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On the Conic Through the Intercepts of the Three Lines Through the Centroid and the Intercepts of a Given Line

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Abstract. Let $\mathscr{L}$ be a line intersecting the sidelines of triangle ABC at X, Y, Z respectively. The lines joining these intercepts to the centroid give rise to six more intercepts on the sidelines which lie on a conic $\mathscr{Q}(\mathscr{L}, G)$. We show that this conic (i) degenerates in a pair of lines if $\mathscr{L}$ is tangent to the Steiner inellipse, (ii) is a parabola if $\mathscr{L}$ is tangent to the ellipse containing the trisection points of the sides, (iii) is a rectangular hyperbola if $\mathscr{L}$ is tangent to a circle $\mathscr{C}_G$ with center G. We give a ruler and compass construction of the circle $\mathscr{C}_G$. Finally, we also construct the two lines each with the property that the conic $\mathscr{Q}(\mathscr{L}, G)$ is a circle.

1. Introduction

In the plane of a triangle ABC, consider a line $\mathscr{L}$ intersecting the sidelines BC, CA, AB respectively at X, Y, Z. Consider also three lines, $\mathcal{L}_a$ through X intersecting CA, AB at Y_a, Z_a, $\mathcal{L}_b$ through Y intersecting AB, BC at Z_b, X_b, and $\mathcal{L}_c$ through Z intersecting BC, CA at X_c, Y_c. The six points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on a conic $\mathscr{Q}$ whose equation can be determined as follows.

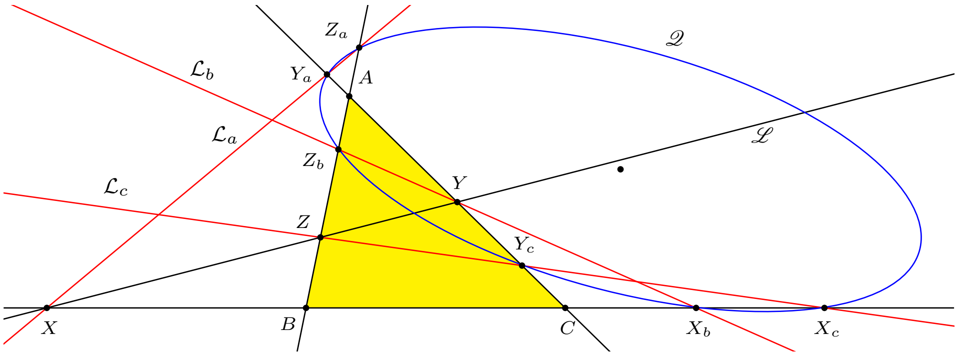

Figure 1

Let $L =: px + qy + rz = 0$ be the equation of $\mathscr{L}$ in homogeneous barycentric coordinates with respect to triangle ABC, so that

$$X = (0 : r : -q), \qquad Y = (-r : 0 : p), \qquad Z = (q : -p : 0).$$

Suppose further that $\mathcal{L}_a$ joins X to a point $P_1 = (f_1 : g_1 : h_1)$, $\mathcal{L}_b$ joins Y to a point $P_2 = (f_2 : g_2 : h_2)$, and $\mathcal{L}_c$ joins Z to a point $P_3 = (f_3 : g_3 : h_3)$,

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so that these lines are represented by linear equations $L_a = 0$, $L_b = 0$, $L_c = 0$ respectively. These linear forms are

$$\begin{aligned} L_a =& \begin{vmatrix} x & y & z \\ 0 & r & -q \\ f_1 & g_1 & h_1 \end{vmatrix} \\ =& \ (qg_1 + rh_1)x - qf_1 y - rf_1 z, \\ L_b =& \ -pg_2 x + (rh_2 + pf_2)y - rg_2 z, \\ L_c =& \ -ph_3 x - qh_3 y + (pf_3 + qg_3)z. \end{aligned}$$

Proposition 1. *The polynomial $L_aL_bL_c - L(P_1)L(P_2)L(P_3)xyz$ is divisible by $px + qy + rz$.*

Proof. If we put $x = -\frac{qy+rz}{p}$, then

$$\begin{aligned} L_a =& \ (qg_1 + rh_1)x - f_1(qy + rz) = (qg_1 + rh_1)x + f_1 \cdot px = L(P_1)x, \\ L_b =& \ L(P_2)y, \\ L_c =& \ L(P_3)z. \end{aligned}$$

It follows that, regarded as a polynomial in x, $L_aL_bL_c - L(P_1)L(P_2)L(P_3)xyz$, evaluated at $x = -\frac{qy+rz}{p}$ is equal to 0. Thus, the cubic polynomial is divisible by $px + qy + rz$. □

Since $L_aL_bL_c - L(P_1)L(P_2)L(P_3)xyz$ is divisible by $px + qy + rz$, the remaining quadratic factor yields a conic $\mathscr{Q}$ containing the six points X_b, X_c, Y_c, Y_a, Z_a, Z_b. For example, if we regard the given line $\mathscr{L}$ as the trilinear polar of $P = (u : v : w)$, and take $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ to be the lines AX, BY, CZ respectively, then $\mathscr{Q}$ is the circumconic $uyz + vzx + wxy = 0$ with perspector P.

2. Preliminaries on conics

We shall make use of the following basic results on conics associated with a triangle. Other preliminary results of triangle geometry can be found in [3]. Consider a conic with barycentric equation

$$\mathscr{C} : \qquad \alpha x^2 + \beta y^2 + \gamma z^2 + 2\lambda yz + 2\mu zx + 2\nu xy = 0.$$

Since this equation can be expressed in the form

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \alpha & \nu & \mu \\ \nu & \beta & \lambda \\ \mu & \lambda & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0,$$

we call $M := \begin{pmatrix} \alpha & \nu & \mu \\ \nu & \beta & \lambda \\ \mu & \lambda & \gamma \end{pmatrix}$ the matrix of the conic $\mathscr{C}$. The adjoint matrix of M, namely,

$$M^{\#} = \begin{pmatrix} \beta\gamma - \lambda^2 & \lambda\mu - \gamma\nu & \nu\lambda - \beta\mu \\ \lambda\mu - \gamma\nu & \gamma\alpha - \mu^2 & \mu\nu - \alpha\lambda \\ \nu\lambda - \beta\mu & \mu\nu - \alpha\lambda & \alpha\beta - \nu^2 \end{pmatrix},$$

defines the dual conic of $\mathscr{C}$. It is easy to verify that

$$MM^{\#} = M^{\#}M = (\det M)I_3.$$

Let $G := \begin{pmatrix}1 & 1 & 1\end{pmatrix}$. [1] The *characteristic* of M is the number

$$\begin{aligned}\chi(M) :=\ & GM^{\#}G^{\mathsf{t}} \\ =\ & 2\mu\nu + 2\nu\lambda + 2\lambda\mu - \lambda^2 - \mu^2 - \nu^2 \\ & - 2(\alpha\lambda + \beta\mu + \gamma\nu) + \beta\gamma + \gamma\alpha + \alpha\beta.\end{aligned}$$

Proposition 2. *The conic defined by the symmetric matrix* M
(a) *degenerates into a pair of lines if* $\det M = 0$*,*
(b) *is a parabola if* $\det M \neq 0$ *and* $\chi(M) = 0$*,*
(c) *has center* $Q = GM^{\#}$ *if* $(\det M)\chi(M) \neq 0$.

