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The Butterfly Theorem Revisited

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Abstract. We start with a proof of the original butterfly theorem, give without proof Mackay's generalization, and finally prove a full generalization of these two versions of the butterfly theorem.

1. Introduction

We give the proof of the original version of the butterfly theorem (Theorem 2 below) with the aid of the following theorem concerning the intersection ratio of two chords in a circle.

Theorem 1. *If the chord BB' in a circle intersects the chord AA' at the point P, then the division ratio*

$$\frac{AP}{PA'} = \frac{AB \cdot AB'}{A'B \cdot A'B'}.$$

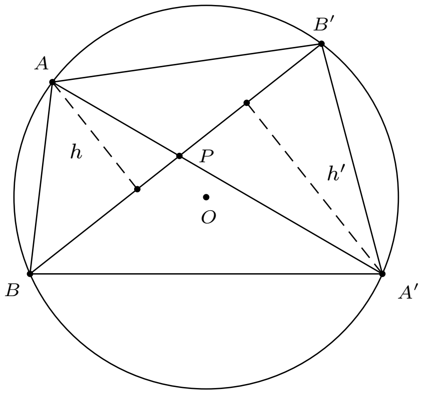

Figure 1

Proof. If R is the radius of the circle (see Figure 1) and h, h' are the heights of triangles ABB', $A'BB'$ from A and A' respectively, then

$$\frac{AP}{PA'} = \frac{h}{h'} = \frac{\frac{AB \cdot AB'}{2R}}{\frac{A'B \cdot A'B'}{2R}} = \frac{AB \cdot AB'}{A'B \cdot A'B'}.$$

□

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Theorem 2 (Butterfly theorem, original version)**.** *If three chords AA', BB', CC' in a circle are concurrent at the midpoint M of AA', then the lines BC and $B'C'$ intersect the line AA' at two points P, P' equidistant from M.*

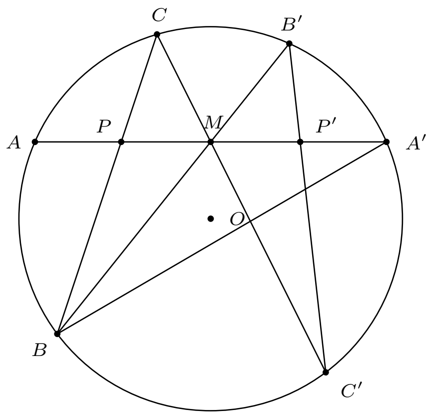

Figure 2

Proof. It is sufficient to prove that $\frac{AP}{PA'} = \frac{A'P'}{P'A'}$ (see Figure 2). From Theorem 1 we have

$$1 = \frac{AM}{MA'} = \frac{AB \cdot AB'}{A'B \cdot A'B'}$$

implying

$$\frac{A'B'}{AB'} = \frac{AB}{A'B}, \tag{1}$$

$$\frac{A'C'}{AC'} = \frac{AC}{A'C}. \tag{2}$$

Hence from Theorem 1 and (1), (2) we have

$$\frac{A'P'}{P'A} = \frac{A'B'}{AB'} \cdot \frac{A'C'}{AC'} = \frac{AB}{A'B} \cdot \frac{AC}{A'C} = \frac{AP}{PA'}.$$

□

Remark. Since $BCB'C'$ with the lines BC, $B'C'$, BC', $B'C$ is a complete quadrangle inscribed in a circle, we may consider AA' as a line that cuts the pair BB', CC' not at M but at two equidistant points from M or from O. So we have the following generalization of the butterfly theorem.

Theorem 3 (Butterfly theorem, Mackay's version)**.** *Given a complete quadrangle inscribed in a circle; if any line cuts two opposite sides at equal distances from the center of the circle, it cuts each pair at equal distances from the center.*

For a proof, see [1, p.105, Theorem 105].

2. A complete generalization of the Butterfly theorem

Since the pairs $(BC,\ B'C')$, $(BB',\ CC')$ and $(BC',\ B'C)$ can be thought of as conics that pass through the four concyclic points B, C, B', C', Theorem 3 and the butterfly theorem can be generalized as in Theorem 5. We first establish a lemma.

Lemma 4. *Two points P and P' are conjugate relative to a circumconic of triangle ABC if and only if the conic passes through the cevian product of P and P'.*

Proof. Let $P = (u : v : w)$ and $P' = (u' : v' : w')$ in barycentric coordinates with respect to triangle ABC. Their cevian product is the point

$$S = \left(\frac{1}{vw' + v'w} : \frac{1}{wu' + w'u} : \frac{1}{uv' + u'v}\right).$$

The two points P and P' are conjugate relative to the circumconic $pyz + qzx + rxy = 0$ with matrix

$$M = \begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix}$$

if and only if $PMP'^{\mathrm{t}} = 0$ (see [2, §10.6.1]). This amounts to

$$p(vw' + v'w) + q(wu' + w'u) + r(uv' + u'v) = 0.$$

Equivalently, the conic passes through S. □

Theorem 5. *Let $ABCD$ be a cyclic quadrilateral, and M be the orthogonal projection of circumcenter O on a line $\mathscr{L}$. If a conic passing through A, B, C, D intersects $\mathscr{L}$ at two points P and Q equidistant from M, then for every conic passing through A, B, C, D and intersecting $\mathscr{L}$, the two intersections are equidistant from M.*

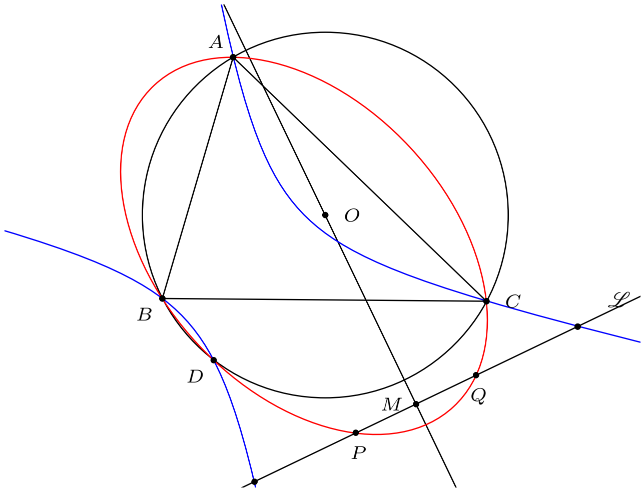

Figure 3

Proof. Let N be the infinite point of the line $\mathscr{L}$, which intersects the conic at P and Q (Figure 3). Since M and N are harmonic conjugate with respect to P and Q, the points M and N are conjugate relative to the conic. The polar of N relative to the circumcircle of ABC is a line perpendicular to NO at O. This is the line OM. So the points M and N are also conjugate relative to the circumcircle of ABC. Hence from Lemma 4 we conclude that D must be the cevian product of M and N relative to ABC. By Lemma 4 again, they must be conjugate relative to every conic that passes through A, B, C, D. If this conic meets $\mathscr{L}$, it must intersect the line at two points equidistant from M. $\square$

References

[1] R. A. Johnson, *Advanced Euclidean Geometry*, Dover reprint, 2007.

[2] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001.

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