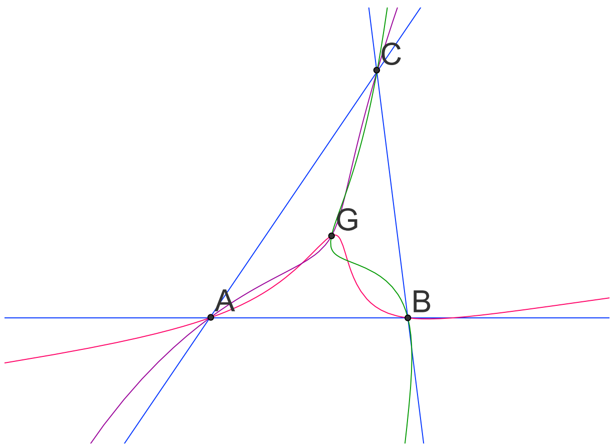

Figure 11. This picture shows the curves $\mathcal{A}(BC, O^*)$ (green), $\mathcal{A}(CA, O^*)$ (purple) and $\mathcal{A}(AB, O^*)$ (red).

2.3.4. *Connection between O^*- and H^*-correspondence.* The point O^* of the triangle $A^*B^*C^*$ is identical with the orthocenter of the pedal triangle of O^* which is the cevian triangle of G^*. Therefore, the O^*-transversal of P agrees with orthotransversal of P for the anticevian triangle of $P = P\sharp G^*$ (with respect to ABC).

2.3.5. *The $\mathcal{S}$-dual of the circumcircle of the triangle $A^*B^*C^*$.* The $\mathcal{S}$-dual of the circumcircle $\mathcal{C}_{A^*B^*C^*}(K^*)$ is the conic $\mathcal{J}_{ABC}(P\sharp K^*)$. The foci of this conic are P and its isogonal conjugate K/P. The line $\mathcal{L}_{ABC}(P\sharp O^*)$ is the polar line of P with respect to $\mathcal{J}_{ABC}(P\sharp K^*)$, so it is a directrix of the conic.

Two examples:

- For $P = O$, $\mathcal{J}_{ABC}(P\sharp K^*)$ is Brocard inellipse of ABC.
- For $P = I_\tau$, $\tau = 0, a, b, c$, we get $P\sharp O^* = G$. Therefore, $\mathcal{L}_{ABC}(P\sharp O^*)$ is the line at infinity, and the conic $\mathcal{J}_{ABC}(P\sharp K^*)$ is a circle. For $\tau = 0$ it is the incircle, for $\tau = a, b, c$ the corresponding excircle of ABC. O^*-correspondence maps the points I_τ , $\tau = 0, a, b, c$, to G. Let us determine the preimage of G under $\sharp O^*$. Obviously, the incenter and the excenters are the only finite points that are mapped to G by $\sharp O^*$. But the equation $P\sharp O^* = G$ is also correct for every point on $\mathcal{L}_\infty$, as can be easily checked.

2.3.6. *The preimage of a point under O^*-correspondence.* There are several possibilities to determine the preimage of a point R under O^*-correspondence. We describe two. Afterwards, we determine the preimage of $\mathcal{L}_\infty$.