Proof. Let $Q = GM^{\#}$. For arbitrary 1×3 matrix P and real number t,

$$\begin{aligned}(Q + tP)M(Q + tP)^{\mathsf{t}} =\ & QMQ^{\mathsf{t}} + (PMQ^{\mathsf{t}} + QMP^{\mathsf{t}})t + (PMP^{\mathsf{t}})t^2 \\ =\ & QMQ^{\mathsf{t}} + 2(QMP^{\mathsf{t}})t + (PMP^{\mathsf{t}})t^2 \\ =\ & (GM^{\#})M(GM^{\#})^{\mathsf{t}} + 2((GM^{\#})MP^{\mathsf{t}})t + (PMP^{\mathsf{t}})t^2 \\ =\ & (\det M)(GM^{\#}G^{\mathsf{t}}) + 2(\det M)(GP^{\mathsf{t}})t + (PMP^{\mathsf{t}})t^2 \\ =\ & (\det M)\chi(M) + 2(\det M)(GP^{\mathsf{t}})t + (PMP^{\mathsf{t}})t^2.\end{aligned}$$

Consider the following possibilities.

(a) If $\det M = 0$, this equation becomes $(Q + tP)M(Q + tP)^{\mathsf{t}} = (PMP^{\mathsf{t}})t^2$. With $t = 0$, this shows that Q is a point on the conic, and for every point P not on the conic, the line PQ intersects $\mathscr{C}$ only at Q (corresponding to $t = 0$). On the other hand, if P lies on the conic, then every point on the line PQ also lies on the conic. It follows that the conic is a union of two lines, possibly identical. The two lines are parallel (possibly identical) if $\chi(M) = 0$. Otherwise, they intersect at a finite point Q.

(b) If $\det M \neq 0$ and $\chi(M) = 0$, then Q is an infinite point on the conic $\mathscr{C}$, and for every finite point P, there is at most one nonzero t for which $tQ + P$ lies on the conic. This shows that $\mathscr{C}$ is a parabola whose axis has infinite point Q.

(c) Suppose $\det M \neq 0$ and $\chi(M) \neq 0$. In this case, $Q = GM^{\#}$ is a finite point. For every infinite point P (satisfying $PG^{\mathsf{t}} = 0$), we have

$$(Q + tP)M(Q + tP)^{\mathsf{t}} = (\det M)\chi(M) + (PMP^{\mathsf{t}})t^2. \tag{1}$$

There are at most two infinite point P satisfying $PMP^{\mathsf{t}} = 0$ (which are the infinite points of the asymptote when $\mathscr{C}$ is a hyperbola). Apart from these infinite points, if the line through Q with infinite point P intersects $\mathscr{C}$ at two real points, these intersections are symmetric with respect to Q. This shows that Q is the center of the conic. □

[1] The symbol G also denotes the centroid of triangle ABC, which has homogeneous barycentric coordinates $(1 : 1 : 1)$.

The equation of the conic $\mathscr{C}$ can be rewritten in the form

$$\theta yz + \varphi zx + \psi xy + (x+y+z)(\alpha x + \beta y + \gamma z) = 0, \tag{2}$$

where

$$\theta = 2\lambda - (\beta+\gamma), \quad \varphi = 2\mu - (\gamma+\alpha), \quad \psi = 2\nu - (\alpha+\beta). \tag{3}$$

Proposition 3. *Suppose* $(\det M)\chi(M) \neq 0$ *and* $\theta\varphi\psi \neq 0$. *The conic* $\mathscr{C}$ *is homothetic to the circumconic*

$$\mathscr{C}_0: \qquad \theta yz + \varphi zx + \psi xy = 0.$$

The ratio of homothety is the square root of $\frac{4\det M}{\theta\varphi\psi}$.

Proof. By Proposition 2(c), the center of the conic $\mathscr{C}$ is the point $Q = GM^{\#}$. The circumconic $\mathscr{C}_0$ (with θ, φ, ψ given by (3)) has matrix

$$M_0 = \frac{1}{2}\begin{pmatrix} 0 & \psi & \varphi \\ \psi & 0 & \theta \\ \varphi & \theta & 0 \end{pmatrix}. \tag{4}$$

Its center is the point

$$Q_0 = (\theta(\varphi+\psi-\theta) : \varphi(\psi+\theta-\varphi) : \psi(\theta+\varphi-\psi)). \tag{5}$$

For the matrix M_0 in (4), we have
(i) $\det(M_0) = \frac{\theta\varphi\psi}{4}$,
(ii) $\chi(M_0) = \frac{1}{4}(2\varphi\psi + 2\psi\theta + 2\theta\varphi - \theta^2 - \varphi^2 - \psi^2) = \chi(M)$ by substitutions using (3).

Note that

$$M = M_0 + \frac{1}{2}\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix} = M_0 + \frac{1}{2}(G^{\mathsf{t}}L + L^{\mathsf{t}}G),$$

where $L = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$.

