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Soddyian Triangles

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Abstract. A Soddyian triangle is a triangle whose outer Soddy circle has degenerated into a straight line. This paper examines some properties of Soddyian triangles, including the facts that no Soddyian triangle can be right angled and all integer Soddyian triangles are Heronian. A generating formula is developed to produce all primitive integer Soddyian triangles. A ruler and compass construction of a Soddyian triangle concludes the paper.

1. The outer Soddy circle

In 1936 the chemist Frederick Soddy re-discovered the Descartes' theorem that relates the radii of two tangential circles to the radii of three touching circles and applied the problem to the three contact circles of a general triangle.

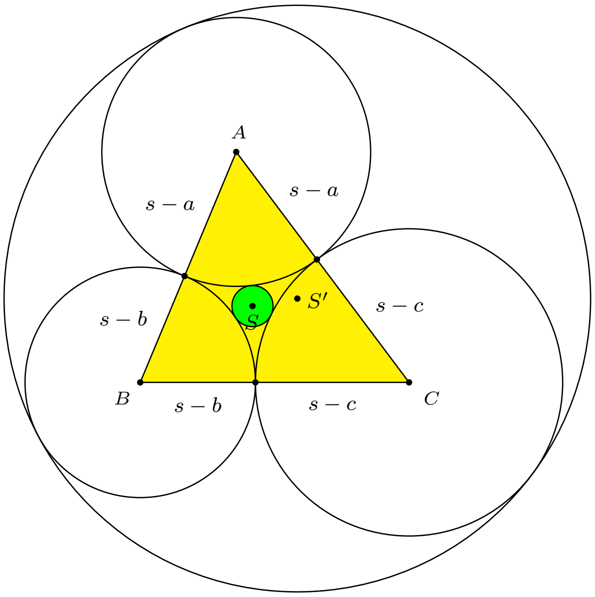

Figure 1

The tangential circles with centers S and S' are called the inner and outer Soddy circles of the reference triangle ABC. If r_i and r_o are the radii of the inner and outer Soddy circles, then

$$r_i = \frac{\Delta}{4R + r + 2s} \quad \text{and} \quad r_o = \frac{\Delta}{4R + r - 2s}. \tag{1}$$

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where Δ is the area of the triangle, R its circumradius, r its inradius, s its semi-perimeter, and $s-a$, $s-b$, $s-c$ the radii of the touching circles (see [1]). By adjusting the side lengths of the reference triangle it is possible to fashion a triangle with contact circles such that the outer Soddy circle degenerates into a straight line. This occurs when $4R+r=2s$ and is demonstrated in Figure 2 below, where it is assumed that $a \leq b \leq c$.

Note that for a Soddyian triangle, the radius of the inner Soddy circle is $\frac{r}{4}$. This follows from

$$r_i = \frac{\Delta}{4R+r+2s} = \frac{\Delta}{4s} = \frac{r}{4}$$

provided $4R+r=2s$.

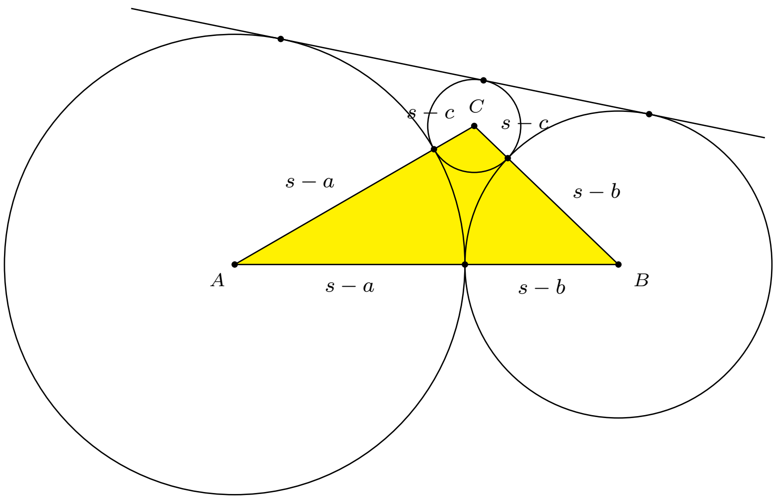

Figure 2

Now the common tangent to the three circles is the outer Soddy circle. Consequently a class of triangles can be defined as Soddyian if their outer Soddy radius is infinite. However from the above diagram it is possible to derive a relationship that is equivalent to the condition that the outer Soddy circle is infinite by considering the length of the common tangents between pairs of touching circles. Given two touching circles of radii u and v, their common tangent has a length of $2\sqrt{uv}$ and applying this to the three touching circle with a common tangent gives

$$\frac{1}{\sqrt{s-c}} = \frac{1}{\sqrt{s-a}} + \frac{1}{\sqrt{s-b}}. \tag{2}$$

2. Can a Soddyian triangle be right angled?

If triangle ABC has a right angle at C, then

$$R = \frac{c}{2} \quad \text{and} \quad r = s-c.$$

If $4R+r=2s$, then $2c+s-c=2s$. This resolves to $c=a+b$, an impossibility. Therefore, no Soddyian triangle is right angled.