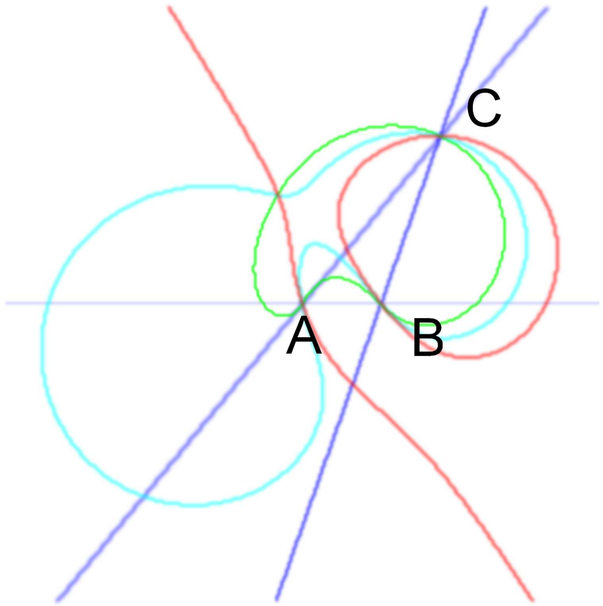

Figure 12. This "insect" consists of the triangle ABC, the (red) Neuberg cubic, the (green) quartic $\mathcal{V}(O^*, X_{647})$ and the (cyan) quartic $\mathcal{V}(O^*, X_{650})$. For the triangle shown here, one real point is (and four more complex points are) mapped to H by O^*-correspondence.

(A) First, we determine the preimages of G and H and the associates of the Gibert point X_{1141} using the way that was described in 1.11. We start with the quartic $\mathcal{V}(O^*, X_{523})$, given by the equation

$$\Sigma_{cyclic}(c^2y^2 + 2yzS_A + b^2z^2)(x(-xS_A + yS_B + zS_C) + yza^2)(b^2 - c^2) = 0.$$

$X_{523} = (b^2 - c^2 : \cdots : \cdots)_{ABC}$ is a point on $\mathcal{L}_\infty$ (the orthopoint of the Euler line). The quartic splits into the line at infinity and a cubic, which is called the Neuberg cubic and we denote by $\mathcal{K}_N$. Since $\mathcal{L}_\infty \sharp O^* = G$, $\sharp O^*$ maps the Neuberg cubic onto the Kiepert hyperbola.

- There are five points on $\mathcal{K}_N$ which are mapped to G by O^*-correspondence, the in- and excenters and the Euler infinity point X_{30}.
- The orthocenter H is the fourth (the non trivial) common point of the Kiepert hyperbola and the Jarabek hyperbola $\mathcal{C}_{ABC}(X_{647})$. Hence, the preimage of H under O^*-correspondence is the intersection of $\mathcal{K}_N$ with the quartic $\mathcal{V}(O^*, X_{647})$. See Figure 12.
- The orthocenter H is the fourth common point of the Kiepert hyperbola and the Feuerbach hyperbola $\mathcal{C}_{ABC}(X_{650})$. Therefore, we can get the preimage of H under O^*- correspondence as the intersection of the Neuberg cubic with the quartic $\mathcal{V}(O^*, X_{650})$. See Figure 12.

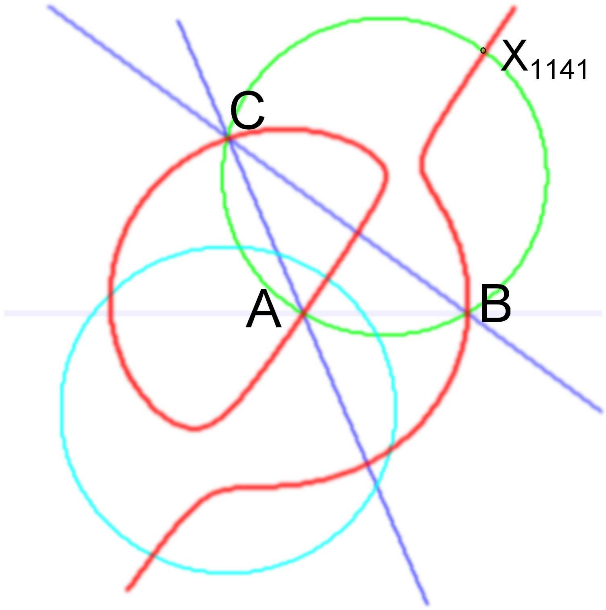

Figure 13. For an obtuse triangle ABC, the quartic $\mathcal{V}(O^*, H)$ splits into two circles, the circum circle (green) and the polar circle (cyan) of the triangle. The red curve is the Neuberg cubic. For the triangle presented here, there are four O^*-associates of X_{1141}, all lying on the polar circle.