Let $P = (u : v : w)$ be an infinite point. By (1), the point $Q + tP$ lies on the conic $\mathscr{C}$ if and only if

$$(PMP^{\mathsf{t}})t^2 + (\det M)\chi(M) = 0. \tag{6}$$

Now,

$$\begin{aligned} PMP^{\mathsf{t}} &= P\left(M_0 + \frac{1}{2}\left(G^{\mathsf{t}}L + L^{\mathsf{t}}G\right)\right)P^{\mathsf{t}} \\ &= PM_0P^{\mathsf{t}} + \frac{1}{2}(PG^{\mathsf{t}})(LP^{\mathsf{t}}) + \frac{1}{2}(PL^{\mathsf{t}})(PG^{\mathsf{t}})^{\mathsf{t}} \\ &= PM_0P^{\mathsf{t}}. \end{aligned}$$

Applying equation (6) to the circumconic $\mathscr{C}_0$, by replacing M by M_0, we conclude that the intersection of $\mathscr{C}_0$ with the line through Q_0 with the same infinite

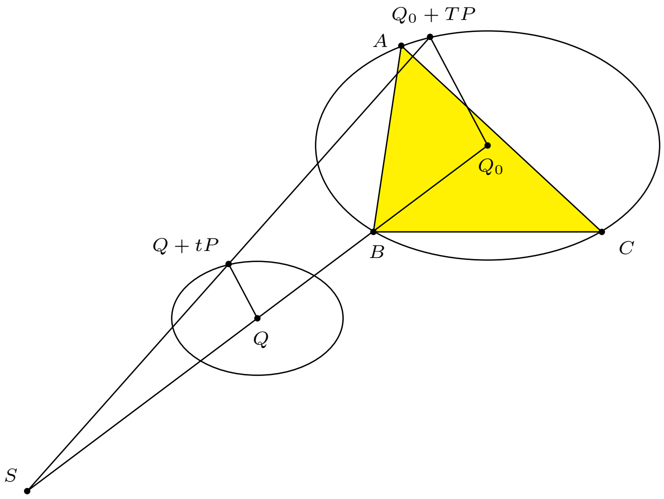

Figure 2.

point P is $Q_0 + TP$ with T given by

$$(PM_0P^{\mathsf{t}})T^2 + (\det M_0)\chi(M_0) = 0,$$

$$(PMP^{\mathsf{t}})T^2 + \frac{\theta\varphi\psi\chi(M)}{4} = 0. \tag{7}$$

Comparing equations (6) and (7), we conclude that parallel lines (with infinite point P) through Q and Q_0 intersect the conics $\mathscr{C}$ and $\mathscr{C}_0$ respectively at points $Q + tP$ and $Q_0 + TP$ with

$$\frac{t^2}{T^2} = \frac{4\det M}{\theta\varphi\psi},$$

which is independent of the infinite point P (for which $PMP^{\mathsf{t}} \neq 0$). Furthermore, the line joining $Q_0 + TP$ to $Q + tP$ intersects the line Q_0Q at a fixed point S such that $\frac{SQ}{SQ_0} = \frac{t}{T}$ (see Figure 2). This shows that the two conics are homothetic at S, with ratio of homothety equal to $\frac{t}{T}$, the square root of $\frac{4\det M}{\theta\varphi\psi}$. $\square$

Remark. The conic $\mathscr{C}$ contains an infinite point if and only if $\chi(M) \leq 0$. [2]

3. The conic $\mathscr{Q}(\mathscr{L}, G)$ associated with the centroid

We shall study the special case when $P_1 = P_2 = P_3 = G$, the centroid of triangle ABC. Here,

$$\begin{aligned} L_a &= (q+r)x - qy - rz, \\ L_b &= -px + (r+p)y - rz, \\ L_c &= -px - qy + (p+q)z, \end{aligned}$$

[2] Proof: Putting $x = -(y+z)$ into the equation of $\mathscr{C}$, we obtain a quadratic in y and z with discriminant $-4\chi(M)$.

and

$$L(P_1) = L(P_2) = L(P_3) = p + q + r.$$

It follows that

$$L_aL_bL_c - (p+q+r)^3xyz = (px+qy+rz)Q(x,y,z)$$

where

$$\begin{aligned}Q(x,y,z) = \ & p(q+r)x^2 + q(r+p)y^2 + r(p+q)z^2 - (p(p+q+r)+2qr)yz \\ & - (q(p+q+r)+2rp)zx - (r(p+q+r)+2pq)xy. \qquad (8)\end{aligned}$$

The conic $\mathscr{Q}(\mathscr{L}, G)$ in question is defined by the equation $Q(x,y,z) = 0$. The matrix of the conic is

$$M(\mathscr{L}) = \frac{1}{2}\begin{pmatrix} 2p(q+r) & -r(p+q+r)-2pq & -q(p+q+r)-2rp \\ -r(p+q+r)-2pq & 2q(r+p) & -p(p+q+r)-2qr \\ -q(p+q+r)-2rp & -p(p+q+r)-2qr & 2r(p+q) \end{pmatrix}. \qquad (9)$$

We shall investigate the possibilities that $\mathscr{Q}(\mathscr{L}, G)$ be (i) degenerate, (ii) a parabola, (iii) a rectangular hyperbola, (iv) a circle.

Lemma 4. *Let $\mathscr{L}$ be the line $px+qy+rz=0$. The matrix $M(\mathscr{L})$ has*
(a) *determinant* $\det M(\mathscr{L}) = -\frac{1}{4}(p+q+r)^4(qr+rp+pq)$,
(b) *adjoint matrix*

$$M(\mathscr{L})^{\#} = \frac{(p+q+r)^2}{4}\begin{pmatrix} -p^2 & 2qr+2rp+pq & 2qr+rp+2pq \\ 2qr+2rp+pq & -q^2 & qr+2rp+2pq \\ 2qr+rp+2pq & qr+2rp+2pq & -r^2 \end{pmatrix}, \qquad (10)$$

and
(c) *characteristic* $\chi(M(\mathscr{L})) = \frac{(p+q+r)^2}{4}(-p^2-q^2-r^2+10qr+10rp+10pq)$.

4. Degenerate $\mathscr{Q}(\mathscr{L}, G)$

From Proposition 2(a), the conic $\mathscr{Q}(\mathscr{L}, G)$ is degenerate when
(i) $p+q+r=0$, or
(ii) $pq+qr+rp=0$.

(i) corresponds to the trivial case when the line $\mathscr{L} : px+qy+rz=0$ contains the centroid $G = (1:1:1)$. The conic $\mathscr{Q}(\mathscr{L}, G)$ is simply the line $\mathscr{L}$ counted twice.

In (ii), $(p:q:r)$ being a point on the Steiner circum-ellipse $xy+yz+zx=0$, the line $px+qy+rz=0$ is tangent to the dual conic, which is the Steiner in-ellipse. This means that *if $\mathscr{L}$ is tangent to the Steiner in-ellipse, then the conic $\mathscr{Q}(\mathscr{L}, G)$ degenerates into two lines*. Actually, these two lines complete with $\mathscr{L}$ a poristic triangle between the two ellipses (see Figure 3).

We shall henceforth assume $p+q+r \neq 0$ and $qr+rp+pq \neq 0$.

