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Pairings of Circles and Sawayama's Theorem

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Abstract. In this article we study pairs of projectively related circles, tangent to a fixed circle, and point out properties, which in a class of particular cases specialize to Sawayama's theorem.

1. Introduction

The well known Sawayama-Thebault theorem ([6]) states that the centers O_1, O_2 of the two circles k_1, k_2 defined by a cevian AW of the triangle ABC are collinear

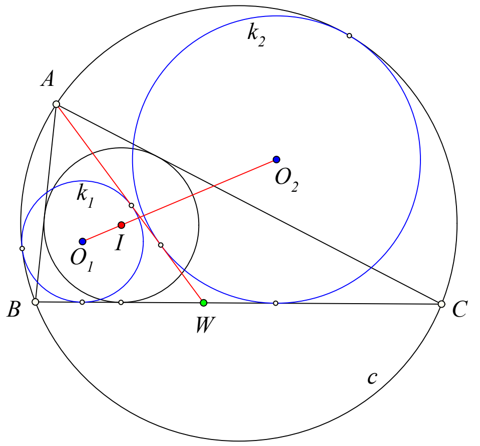

Figure 1. Sawayama-Thebault theorem: O_1, I, O_2 are collinear

with the incenter I of ABC (See Figure 1). The circles k_1, k_2 are, by definition, tangent to AW, BC and the circumcircle c of the triangle. For the history of the problem and a synthetic proof I refer to Ayme's article [1].

In this article we study pairs of projectively related circles (k_1, k_2) called *pairings of circles* and reveal properties, that in some particular cases, called *Sawayama pairings*, lead to Sawayama's theorem.

The general pairings of circles, considered here, result by fixing a triple (c, h, P) of a circle, a line h intersecting c at two non-diametral points B, C and a point inside the circle c but $P \neq O$ and $P \notin h$.

Then we consider all lines through the point P. The intersection points (X, X') of a line through P with the circle, define a pair of circles (k_1, k_2), which are correspondingly tangent to c at X and X' and also tangent to h at points (Z, Z') (See Figure 2). These circles have their centers Y, Y' correspondingly on lines

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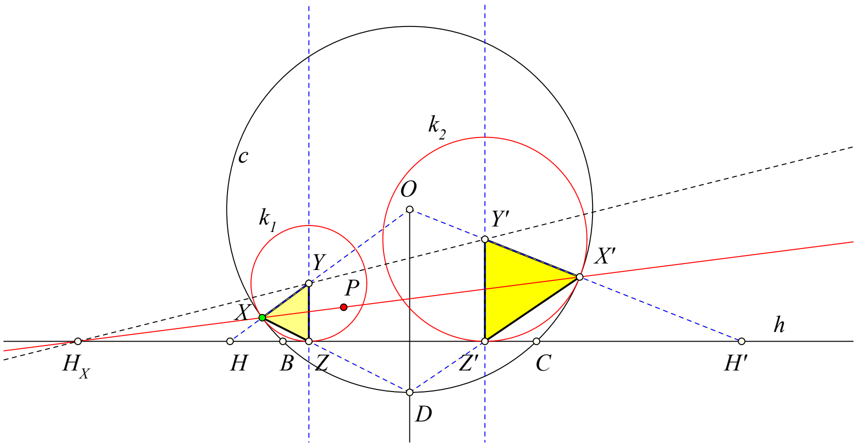

Figure 2. Circle pairing defined by the triple (c, h, P)

OX, OX', and it is easy to see that their respective chords $XZ, X'Z'$ pass always through the middle D of the smaller arc BC, defined on c by h. In fact, using this property, and considering a variable line through P intersecting the circle c at X, X', one can define Z, Z' by correspondingly intersecting DX, DX' with h and defining the two isosceles triangles $XYZ, X'Y'Z'$. The resulting triangles are line perspective, since corresponding sides intersect along line OD. Hence they are also point perspective and define a point H_X on line h, which is collinear with the centers Y, Y', as well as with the contact points X, X' of the circles k_1, k_2. This construct is called in the sequel the *basic configuration*.