3. Can a Soddyian triangle be isosceles?

A Soddy triangle with side lengths $a \le b \le c$ is isosceles only if $a = b$. Since $s = a + \frac{c}{2}$ and $\frac{1}{\sqrt{s-c}} = \frac{2}{\sqrt{s-a}}$, we have $\frac{c}{2} = 4a - 2c$. Hence $a : c = 5 : 8$, and the only primitive integer isosceles Soddyian triangle has sides 5, 5, 8. Note that this has integer area 12.

4. Are all integer Soddyian triangles Heronian?

Now consider the Soddyian constraint $4R + r = 2s$ expressed in terms of the area Δ:

$$\frac{abc}{\Delta} + \frac{\Delta}{s} = 2s.$$

This is quadratic in Δ and

$$\Delta = s^2 \pm \sqrt{s^4 - abcs}$$

Since s is greater than any of the sides, $\Delta < s^2$ and we must have

$$\Delta = s^2 - \sqrt{s^4 - abcs}.$$

By the Heron formula, $16\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$ is an integer. This can only happen if $s^4 - abcs$ is also a square integer. Hence all integer Soddyian triangles are Heronian.

5. Construction of integer Soddyian triangles

It is well known that for a Heronian triangle, the semiperimeter s is an integer. From (2),

$$s - c = \frac{(s-a)(s-b)}{(s-a) + (s-b) + 2\sqrt{(s-a)(s-b)}}.$$

This requires $\sqrt{(s-a)(s-b)}$ to be an integer. We write $s - a = km^2$ and $s - b = kn^2$ for integers k, m, n, and obtain

$$s - c = \frac{km^2n^2}{(m+n)^2}.$$

Therefore,

$$s-a : s-b : s-c : s = m^2(m+n)^2 : n^2(m+n)^2 : m^2n^2 : (m^2+mn+n^2)^2,$$

and we may take

$$\begin{aligned} a =& \ n^2((m+n)^2 + m^2), \\ b =& \ m^2((m+n)^2 + n^2), \\ c =& \ (m+n)^2(m^2+n^2). \end{aligned}$$

For this triangle,

$$\begin{aligned}
\Delta &= m^2n^2(m+n)^2(m^2+mn+n^2),\\
R &= \frac{(m^2+n^2)((m+n)^2+m^2)((m+n)^2+n^2)}{4(m^2+mn+n^2)},\\
r &= \frac{m^2n^2(m+n)^2}{m^2+mn+n^2},\\
s &= (m^2+mn+n^2)^2.
\end{aligned}$$

Here are some examples of integer Soddyian triangles.

m	n	a	b	c	s	Δ	r	R
1	1	5	5	8	9	12	$\frac{4}{3}$	$\frac{25}{6}$
2	1	13	40	45	49	252	$\frac{36}{7}$	$\frac{325}{14}$
3	1	25	153	160	169	1872	$\frac{144}{13}$	$\frac{2125}{26}$
4	1	41	416	425	441	8400	$\frac{400}{21}$	$\frac{9061}{42}$
3	2	136	261	325	361	17100	$\frac{900}{19}$	$\frac{6409}{38}$
5	1	61	925	936	961	27900	$\frac{900}{31}$	$\frac{29341}{62}$
6	1	85	1800	1813	1849	75852	$\frac{1764}{43}$	$\frac{78625}{86}$
5	2	296	1325	1421	1521	191100	$\frac{4900}{39}$	$\frac{56869}{78}$
4	3	585	928	1225	1369	261072	$\frac{7056}{37}$	$\frac{47125}{74}$
7	1	113	3185	3200	3249	178752	$\frac{3136}{57}$	$\frac{183625}{114}$
5	3	801	1825	2176	2401	705600	$\frac{14400}{49}$	$\frac{110449}{98}$
8	1	145	5248	5265	5329	378432	$\frac{5184}{73}$	$\frac{386425}{146}$
7	2	520	4165	4293	4489	1063692	$\frac{15876}{67}$	$\frac{292825}{134}$
5	4	1696	2425	3321	3721	1976400	$\frac{32400}{61}$	$\frac{210781}{122}$
9	1	181	8181	8200	8281	737100	$\frac{8100}{91}$	$\frac{749521}{182}$
7	3	1341	5341	5800	6241	3483900	$\frac{44100}{79}$	$\frac{470989}{158}$

6. Soddyian triangles with a given side

Consider Soddyian triangles with a given base AB in a rectangular coordinate system with origin at A and $B = (c, 0)$ on the x-axis. Suppose the vertex C has coordinates (x, y). Using the expressions in §5 in terms of m and n, allowing them to take on positive real values, we have

$$\begin{aligned}
\frac{x}{c} &= \frac{b^2+c^2-a^2}{2c^2} = \frac{m^3(m^2+mn+2n^2)}{(m+n)(m^2+n^2)^2},\\
\frac{y}{c} &= \frac{2\Delta}{c^2} = \frac{2m^2n^2(m^2+mn+n^2)}{(m+n)^2(m^2+n^2)^2}.
\end{aligned}$$