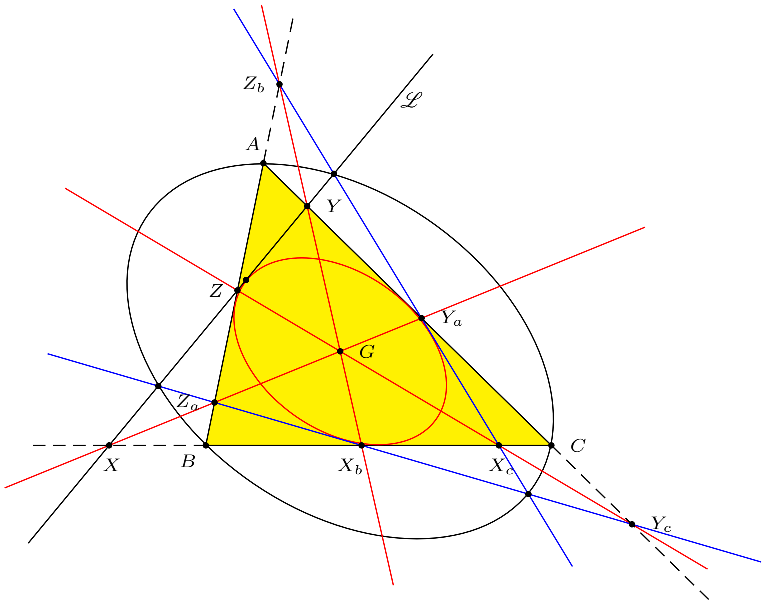

Figure 3.

5. The lines $\mathscr{L}$ for which $\mathscr{Q}(\mathscr{L}, G)$ are parabolas

By Proposition 2(b) and Lemma 4(b), we conclude easily that $\mathscr{Q}(\mathscr{L}, G)$ is a parabola if and only if $(p : q : r)$ is a point on the ellipse

$$\mathscr{E}_0 : \qquad -x^2 - y^2 - z^2 + 10yz + 10zx + 10xy = 0$$

with center G. Equivalently, the line $\mathscr{L}$ is tangent to the conic dual to $\mathscr{E}_0$. This is the ellipse

$$\mathscr{E}_0^* : \qquad -2x^2 - 2y^2 - 2z^2 + 5yz + 5zx + 5xy = 0,$$

also with center G, and containing the trisection points of the sides of the triangle.

Theorem 5 (Dergiades)**.** *The conic $\mathscr{Q}(\mathscr{L}, G)$ is a parabola if and only if the line $\mathscr{L}$ is tangent to the ellipse $\mathscr{E}_0^*$.*

Proposition 6. *If $\mathscr{L} : px + qy + rz = 0$ is the tangent to $\mathscr{E}_0^*$ at P, then the parabola $\mathscr{Q}(\mathscr{L}, G)$ and the ellipse $\mathscr{E}_0^*$ have*
(i) *a common tangent parallel to $\mathscr{L}$ at the antipode of P on $\mathscr{E}_0^*$, and*
(ii) *two remaining common points on the parallel to $\mathscr{L}$ through the centroid G.*

Proof. We take the matrices of the dual ellipses $\mathscr{E}_0$ and $\mathscr{E}_0^*$ to be

$$M = \begin{pmatrix} -1 & 5 & 5 \\ 5 & -1 & 5 \\ 5 & 5 & -1 \end{pmatrix} \quad \text{and} \quad M^* = \begin{pmatrix} -4 & 5 & 5 \\ 5 & -4 & 5 \\ 5 & 5 & -4 \end{pmatrix}$$

respectively. The line $\mathscr{L} : px + qy + rz = 0$ is tangent to the ellipse $\mathscr{E}_0^*$ at the point

$$P = \begin{pmatrix} p & q & r \end{pmatrix} M = \begin{pmatrix} -p + 5q + 5r & 5p - q + 5r & 5p + 5q - r \end{pmatrix}.$$

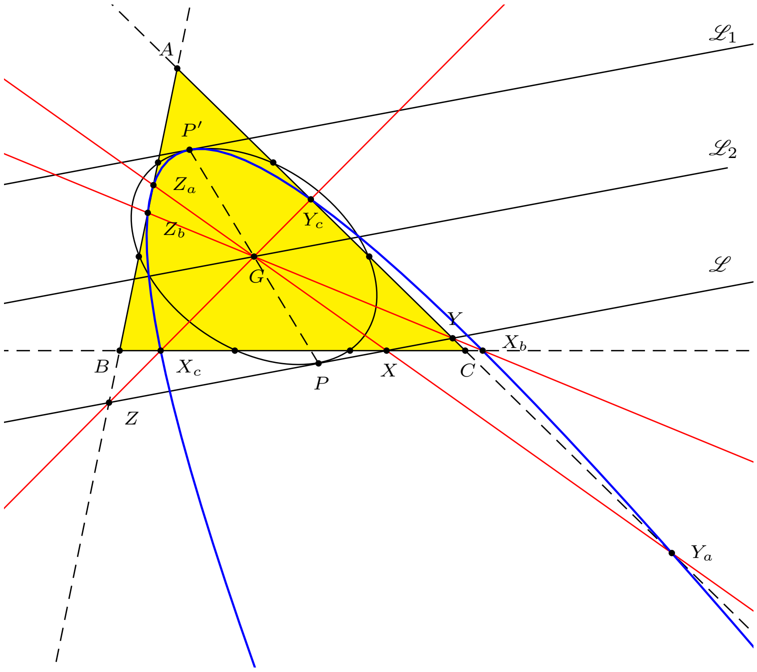

Figure 4

With the matrix $M(\mathscr{L})$ given in (9), and $t = \frac{1}{18}(p+q+r)^2$, it is routine to verify that

$$\begin{pmatrix} x & y & z \end{pmatrix} (M(\mathscr{L}) + tM^*) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{9} L_1(x,y,z) L_2(x,y,z),$$

where

$$\begin{aligned} L_1(x,y,z) = & \ (-p+2q+2r)x + (2p-q+2r)y + (2p+2q-r)z, \\ L_2(x,y,z) = & \ (-2p+q+r)x + (p-2q+r)y + (p+q-2r)z. \end{aligned}$$

Note that the lines $\mathscr{L}_1 : \ L_1(x,y,z) = 0$ and $\mathscr{L}_2 : \ L_2(x,y,z) = 0$ are both parallel to $\mathscr{L}$ (with infinite point $(q-r : r-p : p-q)$). Since the point $(-p+2q+2r : 2p-q+2r : \ 2p+2q-r)$ lies on both conics dual to the ellipse $\mathscr{E}_0^*$ and the parabola $\mathscr{Q}(\mathscr{L}, G)$, the line $\mathscr{L}_1$ is a common tangent of the two conics. The point of tangency is

$$\begin{aligned} & \begin{pmatrix} -p+2q+2r & 2p-q+2r & 2p+2q-r \end{pmatrix} M \\ = & 3 \begin{pmatrix} 7p+q+r & p+7q+r & p+q+7r \end{pmatrix}. \end{aligned}$$

This is the antipode of P on the ellipse $\mathscr{E}_0^*$.