- Apart from A, B, C, the Gibert point X_{1141} is the only common point of the circumcircle and the Neuberg cubic $\mathcal{K}_N$, see [3]. The O^*-correspondence maps the circumcircle to the circumconic $\mathcal{C}_{ABC}(H)$ (see 1.10) and the Neuberg cubic to $\mathcal{C}_{ABC}(X_{523})$. Therefore, $X_{1141}\sharp O^*$ is the fourth common point of $\mathcal{C}_{ABC}(H)$ and $\mathcal{C}_{ABC}(X_{523})$. The line $\mathcal{L}_{ABC}(X_{1141}\sharp O^*)$ is a line trough H, perpendicular to the Euler line. (The point $X_{1141}\sharp O^*$ is not in the current edition of [7].) The quartic $\mathcal{V}(O^*, H)$ is the union of the circumcircle and the algebraic set $\{(p_a : \cdots : \cdots)_{ABC} | S_A p_a^2 + S_B p_b^2 + S_C p_c^2 = 0\}$. This set is the polar circle of ABC (the circle with center H and radius $\rho = \sqrt{-S_A S_B S_C}/(\sqrt{8}S)$ if ABC is obtuse, the set $\{H\}$ if ABC is right-angled, and the empty set (set without any real point) if ABC is acute. See Figure 13.

Another example: The preimage of the vertices A, B, C. The quartic $\mathcal{V}(Q^*, A)$ consists of the circle with diameter BC and the point A. Therefore, the preimage of A consists of all points lying on the circle with diameter BC but not on a sideline of ABC.

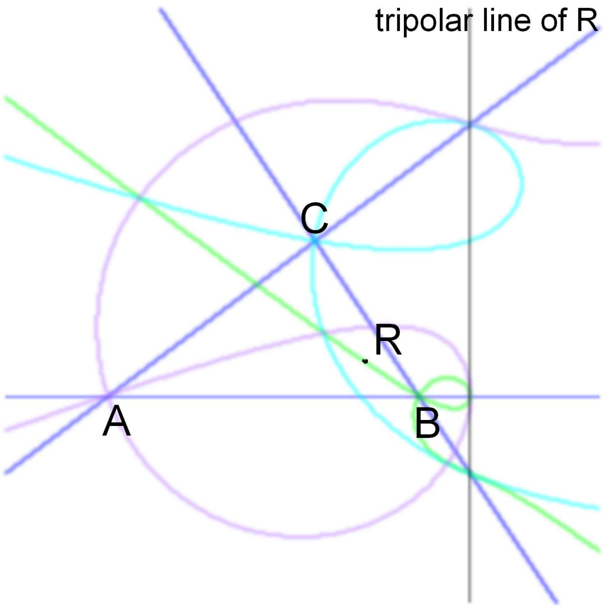

Figure 14. This shows the curves $\mathcal{K}(A; R_B, R_C)$ (purple), $\mathcal{K}(B; R_C, R_A)$ (green) and $\mathcal{K}(C; R_A, R_A)$ (light blue) and the (black) line $\mathcal{L}_S(R)$. For the triangle ABC drawn here, the preimage of R under O^*-correspondence consists of three (real and two nonreal/complex) points. See 2.3.6.(B).

(B) A second way to determine the preimage of a point. The tripolar line $\mathcal{L}_{ABC}(R)$ of a point R intersects the triangle lines BC, CA, AB in $R_A := (0 : -r_b : r_c)_{ABC}$, $R_B := (r_a : 0 : -r_c)_{ABC}$, $R_C := (-r_a : r_b : 0)_{ABC}$, respectivly. Supposing that a point P is neither an edge-point nor a point on the line of infinity, this point P can be in the preimage of R only if the corresponding polar triangle $B^*C^*Q^*$ of $R_B R_C A$ is an isosceles triangle with $\mathrm{d}(Q^*, B^*) = \mathrm{d}(Q^*, C^*)$. Here, $Q^* = (pa/ra : \cdots : \cdots)_{ABC}$ is the pole of $\mathcal{L}_{ABC}(R)$ with respect to $\mathcal{S}$.

