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A Triad of Circles Tangent Internally to the Nine-Point Circle

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Abstract. Given an acute triangle, we construct the three circles each tangent to two sides and to the nine point circle internally. We show that the centers of these three circles are collinear.

In this note we construct, for a given acute triangle, the three circles each tangent to two sides of the triangle and tangent to the nine point circle internally. We show that the centers of these three circles are collinear ([1, 3]).

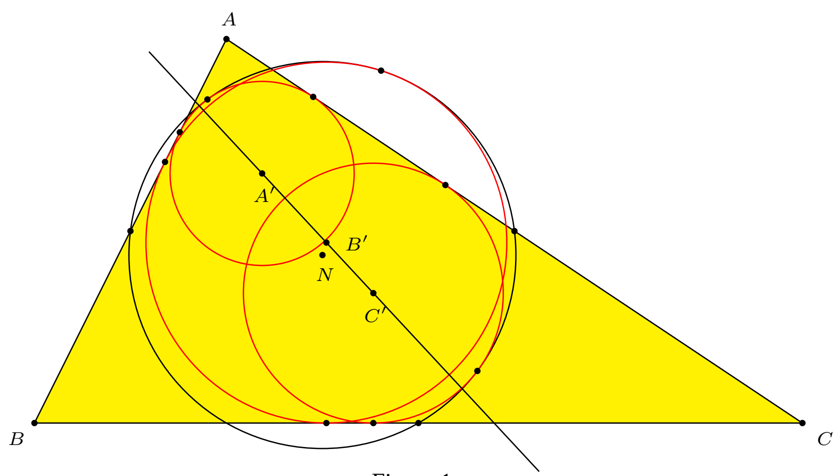

Figure 1.

Let ABC be the given triangle with incenter I. For three points A', B', C' on the respective angle bisectors, write the vectors

$$\mathbf{IA}' = p\mathbf{IA}, \quad \mathbf{IB}' = q\mathbf{IB}, \quad \mathbf{IC}' = r\mathbf{IC}.$$

Since

$$\left(\frac{a}{p}\right)\mathbf{IA}' + \left(\frac{b}{q}\right)\mathbf{IB}' + \left(\frac{c}{r}\right)\mathbf{IC}' = a\mathbf{IA} + b\mathbf{IB} + c\mathbf{IC} = 0,$$

the three points A', B', C' are collinear if and only if $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0$.

Now consider the nine-point circle of triangle ABC. This is tangent to the incircle at the Feuerbach point F_{e}. The power of A is $d^2 = \frac{1}{2}S_A$. If we apply

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inversion with center A and power d^2, the inverse of the incircle is a circle (A') tangent to AB, AC and the nine-point circle at the second intersection F_1 of the line AF_e. We have

$$\frac{AA'}{AI} = \frac{d^2}{(s-a)^2}.$$

Hence, $p = \frac{IA'}{IA} = \frac{2(a-b)(a-c)}{(b+c-a)^2}$.

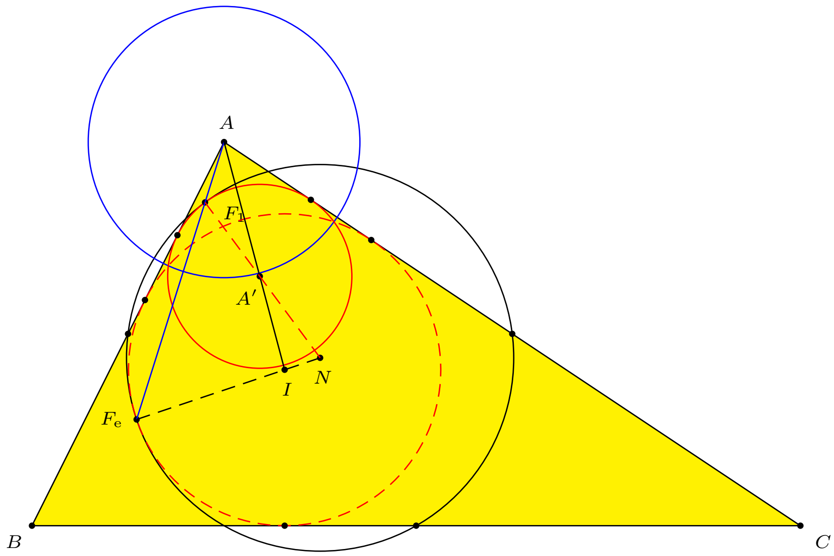

Figure 2.

Similarly for the other centers B', C' we have

$$q = \frac{2(b-c)(b-a)}{(c+a-b)^2}, \qquad r = \frac{2(c-a)(c-b)}{(a+b-c)^2}.$$

It is easy to prove that

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0.$$

Therefore, the three centers A', B', C' are collinear.