In §2 we discuss a fundamental property of the basic configuration coming from the projective geometry aspect. In §3 we deduce that the line of centers YY' always passes through a fixed point I and locate its position. In addition we show a property of the map $Z' = f(Z)$, defined on line h (Theorem 5). In §4 we investigate two pencils of circles, naturally connected to the basic configuration and giving the geometric meaning and consequences of the calculations of the preceding section. In §5 we establish the existence of a certain variable line, associated to pairing and pivoting about a fixed point (Theorem 13). This line, called *pivoting line*, specializes in Sawayama's theorem to the cevian AW from the vertex of the triangle of reference ABC. In §6 we characterize the Sawayama pairings and note a resulting proof of Sawayama's theorem. Finally in §7 we supply a few details concerning the case of Sawayama's theorem.

2. The projective aspect

In the sequel (AB, CD) denotes the intersection of the lines AB and CD, $D = C(A, B)$ denotes the harmonic conjugate of C with respect to (A, B), and $(ABCD) = \frac{CA}{CB} : \frac{DA}{DB}$ denotes the cross ratio of the four points. Next construction is a typical one of the definition of a homology of the projective plane [3, p.9]. Given two

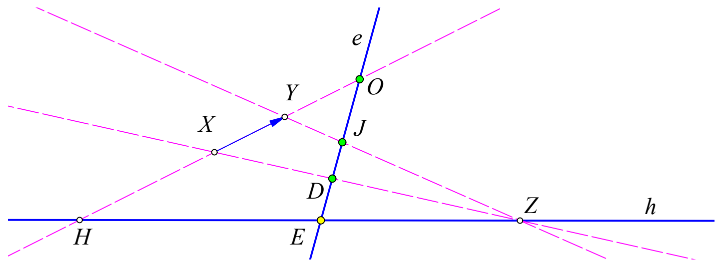

Figure 3. A typical homology $Y = f(X)$

lines e, h, intersecting at point E, and three points D, J, O on e, define the transformation f as follows: For each point X of the plane, let Z be the intersection $Z = (XD, h)$ and set $Y = f(X) = (XO, ZJ)$. Defining $H = (h, XO)$, the equality of cross ratios $(OHXY) = (OEDJ)$ proves next lemma.

Lemma 1. *The map f is a homology with center O, axis h and homology coefficient equal to the cross ratio $k = (OEDJ)$.*

Of particular interest for our subject is the case in which point J goes to infinity,

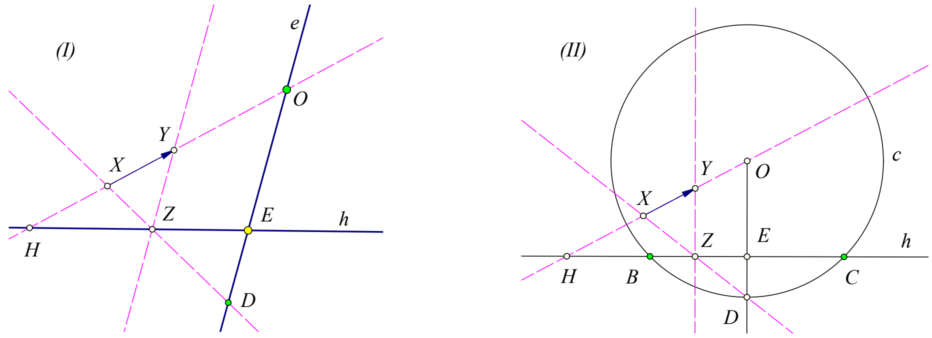

Figure 4. Specializing twice the previous homology

then the configuration becomes like the one of figure $4(I)$ and the cross ratio has the value $k = \frac{DO}{DE}$. A further specialization of the previous figure to 4(II), in which points D, E, O are defined by a circle $c(O, r)$ and a line h, intersecting the circle at two non-diametral points B, C, leads to the projective aspect of our basic configuration. In this case E is the projection of the center O on h and D is the intersection of the half-line OE with c. Using these notations and conventions the following lemma is valid.