The locus of points P satisfying the last equation is (after completion) the cubic $\mathcal{K}(A; R_B, R_C)$. See 1.9 for a definition of the cubics $\mathcal{K}$ and Figure 14 for a picture.

(C) The preimage of $\mathcal{L}_\infty$. The points P whose coordinates satisfy the equation

$$\Sigma_{cyclic}\, p_a/[(a^{*2}(b^{*2}+c^{*2}-a^{*2})] = 0,\ a^* = a^*(p_a, p_b, p_c),\ \cdots,$$

are points on one of the sidelines of ABC or points on an octic which passes twice through each of the vertices A, B, C and also passes through the vertices of the orthic triangle, see Figure 15.

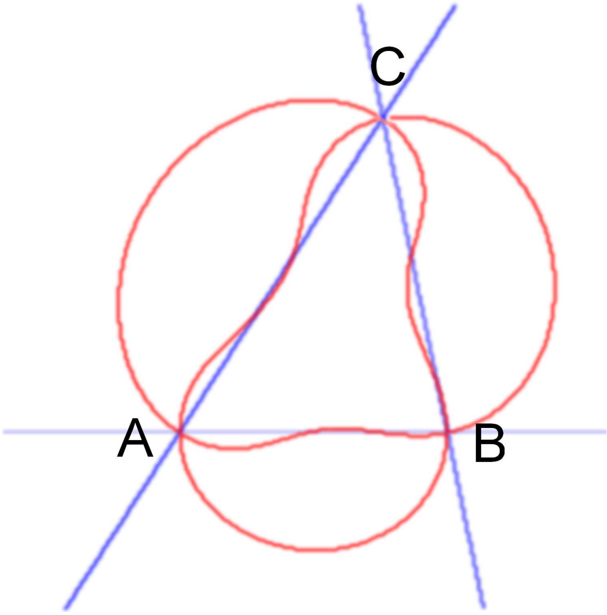

Figure 15. The preimage of $\mathcal{L}_\infty$ under $\sharp O^*$ consists of all points of dom(O^*) lying on the (red) octic, see 2.3.6.(C).

2.4. *H^*-correspondence.* For a nearly complete analysis of orthocorrespondence, see [3] and [4].

2.5. *N^*-correspondence.*

2.5.1. *Calculation of dom(N^*).* $N^{[P]} = (a^{*2}(b^{*2}+c^{*2}) - (b^{*2}-c^{*2})^2) : \cdots : \cdots)_{ABC}$, $a^* = a^*(p_a, p_b, p_c), \cdots$. The algebraic set $\{(p_a : p_b : p_c)_{ABC} \mid a^{*2}(b^{*2}+c^{*2}) - (b^{*2}-c^{*2})^2) = 0\}$ splits into the line BC and the quartic $\mathcal{V}(N^*, A)$ which passes through all the vertices of ABC (A being a dubble point) and the vertices H_B and H_C of the orthic triangle. H_B and H_C are the only intersection points of $\mathcal{V}(N^*, A)$ with AC resp. AB. The two quartics $\mathcal{V}(N^*, B)$ and $\mathcal{V}(N^*, C)$ meet at six points, the vertices A, B, C, the point H_A and two more points, one of type 0 and one of type a, see Figure 15. If the triangle ABC is neither perpendicular nor equilateral, we have dom$(N^*) = \mathcal{E} -$ 12 points.

Special images

- $N^{[I]} = X_{10},\ I\sharp N^* = X_{81}$.