These centers are

$$A' = pA + (1-p)I = \left(\frac{p(a+b+c)}{1-p} + a : b : c\right),$$

$$B' = qB + (1-q)I = \left(a : \frac{q(a+b+c)}{1-q} + b : c\right),$$

$$C' = rC + (1-r)I = \left(a : b : \frac{r(a+b+c)}{1-r} + c\right).$$

If the line containing these centers has barycentric equation $ux + vy + wz = 0$ with reference to triangle ABC, then

$$A' = \left(-\frac{bv+cw}{u} : b : c\right), \quad B' = \left(a : -\frac{au+cw}{v} : c\right), \quad C' = \left(a : b : -\frac{au+bv}{w}\right).$$

It follows that

$$\frac{p-1}{p} = \frac{(a+b+c)u}{au+bv+cw}, \qquad \frac{q-1}{q} = \frac{(a+b+c)v}{au+bv+cw}, \quad \frac{r-1}{r} = \frac{(a+b+c)w}{au+bv+cw},$$

and

$$\begin{aligned} u : v : w = & \frac{p-1}{p} : \frac{q-1}{q} : \frac{r-1}{r} \\ = & \frac{b^2+c^2-a^2}{2(c-a)(a-b)} : \frac{c^2+a^2-b^2}{2(a-b)(b-c)} : \frac{a^2+b^2-c^2}{2(b-c)(c-a)} \\ = & (b-c)S_A : (c-a)S_B : (a-b)S_C. \end{aligned}$$

The line containing these points has equation

$$(b-c)S_A x + (c-a)S_B y + (a-b)S_C z = 0.$$

This line contains the orthocenter $\left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$ and the Spieker center. As such, it is the Soddy line of the inferior triangle. It is perpendicular to the Gergonne axis, and is the trilinear polar of X_{1897}. Randy Hutson [2] has remarked that this is also the Brocard axis of the excentral triangle.

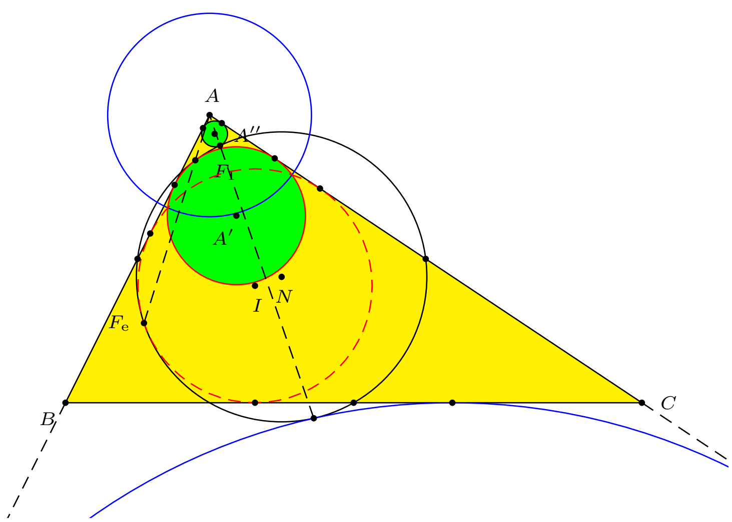

Figure 3.

We conclude with two remarks about the constructions in this note.

(1) If angle A is acute, then the circle (A') is tangent internally to the nine-point circle, and the circle (A'') inverse to the A-excircle is tangent externally to the nine-point circle (see Figure 3).

(2) The constructions apply also to obtuse triangles. If angle A is obtuse, the points A' , A'' are on the extention of IA. The circle (A') is tangent externally to the nine-point circle, and the inverse of the A-excircle is a circle tangent internally to the nine-point circle (see Figure 4).

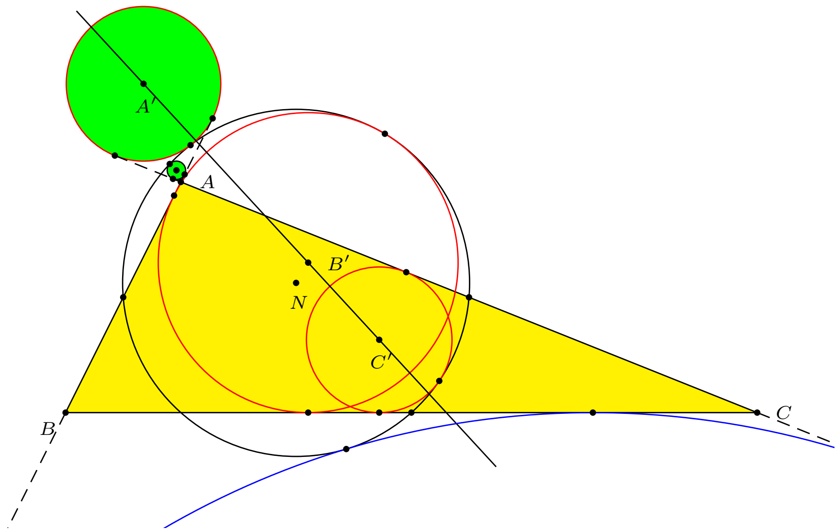

Figure 4.

References

[1] N. Dergiades, Hyacinthos, message 21408, January 12, 2013.
[2] R. Hutson, Hyacinthos, message 21411, January 12, 2013.
[3] A. Myakishev, Hyacinthos, messages 21396, 21404, January 11, 2013.

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