Lemma 2. *For each point X of the circle c the triangle XYZ is isosceles and the circle $k_1(Y, |XY|)$ is tangent to the circle c at X and to line h at Z. As X varies on the circle c the center Y of k_1 describes a parabola.*

In fact, the first claim follows from an easy angle chasing argument (See Figure 5). From this, or considering the locus as the image $f(c)$ of the circle via the

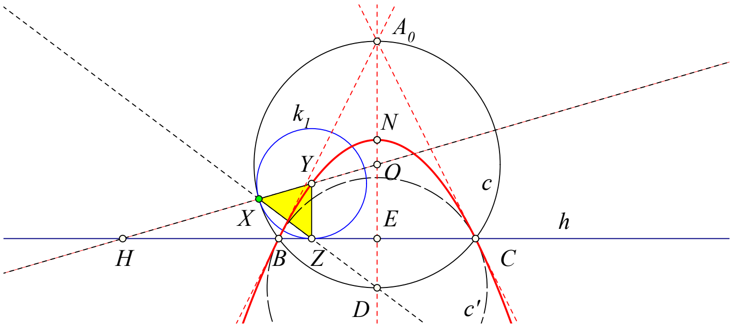

Figure 5. Tangent circles and parabola of centers

homology, follows also easily that the geometric locus of the centers Y of the circles k_1 is a parabola tangent to the triangle A_0BC at B, C and point A_0 being the diametral of D. The parabola passes through the middle N of A_0E and is also tangent to the circle $c'(D, |DB|)$.

3. Pairs of circles

The *circle pairing* of our basic configuration results by considering all lines through the fixed point P. Joining each point X of the circle c with P and considering the second intersection point X' with c one defines an involution ([7, I p.221]) of the points of c denoted by g_P. Point X on the circle and the corresponding

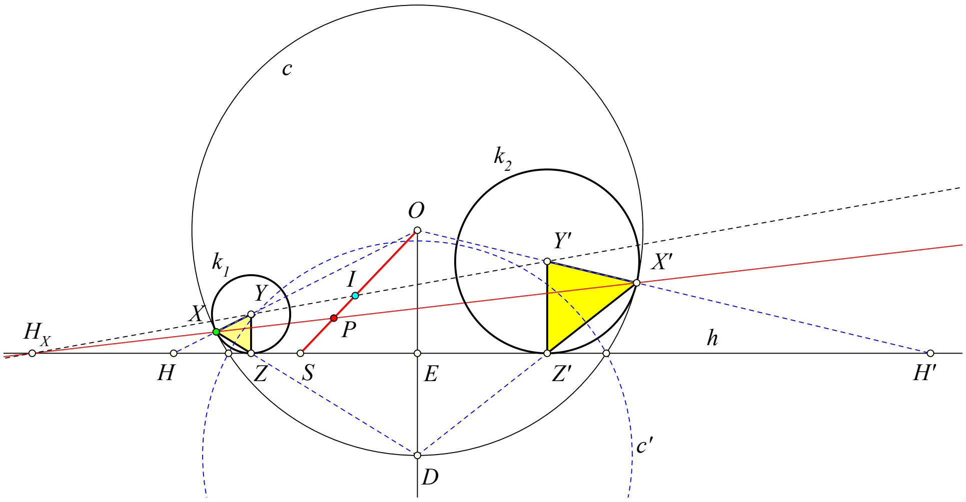

Figure 6. Pairing of circles for the triple (c, h, P)

point $X' = g_P(X)$ define, by the recipe of the previous section, two triangles $XYZ, X'Y'Z'$ and their corresponding circumcircles k_1, k_2 (See Figure 6).

Lemma 3. *Circle c' is orthogonal to circles k_1 and k_2 and points $X, Z, X'Z'$ are concyclic.*

This follows by considering the inversion with respect to circle c'. It is easily seen that this interchanges circle c and line h and fixes, as a whole, circles k_1 and k_2. Hence these two circles are orthogonal to c'. The second claim is a consequence of the first one.