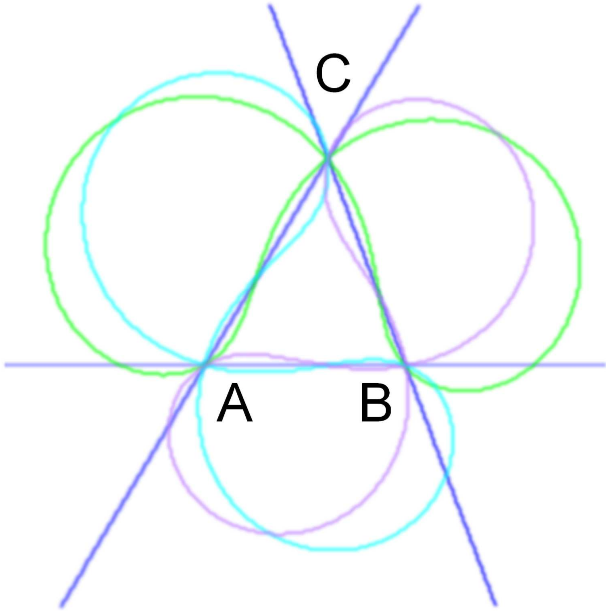

Figure 16. The picture shows the curves $\mathcal{V}(N^*, A)$ (light blue), $\mathcal{V}(N^*, B)$ (purple) and $\mathcal{V}(N^*, C)$ (green).

- $O\sharp N^* = (1/(S_A((a^2(b^2+c^2)-(b+c)^2(b-c)^2)^2 - 2a^4b^2c^2) : \cdots : \cdots)_{ABC}$.
- $G\sharp N^* = (1/(2a^4 - 18b^2c^2 + 7S_A(b^2+c^2)^2) : \cdots : \cdots)_{ABC}$.
- $N^{[H]} = H,\ H\sharp N^* = H/N = X_{275}$.
- $\mathcal{L}_\infty \sharp N^* = G$.
- $\sharp N^*$ maps a point $P = (0 : p_b : p_c)_{ABC}$, $p_b p_c \neq 0$ onto the point $(p_b p_c((p_b - p_c)a^2 - (b^2 - c^2)) : p_b f_a(p_a, p_b, p_c) : p_c f_a(p_a, p_b, p_c))_{ABC}$, with $f_a(p_a, p_b, p_c) = ((p_b^2 + p_c^2)a^4 - 2(p_b b^2 + p_c c^2) - (b^2 - c^2)^2)$.

2.5.2. *The $\mathcal{S}$-dual of the nine-point-circle of the triangle $A^*B^*C^*$.* We start from the well known fact that for any two different points P and Q in the plane of a triangle Δ, both not lying on $\partial\Delta$, there exists a conic which passes through the vertices of the cevian triangles of P and of Q, see [5] (for instance). This conic is uniquely determined by P and Q and we denote it by $\mathcal{C}_\Delta(P, Q)$.

Of course, the dual of this statement is also true: Given two different points P and Q, both not lying on $\partial\Delta$, there exists exactly one conic which is an inconic of the anticevian triangles of P and of Q. This conic we denote by $\mathcal{J}_\Delta(P, Q)$. We now specialize in the nine-point-circle $\mathcal{C}_{A^*B^*C^*}(G^*, H^*)$ and its $\mathcal{S}$-dual $\mathcal{J}_{ABC}(P, P\sharp H^*)$. Figure 17 shows a picture of this conic.

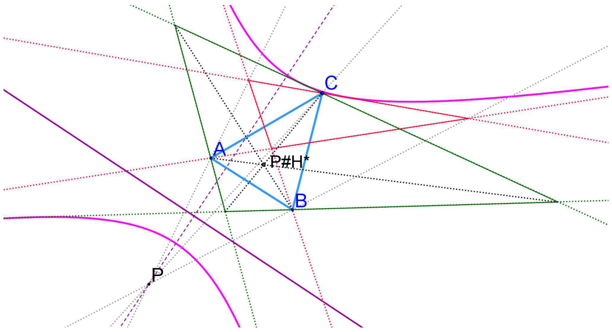

Figure 17. For the triangle ABC and the point P, the picture shows the (purple) conic $\mathcal{J}_{ABC}(P, P\sharp H^*)$, which is an inconic of the (red) anticevian triangle of $P\sharp G^* = P$ and of the (green) anticevian triangle of $P\sharp H^*$. The point P is a focus of this conic, and the purple line is the corresponding directrix which is also the tripolar line of the point $P\sharp N^*$.