The other line $\mathscr{L}_2$ clearly contains the centroid G, and therefore a pair of antipodal points on the ellipse $\mathscr{E}_0^*$. These two points also lies on the parabola $\mathscr{Q}(\mathscr{L}, G)$. $\square$

6. The lines $\mathscr{L}$ for which $\mathscr{Q}(\mathscr{L}, G)$ are rectangular hyperbolas

Apart from the cases of pairs of lines and parabolas, the type of the conic $\mathscr{Q}(\mathscr{L}, G)$ can be easily determined by an application of Proposition 2(c) and Proposition 3.

Proposition 7. *If $p+q+r \neq 0$ and $qr+rp+pq \neq 0$, the conic $\mathscr{Q}(\mathscr{L}, G)$ has center*

$$Q = (-p^2+3p(q+r)+4qr : \; -q^2+3q(r+p)+4rp : \; -r^2+3r(p+q)+4pq).$$

It is homothetic to the circumconic

$$(p+2q)(p+2r)yz+(q+2r)(q+2p)zx+(r+2p)(r+2q)xy = 0, \tag{11}$$

with ratio of homothety τ given by

$$\tau^2 = \frac{(p+q+r)^4(pq+qr+rp)}{(p+2q)(p+2r)(q+2r)(q+2p)(r+2p)(r+2q)}, \tag{12}$$

provided $(p+2q)(p+2r)(q+2r)(q+2p)(r+2p)(r+2q) \neq 0$.

Since the conic $\mathscr{Q}(\mathscr{L}, G)$ is homothetic to the circumconic defined by (11), it is a rectangular hyperbola if and only if this circumconic contains the orthocenter $H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$, *i.e.*,

$$S_A(p+2q)(p+2r)+S_B(q+2r)(q+2p)+S_C(r+2p)(r+2q) = 0.$$

Equivalently, the line $\mathscr{L} : px+qy+rz = 0$ is tangent to the conic dual to

$$\mathscr{E}_1 : \quad S_A(x+2y)(x+2z)+S_B(y+2z)(y+2x)+S_C(z+2x)(z+2y) = 0.$$

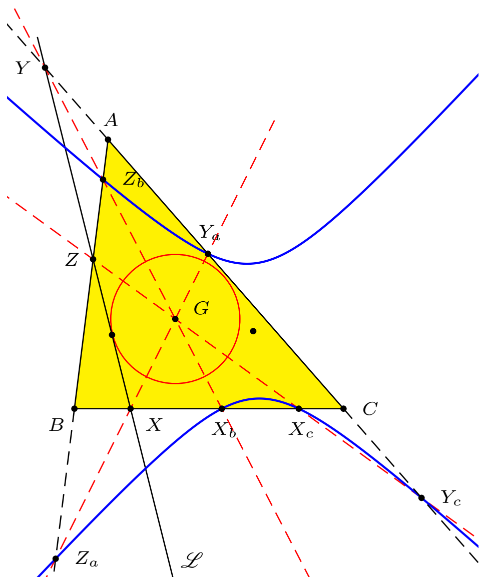

Figure 5

Proposition 8. *The conic $\mathscr{E}_1$ is an ellipse with center G. Its dual conic is a circle also with center G.*

Proof. The matrix of the conic $\mathscr{E}_1$ being

$$M_1 = \begin{pmatrix} S_A & S_A+S_B+2S_C & S_A+2S_B+S_C \\ S_A+S_B+2S_C & S_B & 2S_A+S_B+S_C \\ S_A+2S_B+S_C & 2S_A+S_B+S_C & S_C \end{pmatrix},$$

with

$$\det M_1 = 9(S_A+S_B+S_C)(S_{BC}+S_{CA}+S_{AB}) = \frac{9}{2}(a^2+b^2+c^2)S^2 \neq 0,$$

the conic is nondegenerate. The adjoint matrix is

$$M_1^{\#} = \begin{pmatrix} m_{aa} & m_{ab} & m_{ac} \\ m_{ab} & m_{bb} & m_{bc} \\ m_{ac} & m_{bc} & m_{cc} \end{pmatrix}$$

where

$$\begin{aligned} m_{aa} =& \ -4S_A(S_A+S_B+S_C) - (S_{BB}+S_{BC}+S_{CC}), \\ m_{ab} =& \ -S_{CC}+2S_{BC}+2S_{AC}+2S_{AA}+5S_{AB}+2S_{BB}, \\ m_{ac} =& \ -S_{BB}+2S_{AB}+2S_{BC}+2S_{CC}+5S_{AC}+2S_{AA}, \\ m_{bc} =& \ -S_{AA}+2S_{AB}+2S_{AC}+2S_{BB}+5S_{BC}+2S_{CC}, \end{aligned}$$

and m_{bb}, m_{cc} are analogously defined. It is easy to check that

$$\begin{aligned} & m_{aa}+m_{ab}+m_{ac} = m_{ab}+m_{bb}+m_{bc} = m_{ac}+m_{bc}+m_{cc} \\ =& \ 3(S_{BC}+S_{CA}+S_{AB}) = 3S^2. \end{aligned}$$

From this we conclude that the conic $\mathscr{E}_1$ has center G. Also, $\chi(M_1) = 9S^2 > 0$. It follows that $\mathscr{E}_1$ is an ellipse. [3]

The equation of the dual conic can be written in the form

$$\begin{aligned} & 9(S_A+S_B+S_C)((S_B+S_C)yz+(S_C+S_A)zx+(S_A+S_B)xy) \\ & +(x+y+z)(m_{aa}x+m_{bb}y+m_{cc}z) \\ =& \ 0. \end{aligned}$$

From this it is clear that this dual conic is a circle. The center has coordinates given by $G(M_1^{\#})^{\#}$, which we may simply take as $GM_1 = 3(S_A+S_B+S_C)G$. This is the centroid G. □

We denote this circle by $\mathscr{C}_G$.

Theorem 9. *The conic $\mathscr{Q}(\mathscr{L}, G)$ is a rectangular hyperbola if and only if the line $\mathscr{L}$ is tangent to the circle $\mathscr{C}_G$.*

[3] See Remark at the end of §2.