As noticed in the introduction, the two triangles are perspective and their perspectivity center H_X is on line h. With the notations adopted so far the following is true.

Lemma 4. *The lines YY', joining the centers of the circles k_1, k_2, pass through a fixed point I lying on line OP. The points P, I are related by the homology f defined by the triple (c, h, P):*

$$\frac{PO}{PS} = \frac{IO}{IS} \cdot \frac{DO}{DE}.$$

In fact, lines YY' are images $YY' = f(XX')$ of lines XX' under the homology f, hence they pass all through $I = f(P)$. As a result, I lies on OP and taking the intersection $S = (OP, h)$ we have $(OSPI) = \frac{DO}{DE}$. The formula results by expanding the cross ratio $(OSPI) = \frac{PO}{PS} : \frac{IO}{IS}$.

Theorem 5. *With the previous notations the map $Z' = f(Z)$ on line h is an involution, which in line coordinates with origin at the projection Q of I on h obtains the form, for a constant w:*

$$z' = \frac{-w^2}{z}.$$

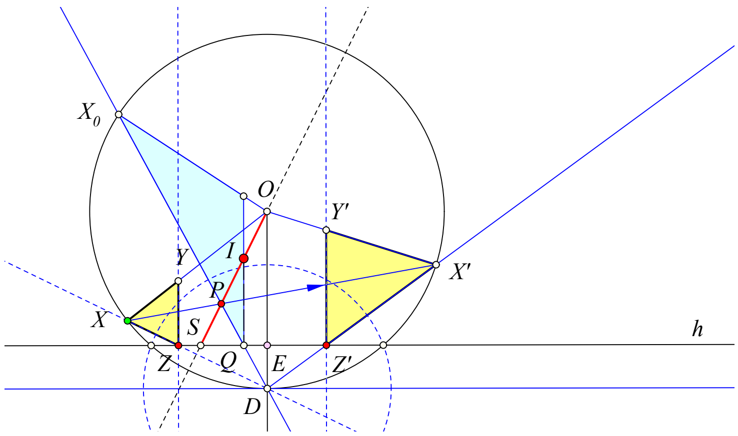

Figure 7. The involution $Z' = g(Z)$

In fact, using the involution $X' = g_P(X)$ on the circle and intersecting h with the rays of the pencil through D one defines a corresponding involution $Z' = g(Z)$

on line h. This, taking coordinates z, z' with origin at Q, is described by a Moebius transformation ([7, I,p.157]) of the kind (See Figure 7).

$$z' = \frac{az+b}{cz-a}.$$

The rest of the theorem, on the form of this involution (vanishing of a), follows immediately from the collinearity of D, P, Q, which is a consequence of Lemma 4. In fact, if the line XX' passing through P, obtains the position of PD, then points X', D become coincident and the corresponding circle k_2 becomes the tangent to c at D, the line of centers YY' becomes orthogonal to h and passes through I (See Figure 7). For this particular position of XX' we see also that as point X' converges to D, the corresponding Z' goes to infinity. Thus for $z = 0$ the value of $f(z)$ must be infinity, hence $a = 0$, thereby proving the theorem by setting the constant $w^2 = -b/c$. The value of the constant can be easily calculated by

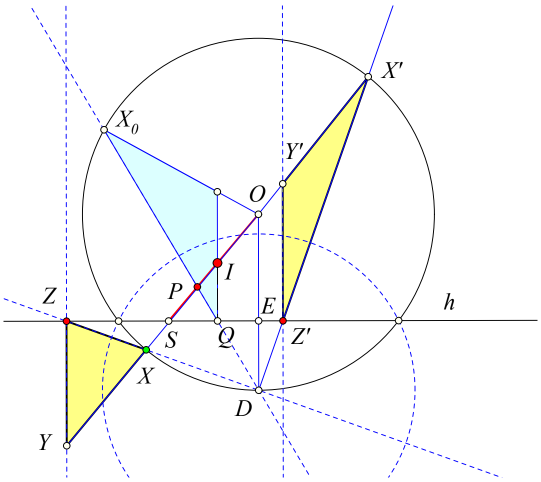

Figure 8. The value of the constant

letting X take the place of the intersection point of the line OP with c. Then XX' becomes a diameter of c and DZZ', $DX'X$ become similar right angled triangles. Setting for D the coordinates $D(e, d)$ and denoting the coordinates of all other points by corresponding small letters, we obtain