3. Description of the algebraic set $P\sharp q*$ for $q^* = G^*O^*$.

We refer to results given in 1.5.5 and look at two special cases for P, $P = I$ and $P = H$.

3.1. $P = I$. We take $P = I$. Let $G^*O^* = \mathcal{L}_{A^*B^*C^*}(X^*_{648})$ be the Euler line of the triangle $A^*B^*C^*$. The lines $G^*O^* = \mathcal{L}_{A^*B^*C^*}(X^*_{648})$ and $IO = \mathcal{L}_{ABC}(X_{651})$ are identical lines because we have $O^* = I$ and the orthopoint X^*_{523} of G^*O^* agrees with the orthopoint X_{513} of IO. The $\mathcal{S}$-dual of the line G^*O^* is the point X_{513}, so the lines $\mathcal{L}_\mathcal{S}(Q^*)$ with Q^* a point on G^*O^* form a pencil through X_{513}. The $\mathcal{S}$-dual of O^* is the line at infinity, and for a point on G^*O^*, different from O^*, the $\mathcal{S}$-dual $\mathcal{L}_\mathcal{S}(Q^*)$ is perpendicular to IO. As a special case we have the line $\mathcal{L}_\mathcal{S}(X^*_{30})$ which passes through $I = O^*$. Because of the equation $\mathrm{d}(O^*, N^*) = \mathrm{d}(N^*, H^*)$, the quadruplet $(O^*, H^*; N^*, X^*_{30})$ is an harmonic range of points. Therefore, $(\mathcal{L}_\mathcal{S}(X^*_{30}), \mathcal{L}_\mathcal{S}(N^*); \mathcal{L}_\mathcal{S}(H^*), \mathcal{L}_\mathcal{S}(O^*))$ is an harmonic range of lines, and we get $\mathrm{d}(I, \mathcal{L}_\mathcal{S}(H^*)) = d(\mathcal{L}_\mathcal{S}(H^*), \mathcal{L}_\mathcal{S}(N^*))$. We also have an harmonic range $(O^*, N^*; G^*, H^*)$ which implies that the quadruplet $(\mathcal{L}_\mathcal{S}(H^*), \mathcal{L}_\mathcal{S}(G^*); \mathcal{L}_\mathcal{S}(N^*), \mathcal{L}_\mathcal{S}(O^*))$ is harmonic and we have equal distances between the lines $\mathcal{L}_\mathcal{S}(H^*), \mathcal{L}_\mathcal{S}(N^*)$ and the lines $\mathcal{L}_\mathcal{S}(N^*), \mathcal{L}_\mathcal{S}(G^*)$. After all, we involve the DeLongchamps point L. Because of the harmonic range $(H^*, L^*; O^*, X^*_{30})$, we have equal distances between the lines $\mathcal{L}_\mathcal{S}(H^*)$, $\mathcal{L}_\mathcal{S}(X^*_{30})$ and the lines $\mathcal{L}_\mathcal{S}(X^*_{30})$, $\mathcal{L}_\mathcal{S}(L^*)$. The constellation of these lines is shown in Figure 18.