6.1. *Construction of the centroidal circle $\mathscr{C}_G$.* Since

$$\det M_1 = 9(S_A + S_B + S_C)(S_{BC} + S_{CA} + S_{AB}) = 9S^2(S_A + S_B + S_C),$$

we have

$$\det M_1^{\#} = (\det M_1)^2 = 81S^4(S_A + S_B + S_C)^2. \tag{13}$$

Making use of Proposition 3 and (13), we determine the ratio of homothety of $\mathscr{C}_G$ and the circumcircle as the square root of

$$\frac{4\det M_1^{\#}}{(9(S_A + S_B + S_C))^3(S_B + S_C)(S_C + S_A)(S_A + S_B)} = \frac{8S^4}{9a^2b^2c^2(a^2 + b^2 + c^2)}.$$

This means that the square of the radius of the circle $\mathscr{C}_G$ is

$$\frac{8S^4}{9a^2b^2c^2(a^2 + b^2 + c^2)} \cdot \frac{a^2b^2c^2}{4S^2} = \frac{2S^2}{9(a^2 + b^2 + c^2)} = \frac{S}{3a} \cdot \left(\frac{2}{3} \cdot \frac{a^2}{a^2 + b^2 + c^2} \cdot \frac{S}{a}\right).$$

In the last expression, the first factor is the height of the centroid G above the line BC. The second factor is $\frac{2}{3}$ of the height of the symmedian point K above the same line. This leads to the following construction of the circle.

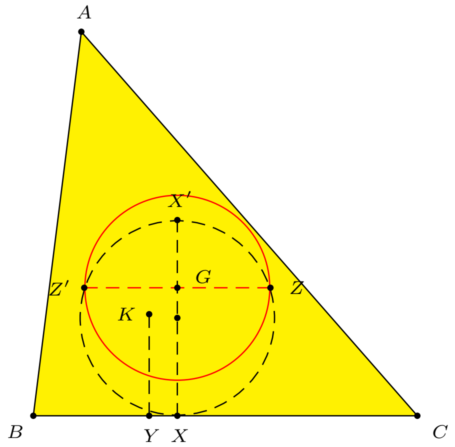

Figure 6

Construction 10. *Given triangle ABC with centroid G and symmedian point K, construct*
(1) *the pedals X and Y of G and K on the line BC,*
(2) *a point X' on the extension of XG such that $GX' = \frac{2}{3}YK$,*
(3) *the circle with diameter XX',*
(4) *the perpendicular through G to the line XX' to intersect the circle in* (3) *at Z and Z'.*

The circle with center G and diameter ZZ' is the circle $\mathscr{C}_G$.

7. The lines $\mathscr{L}$ for which $\mathscr{Q}(\mathscr{L}, G)$ are circles

The conic $\mathscr{Q}(\mathscr{L}, G)$ is a circle if and only if

$$(p+2q)(p+2r) \;:\; (q+2r)(q+2p) \;:\; (r+2p)(r+2q) \;=\; a^2 : b^2 : c^2. \quad (14)$$

We may regard $(p : q : r)$ as a common point of conics defined by

$$\frac{(x+2y)(x+2z)}{a^2} = \frac{(y+2z)(y+2x)}{b^2} = \frac{(z+2x)(z+2y)}{c^2},$$

or equivalently,

$$\frac{(x+y+z)^2-(y-z)^2}{a^2} = \frac{(x+y+z)^2-(z-x)^2}{b^2} = \frac{(x+y+z)^2-(x-y)^2}{c^2}.$$

The equation of the second and third expression can be rewritten as

$$(b^2-c^2)(x+y+z)^2 = b^2(x-y)^2 - c^2(z-x)^2; \quad (15)$$

similarly for the other two equations. Thus, $(p : q : r)$ is a common point of the conics

$$\begin{aligned}
F_a &:= c^2y^2 - b^2z^2 - 2(b^2-c^2)yz - 2(b^2-2c^2)zx - 2(2b^2-c^2)xy = 0,\\
F_b &:= a^2z^2 - c^2x^2 - 2(c^2-a^2)zx - 2(c^2-2a^2)xy - 2(2c^2-a^2)yz = 0,\\
F_c &:= b^2x^2 - a^2y^2 - 2(a^2-b^2)xy - 2(a^2-2b^2)yz - 2(2a^2-b^2)zx = 0.
\end{aligned}$$

We easily determine, by Proposition 2(c), that these conics all have center G. Since

$$\begin{aligned}
F_a + F_b + F_c = 3((b^2-c^2)yz + (c^2-a^2)zx + (a^2-b^2)xy)&\\
+ (x+y+z)((b^2-c^2)x + (c^2-a^2)y + (a^2-b^2)z) &= 0
\end{aligned}$$

is degenerate, this represents the two asymptotes of a hyperbola with center G, homothetic to the Kiepert hyperbola. These asymptotes, as is well known, are parallel to the axes of the Steiner circum-ellipse. We therefore conclude that the common points of the conics $F_a = F_b = F_c = 0$ are on an axis of the Steiner ellipse, which turns out to be the minor axis. Each of these two points leads to a conic $\mathscr{Q}(\mathscr{L}, G)$ which is a circle.

We shall make use of the following notations:

$$\begin{aligned}
\mathsf{P} &:= a^2 + b^2 + c^2;\\
\mathsf{P}_a &:= b^2 + c^2 - 2a^2, \quad \mathsf{P}_b := c^2 + a^2 - 2b^2, \quad \mathsf{P}_c := a^2 + b^2 - 2c^2;\\
\mathsf{Q} &:= a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2.
\end{aligned}$$

Lemma 11. *The infinite points of the Kiepert hyperbola are*

$$((b^2-c^2)(\mathsf{P}_a + \varepsilon\sqrt{\mathsf{Q}}) : (c^2-a^2)(\mathsf{P}_b + \varepsilon\sqrt{\mathsf{Q}}) : (a^2-b^2)(\mathsf{P}_c + \varepsilon\sqrt{\mathsf{Q}}))$$

for $\varepsilon = \pm 1$.

The point for $\varepsilon = -1$ is the infinite point of the minor axis of the Steiner ellipse. It is the point X_{3414} of [2]. The infinite point corresponding to $\varepsilon = +1$ is X_{3413}.