$$(z-e)(z'-e) = -d^2 \qquad \text{and} \qquad (z-s)(z'-s) = \delta,$$

where $\delta = -|SX||SX'|$ is the power of S with respect to the circle c. The second equation follows from the concyclicity of points Z, X, Z', X' (Lemma 3). From these we obtain the value

$$-w^2 = zz' = \frac{s(d^2+e^2)+e(\delta-s^2)}{e-s}.$$

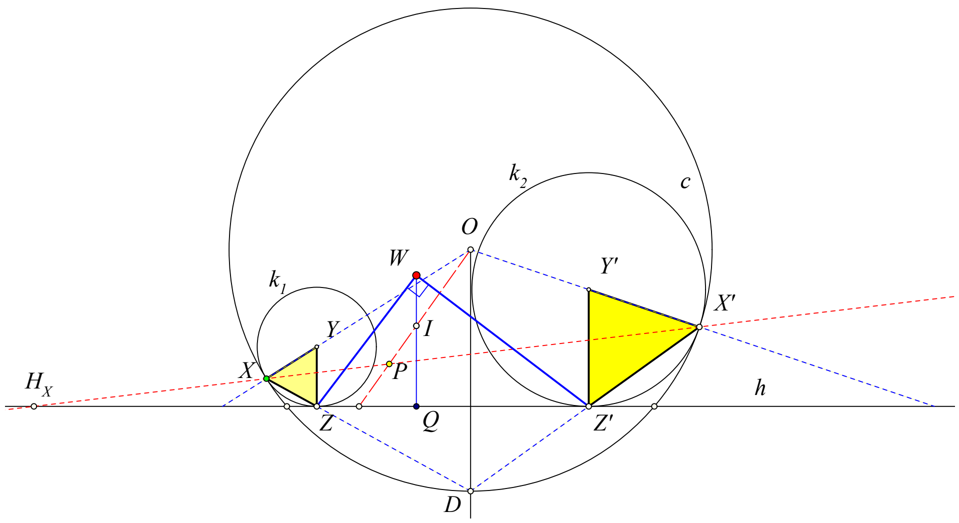

Figure 9. Involution $Z' = g(Z)$ by orthogonals through W

Interpreting geometrically the theorem, we deduce that there is a fixed point $W(0, w)$ on the axis QI, such that the involution $Z' = g(Z)$ is defined by means of the intersections of line h by two orthogonal lines rotating about W ([5, Involution, p.3]), this is formulated in the following.

Corollary 6. *The circle pairing induces an involution $Z' = g(Z)$ on line h, which coincides with the one defined by means of pairs of orthogonal lines rotating about a fixed point W lying on line IQ.*

4. Two auxiliary pencils of circles

Here we continue the exploration of the basic configuration using the notation and conventions of the previous sections.

Theorem 7. *The following properties of the basic configuration are true.*
(1) *The radical axis DN of circles k_1 and k_2 passes through D and the middle N of ZZ'.*
(2) *If $M = (DN, IQ)$, then the circle m_X centered at M and orthogonal to k_1 and k_2, intersects the parallel h' to h from I at two fixed points V_1 and V_2.*
(3) *The circles m_X are members of a pencil with base points V_1 and V_2.*

Property (1) immediately follows from Lemma 3, since then D is on the radical axis of the two circles, and because of the tangency to line h, the middle N of ZZ' is also on the radical axis of k_1 and k_2 (See Figure 10).

Property (2) follows from the fact that both circles c' and m_X being orthogonal to k_1 and k_2, their radical axis coincides with YY', which passes through I. The power of I with respect to c' and m_X is constant and equal in absolute value to $|IV_1|^2$, as claimed.