IO intersects

- $\mathcal{L}_\mathcal{S}(H^*)$ in X_{1319} (Bevan-Schröder-Point, midpoint of I and X_{36} , see [6], [7], [8],
- $\mathcal{L}_\mathcal{S}(N^*)$ in X_{36} (inverse in circumcircle of the incenter; midpoint of I and X_{484}, see [7]) ,
- $\mathcal{L}_\mathcal{S}(G^*)$ in X_{1155} (Schröder-Point; midpoint of X_{36} and X_{484} and intersection of $\mathcal{L}_{ABC}(I)$ with IO, see [6], [7]),

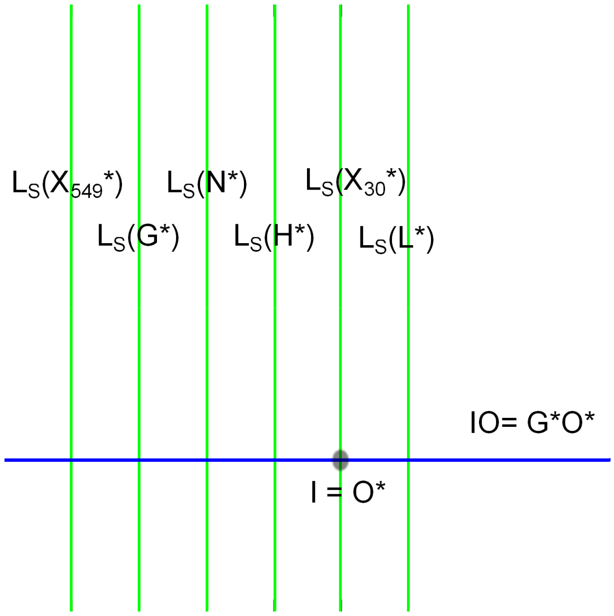

Figure 18. This shows the constellation of the lines $\mathcal{L}_\mathcal{S}(Q^*)$, $Q^* = H^*$, N^*, G^*, L^*, $X_{30}{}^*$, X_{549}^*, in case of $P = I$.

- $\mathcal{L}_\mathcal{S}(L^*)$ in $(3a^4(b+c)+2a^3(b^2-13bc+c^2)+4a^2(-b^3+4b^2c+4bc^2-c^3)+2a(-b^4+5b^3c-12b^2c^2+5bc^3-c^4)+(b+c)(b-c)^4 : \cdots : \cdots)_{ABC}$,
- $\mathcal{L}_\mathcal{S}(X_{549}^*)$ in X_{3245} (X_{549}^* is the midpoint of I* and O*; X_{3245} is the reflection of I in X_{1155}, see [7]).

I propose to call the point I/Q the *I-conjugate* of Q. The set of I-conjugates of points on IO is the circumconic $\mathcal{C}_{ABC}(X_{513})$, for short: The I-conjugate of IO is $\mathcal{C}_{ABC}(X_{513})$. This conic passes through the points $I = I\sharp G^*$ and $G = I\sharp O^*$, so it is a hyperbola. It also passes through the points $I\sharp H^* = X_{57}, I\sharp N^* = X_{81}$ and $I\sharp L^* = X_{145}$. The center of the circumconic is the point $X_{1015} = X_{513}^2$. It should not be too difficult (but quite a bit of work) to calculate the center functions of $I\sharp Q^*$ for all known centers Q^* on the Euler line q^*. A few of the points $I\sharp Q^*$ are listed in [7], many are not, even though some of them have relatively simple center functions.

The circle $\mathcal{S}$ is concentric with the incircle $\mathcal{J}_{ABC}(Ge)$ of ABC, so we can choose $\mathcal{S} = \mathcal{J}_{ABC}(Ge)$. In this case, the triangle $A^*B^*C^*$ is the intouch triangle of ABC. The line IO intersects the incircle in X_{2446} and in X_{2447}, see [7]. In [7] we also can find $X_{30}{}^* = X_{517}$, $H^* = X_{65}$, $N^* = X_{942}$, $G^* = X_{354}$ (Weill-point), $L^* = X_{3057}$.

Note. Choosing $P = {}_\tau I$ for $\tau \in \{a, b, c\}$, the Euler line of the triangle $A^*B^*C^*$ is identical with the line ${}_\tau IO$ of ABC.