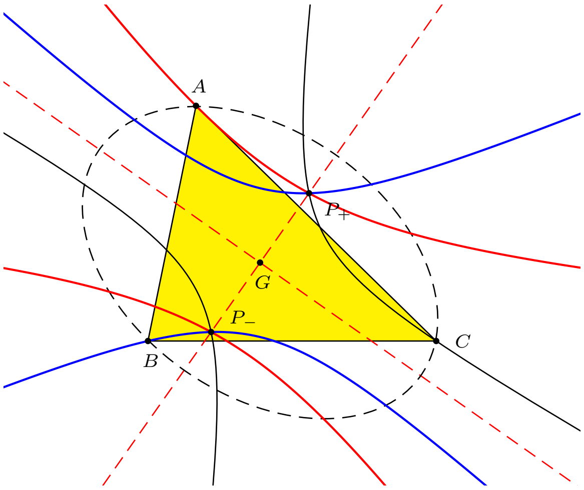

Figure 7

Proposition 12. *The two real common points of the conics $F_a = 0$, $F_b = 0$, $F_c = 0$ are*

$$P_\varepsilon = (\varepsilon t + (b^2 - c^2)(\mathsf{P}_a - \sqrt{\mathsf{Q}}) : \cdots : \cdots) \tag{16}$$

for $\varepsilon = \pm 1$, where t is the square root of

$$\frac{1}{9}(\mathsf{P} + 2\sqrt{\mathsf{Q}})(\mathsf{P}_a - \sqrt{\mathsf{Q}})(\mathsf{P}_b - \sqrt{\mathsf{Q}})(\mathsf{P}_c - \sqrt{\mathsf{Q}}). \tag{17}$$

Proof. A point on the minor axis of the Steiner ellipse is of the form (16) for some t. Substituting into the equation $F_a = 0$, or equivalently (15), and solving, we obtain

$$t^2 = \frac{1}{9}(\lambda + \mu\sqrt{\mathsf{Q}}),$$

where

$$\lambda = 4\mathsf{Q}^2 + \mathsf{P}\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c, \tag{18}$$
$$\mu = 2(\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c + \mathsf{P}\mathsf{Q}). \tag{19}$$

Since $\mathsf{P}_a + \mathsf{P}_b + \mathsf{P}_c = 0$ and $\mathsf{P}_a\mathsf{P}_b + \mathsf{P}_b\mathsf{P}_c + \mathsf{P}_c\mathsf{P}_a = -3\mathsf{Q}$, we have

$$\begin{aligned}
\lambda + \mu\sqrt{\mathsf{Q}} &= 2\mathsf{Q}\sqrt{\mathsf{Q}}(\mathsf{P} + 2\sqrt{\mathsf{Q}}) + \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c(\mathsf{P} + 2\sqrt{\mathsf{Q}}) \\
&= (\mathsf{P} + 2\sqrt{\mathsf{Q}})(2\mathsf{Q}\sqrt{\mathsf{Q}} + \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c) \\
&= (\mathsf{P} + 2\sqrt{\mathsf{Q}})(-\sqrt{\mathsf{Q}}^3 + (\mathsf{P}_a + \mathsf{P}_b + \mathsf{P}_c)\mathsf{Q} \\
&\qquad - (\mathsf{P}_a\mathsf{P}_b + \mathsf{P}_b\mathsf{P}_c + \mathsf{P}_c\mathsf{P}_a)\sqrt{\mathsf{Q}} + \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c) \\
&= (\mathsf{P} + 2\sqrt{\mathsf{Q}})(\mathsf{P}_a - \sqrt{\mathsf{Q}})(\mathsf{P}_b - \sqrt{\mathsf{Q}})(\mathsf{P}_c - \sqrt{\mathsf{Q}})
\end{aligned}$$

leading to the factorization of t^2 given in (17) above. □

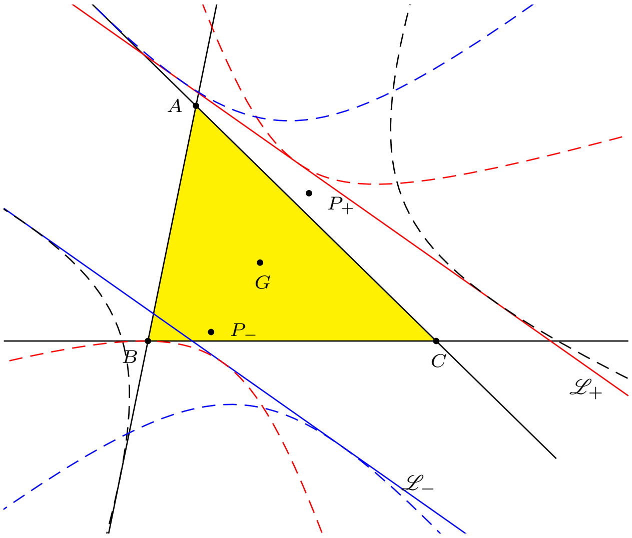

Figure 8.

The points P_ε, $\varepsilon = \pm 1$, correspond to two lines $\mathscr{L}_\varepsilon$ which are common tangents to the conics dual to $F_a = 0$, $F_b = 0$, and $F_c = 0$ (see Figure 8). For these two lines, the conics $\mathscr{Q}(\mathscr{L}_\varepsilon, G)$ are circles, which we simply denote by $\mathscr{C}_\varepsilon$ (see Figure 9).

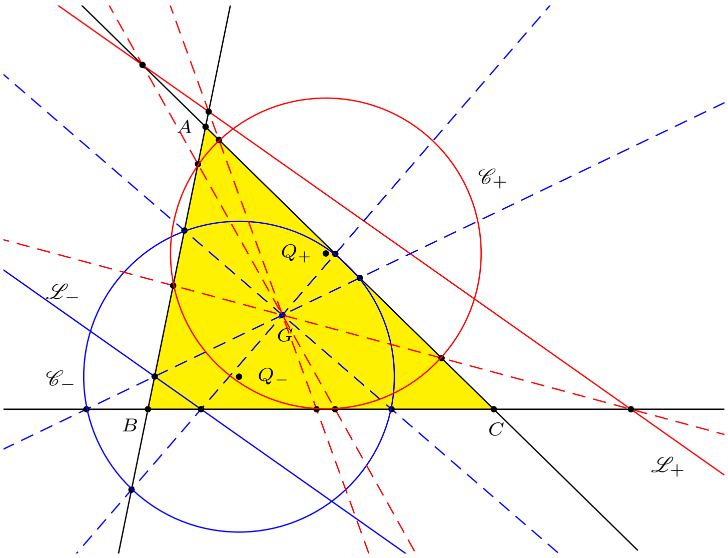

Figure 9.