Property (3) is an immediate consequence of (2).

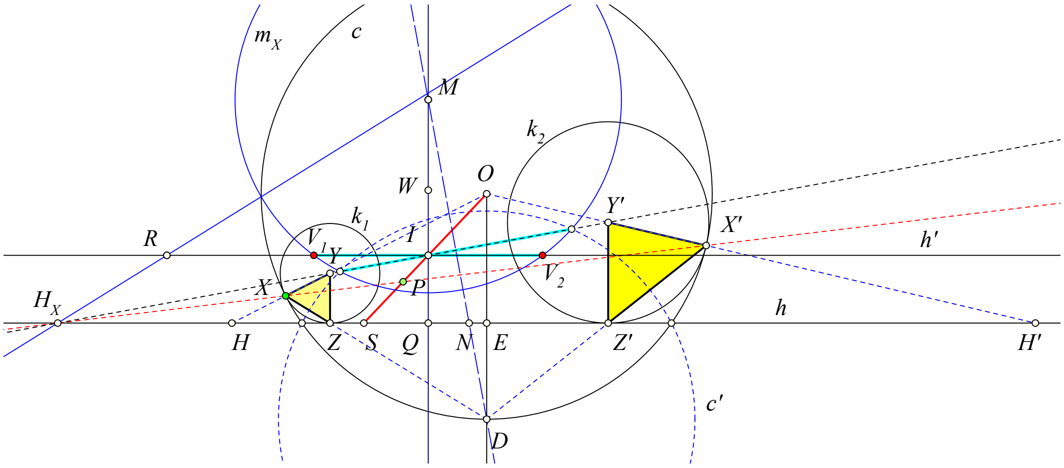

Figure 10. The pencil of circles m_X with base points V_1, V_2

Corollary 8. *The radical axis of circle m_X and c passes through a fixed point R on V_1V_2 and passes also through H_X.*

The first claim is a well known property for the radical axis of members of a pencil and a fixed circle ([2, p.210]). The second claim follows from the fact that H_X is the radical center of circles c', c and m_X.

Next theorem deals with the circles having diameter ZZ'. Since by Theorem 5 the map $Z' = g(Z)$ is an involution, we know that this is a pencil of circles ([4, vol.II,p.27]). Here we make precise the kind of the pencil and the location of its base points. Note that this pencil and the one of the preceding theorem have a common member m_0 carrying the base points of both pencils.

Theorem 9. *The following properties are valid.*
(1) *The circle n_X with diameter ZZ' is orthogonal to k_1 and k_2.*
(2) *The circle n_X intersects the circle m_0 centered at Q and passing through V_1 and V_2, at its diametral points W and W'.*
(3) *The circles n_X are members of a pencil with base points W and W'.*

Property (1) follows from the fact that n_X has its center at the middle N of ZZ', from which the tangents to k_1, k_2 are equal (See Figure 11).

Property (2) results from Theorem 5. Another proof results from the fact that circle n_X belongs to the pencil of orthogonal circles to k_1 and k_2. From this it follows that circles m_X, c' and n_X pass through the two base points U_1, U_2 lying on line YY', which passes through I. Let W and W' be the diametral points of circle m_0 on the diameter passing through I. Then $|IW||IW'| = |IV_1||IV_2|$ but also, since V_1V_2, U_1U_2 are chords of the circle m_X it is $|IV_1||IV_2| = |IU_1||IU_2|$. This implies that I is on the radical axis of m_0, n_X, which, since both circles have their center on h, is orthogonal to h. Thus it coincides with IQ or equivalently with WW'.