3.2. $P = H$. We assume that ABC is an oblique triangle. Taking $P = H$, the triangles ABC and $A^*B^*C^*$ are homothetic with center H, and we have $(a^* : b^* : c^*) = (a : b : c)$. The point H is an inner center if ABC is acute, and it is an outer center if ABC is obtuse. If we put the radius of $\mathcal{S}$ to $\sqrt{|S_A S_B S_C|}/(\sqrt{8}S)$, the triangle $A^*B^*C^*$ agrees with ABC in case of an obtuse triangle ABC ($\mathcal{S}$ is the polar circle of ABC), while for an acute triangle we get $A^*B^*C^*$ by reflecting ABC in H.

We can state the following

Lemma. Real version: For every point Q in the plane of an obtuse triangle ABC, the line $\mathcal{L}_{ABC}(H/Q)$ agrees with the polar line of Q with respect to the polar circle S. For every point Q in the plane of an acute triangle ABC, one gets the line $\mathcal{L}_{ABC}(H/Q)$ by reflecting the polar line of Q (with respect to the circle $\mathcal{S}$) in H. Complex version: For every point Q in the plane of an oblique triangle ABC, the line $\mathcal{L}_{ABC}(H/Q)$ agrees with the polar line of Q with respect to the quadric $S_A x^2 + S_B y^2 + S_C z^2 = 0$.

I propose to call the point H/Q the *H-conjugate of Q*. The H-conjugate of the Euler line is the Kiepert hyperbola.

The constellation of the lines $\mathcal{L}_{\mathcal{S}}(Q^*),\ Q^* = N^*, G^*, O^*, L^*, X_{30}^*$, is shown in Figure 19. The proof of this is quite similar to the proof of the constellation of lines given in the previous subsection.

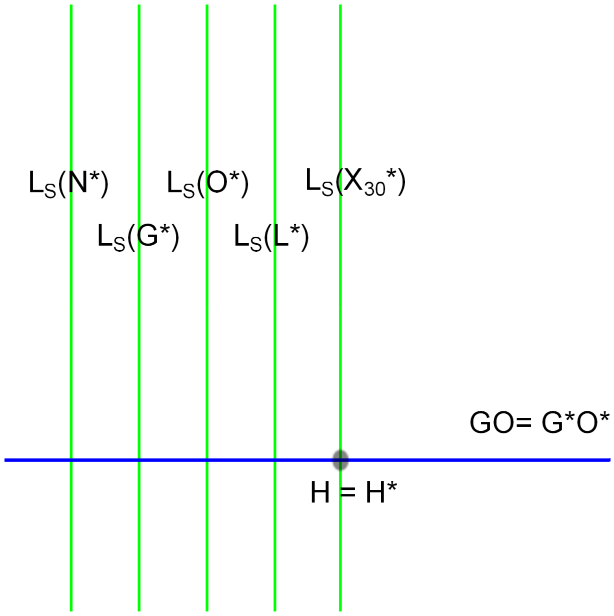

Figure 19.

GO intersects

- $\mathcal{L}_{\mathcal{S}}(G^*) = \mathcal{L}_{ABC}(H)$ in X_{486} (inner Vecten point),
- $\mathcal{L}_{\mathcal{S}}(O^*) = \mathcal{L}_{ABC}(X_{2052})$ in X_{403} (X_{403} is the point X_{36} of the orthic triangle, see [7]),
- $\mathcal{L}_{\mathcal{S}}(N^*) = \mathcal{L}_{ABC}(X_{275})$ in X_{186} (inverse in circumcircle of H, see [7]),
- $\mathcal{L}_{\mathcal{S}}(L^*) = \mathcal{L}_{ABC}(K/L)$ in $((2a^6 - a^4(b^2+c^2) - 4a^2(b^2-c^2)^2 + 3(b^2-c^2)^2(b^2+c^2))/S_A : \cdots : \cdots)_{ABC}$.

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Manfred Evers: Bendenkamp 21, Ratingen, 40880 Germany
E-mail address: `manfred_evers@yahoo.com`