Writing $n = tm$, we obtain a parametrization of the locus of C as follows (see Figure 3).

$$(x, y) = c\left(\frac{1+t+2t^2}{(1+t)(1+t^2)^2}, \frac{2t^2(1+t+t^2)}{(1+t)^2(1+t^2)^2}\right).$$

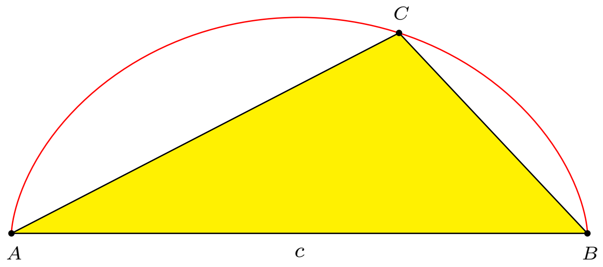

Figure 3.

In fact, given $s-a$ and $s-b$, there is a simple ruler and compass construction for the Soddyian triangle.

Construction 1. *Given a segment AB and a point Z on it (with $AZ = s-a$ and $BZ = s-b$),*
(1) *construct the perpendicular to AB at Z, to intersect the semicircle with diameter AB at P;*
(2) *let A' and B' be points on the same side of AB such that AA', $BB' \perp AB$ and $AA' = AZ$, $BB' = BZ$;*
(3) *join PA' and PB' to intersect AB at X and Y respectively;*
(4) *construct the circle through P, X, Y to intersect the line PZ again at Q;*
(5) *let X' and Y' be point on AZ and ZB such that $X'Z = ZY' = ZQ$;*
(6) *construct the circles centers A and B, passing through Y' and X' respectively, to intersect at C.*

The triangle ABC is Soddyian with incircle touching AB at Z (see Figure 4).

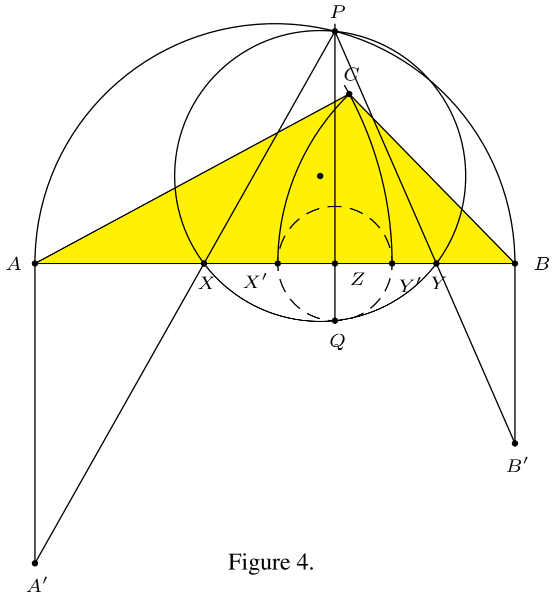

Figure 4.

Proof. Let $AZ = u$ and $BZ = v$. From (1), $ZP = \sqrt{uv}$. From (2),

$$ZX = ZA \cdot \frac{ZP}{ZP + AA'} = u \cdot \frac{\sqrt{uv}}{u + \sqrt{uv}} = \frac{u\sqrt{v}}{\sqrt{u} + \sqrt{v}}.$$

Similarly, $ZY = \frac{v\sqrt{u}}{\sqrt{u}+\sqrt{v}}$. By the intersecting chords theorem,

$$ZQ = \frac{ZX \cdot ZY}{ZP} = \frac{uv}{(\sqrt{u}+\sqrt{v})^2}.$$

It follows that

$$\frac{1}{\sqrt{ZQ}} = \frac{\sqrt{u}+\sqrt{v}}{\sqrt{uv}} = \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} = \frac{1}{\sqrt{AZ}} + \frac{1}{\sqrt{ZB}}.$$

Therefore, triangle ABC satisfies

$$\begin{aligned} BC = &\ BX' = BZ + ZX' = BZ + ZQ, \\ AC = &\ AY' = AZ + ZY' = AZ + ZQ, \\ AB = &\ AZ + ZB, \end{aligned}$$

with

$$\frac{1}{\sqrt{s-c}} = \frac{1}{\sqrt{ZQ}} = \frac{1}{\sqrt{AZ}} + \frac{1}{\sqrt{ZB}} = \frac{1}{\sqrt{s-a}} + \frac{1}{\sqrt{s-b}}.$$

It is Soddyian and with incircle tangent to AB at Z. $\square$

Reference

[1] N. Dergiades, The Soddy circles, *Forum Geom.*, 7 (2007) 191–197.

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