Proposition 13. *The two circles $\mathscr{C}_\varepsilon$, $\varepsilon = \pm 1$, have centers*

$$Q_\varepsilon = \left(\frac{f + g\sqrt{\mathsf{Q}}}{\varepsilon \cdot 3t} + (b^2 - c^2)(\mathsf{P}_a - \sqrt{\mathsf{Q}}) : \cdots : \cdots \right), \tag{20}$$

where

$$f = 4\mathsf{Q}^2 - \mathsf{P}\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c = 8\mathsf{Q}^2 - \lambda, \tag{21}$$

$$g = 2(\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c - \mathsf{P}\mathsf{Q}) = 4\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c - \mu. \tag{22}$$

They are congruent and have radius

$$\rho = \frac{\mathsf{P} + 2\sqrt{\mathsf{Q}}}{18S} \cdot \sqrt{\mathsf{P} - \sqrt{\mathsf{Q}}}.$$

Proof. With (p, q, r) given by (14), and λ, μ by (18), (19), we have

$$(p+2q)(p+2r) = a^2W, \quad (q+2r)(q+2p) = b^2W, \quad (r+2p)(r+2q) = c^2W$$

for

$$\begin{aligned} W = \ & \frac{(p-q)^2 - (r-p)^2}{b^2 - c^2} = 3(\mathsf{P}_a - \sqrt{\mathsf{Q}})(\mathsf{P}_a^2 - \mathsf{P}_a\sqrt{\mathsf{Q}} - 2\mathsf{Q}) \\ = \ & 3(\mathsf{P}_a - \sqrt{\mathsf{Q}})^2(\mathsf{P}_a + 2\sqrt{\mathsf{Q}}) \\ = \ & 3(\mathsf{P}_b - \sqrt{\mathsf{Q}})^2(\mathsf{P}_b + 2\sqrt{\mathsf{Q}}) \\ = \ & 3(\mathsf{P}_c - \sqrt{\mathsf{Q}})^2(\mathsf{P}_c + 2\sqrt{\mathsf{Q}}). \end{aligned}$$

Note that

$$\begin{aligned} p + q + r = \ & \varepsilon \cdot 3t, \\ pq + qr + rp = \ & 3t^2 - 2\mathsf{Q}^2 - \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c\sqrt{\mathsf{Q}} \\ = \ & \frac{1}{3}\left(\lambda + \mu\sqrt{\mathsf{Q}} - 6\mathsf{Q}^2 - 3\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c\sqrt{\mathsf{Q}}\right) \\ = \ & \frac{1}{3}\left(2\mathsf{P}\mathsf{Q}\sqrt{\mathsf{Q}} - 2\mathsf{Q}^2 + \mathsf{P}\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c - \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c\sqrt{\mathsf{Q}}\right) \\ = \ & \frac{1}{3}\left(2\mathsf{Q}\sqrt{\mathsf{Q}}(\mathsf{P} - \sqrt{\mathsf{Q}}) + \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c(\mathsf{P} - \sqrt{\mathsf{Q}})\right) \\ = \ & \frac{1}{3}(\mathsf{P} - \sqrt{\mathsf{Q}})\left(8\mathsf{Q}\sqrt{\mathsf{Q}} + 4(\mathsf{P}_a + \mathsf{P}_b + \mathsf{P}_c)\mathsf{Q}\right. \\ & \qquad \left. + \ 2(\mathsf{P}_a\mathsf{P}_b + \mathsf{P}_b\mathsf{P}_c + \mathsf{P}_c\mathsf{P}_a)\sqrt{\mathsf{Q}} + \mathsf{P}_a\mathsf{P}_b\mathsf{P}_c\right) \\ = \ & \frac{1}{3}(\mathsf{P} - \sqrt{\mathsf{Q}})(\mathsf{P}_a + 2\sqrt{\mathsf{Q}})(\mathsf{P}_b + 2\sqrt{\mathsf{Q}})(\mathsf{P}_c + 2\sqrt{\mathsf{Q}}). \end{aligned}$$

From these,

$$\begin{aligned}
&-p^2+3p(q+r)+4qr\\
=&\,4(pq+qr+rp)-p(p+q+r)\\
=&\,4(3t^2-2\mathsf{Q}^2-\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c\sqrt{\mathsf{Q}})-(\varepsilon t+(b^2-c^2)(\mathsf{P}_a-\sqrt{\mathsf{Q}}))\varepsilon\cdot 3t\\
=&\,9t^2-8\mathsf{Q}^2-4\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c\sqrt{\mathsf{Q}}-3\varepsilon(b^2-c^2)(\mathsf{P}_a-\sqrt{\mathsf{Q}})t\\
=&\,\lambda-8\mathsf{Q}^2+(\mu-4\mathsf{P}_a\mathsf{P}_b\mathsf{P}_c)\sqrt{\mathsf{Q}}-3\varepsilon(b^2-c^2)(\mathsf{P}_a-\sqrt{\mathsf{Q}})t\\
=&\,-f-g\sqrt{\mathsf{Q}}-3\varepsilon(b^2-c^2)(\mathsf{P}_a-\sqrt{\mathsf{Q}})t\\
=&-\varepsilon\cdot 3t\left(\frac{f+g\sqrt{Q}}{\varepsilon\cdot 3t}+(b^2-c^2)(\mathsf{P}_a-\sqrt{\mathsf{Q}})\right),
\end{aligned}$$

where f and g are given in (21) and (22) above. Similarly,

$$\begin{aligned}
-q^2+3q(r+p)+4rp=&\,-\varepsilon\cdot 3t\left(\frac{f+g\sqrt{Q}}{\varepsilon\cdot 3t}+(c^2-a^2)(\mathsf{P}_b-\sqrt{\mathsf{Q}})\right),\\
-r^2+3r(p+q)+4pq=&\,-\varepsilon\cdot 3t\left(\frac{f+g\sqrt{Q}}{\varepsilon\cdot 3t}+(a^2-b^2)(\mathsf{P}_c-\sqrt{\mathsf{Q}})\right).
\end{aligned}$$

By Proposition 2(c), the center of the circle $\mathscr{C}_\varepsilon$ is the point Q_ε given by (20) above. Furthermore, the homothetic ratio of $\mathscr{C}_\varepsilon$ and the circumcircle is the square root of

$$\frac{(p+q+r)^4(pq+qr+rp)}{(p+2q)(p+2r)(q+2r)(q+2p)(r+2p)(r+2q)}=\frac{(\mathsf{P}+2\sqrt{\mathsf{Q}})^2(\mathsf{P}-\sqrt{\mathsf{Q}})}{3\cdot 27a^2b^2c^2}.$$

From these, it follows that the common radius of the circles $\mathscr{C}_\varepsilon$ is

$$\frac{\mathsf{P}+2\sqrt{\mathsf{Q}}}{9abc}\sqrt{\mathsf{P}-\sqrt{\mathsf{Q}}}\cdot\frac{abc}{2S}=\frac{\mathsf{P}+2\sqrt{\mathsf{Q}}}{18S}\sqrt{\mathsf{P}-\sqrt{\mathsf{Q}}}.$$

□

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