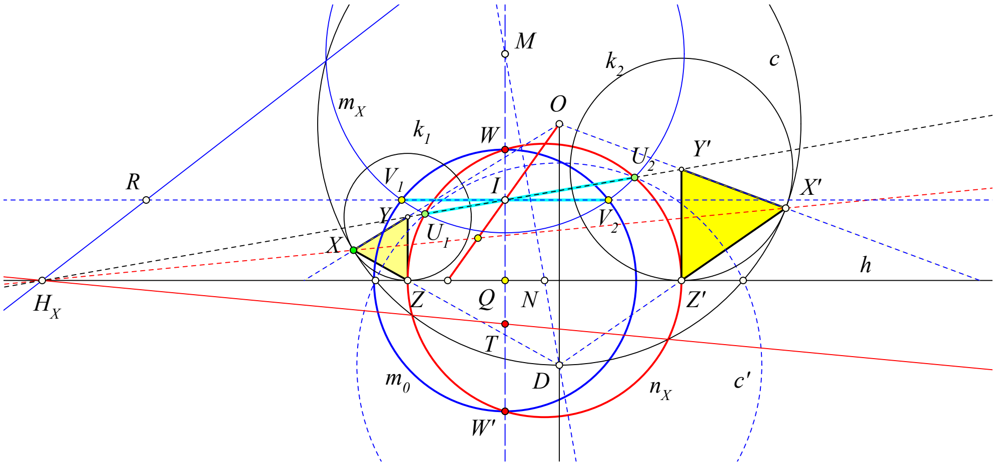

Figure 11. The pencil of circles n_X with base points W, W'

Property (3) is an immediate consequence of (2). Note that point W coincides with the one defined in Corollary 6.

Corollary 10. *The radical axis of circle n_X and c passes through a fixed point T on WW', and passes also through point H_X.*

The first claim is a well known property for the radical axis of members of a pencil and a fixed circle ([2, p.210]). The second claim follows from the fact that H_X is the radical center of circles c, m_X and n_X.

5. The pivoting line of the pairing of circles

Next figure results by drawing orthogonals respectively to lines WZ and WZ' from the centers Y and Y' of the circles k_1 and k_2. These intersect at right angles at point W_X and intersect also the parallel h' to h from I at points W_1, W_2.

Theorem 11. *Lines ZW_1 and $Z'W_2$ are parallel and W_X varies on a fixed line h'' parallel to h at distance equal to $|WI|$.*

To prove the theorem a short calculation seems unavoidable. For this, we use coordinates along line h and its orthogonal through Q, as in §3. Denoting coordinates with respective small letters and by r, r', r_1, r_2 respectively the radii of circles c, c', k_1, k_2, we find easily the relations

$$\begin{aligned} zz' &= -w^2, & r'^2 &= -2rd, & w^2 &= 2d(i-r) - d^2 - e^2, \\ r_1 &= \frac{1}{2d}(2rd + (z-e)^2 + d^2), & & & r_2 &= \frac{1}{2d}(2rd + (z'-e)^2 + d^2), \\ w_1 &= \frac{wz + (r_1 - i)z'}{w}, & & & w_2 &= \frac{wz' + (r_2 - i)z}{w}. \end{aligned}$$

From these, by a short calculation, we see that

$$w_2 - w_1 = (z' - z).$$

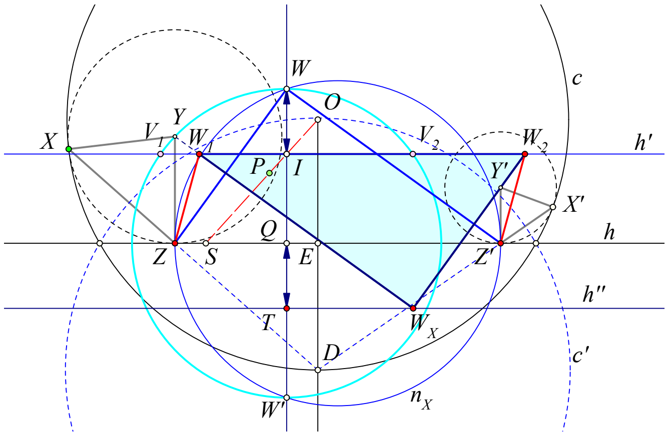

Figure 12. Pair of parallels $ZW_1, Z'W_2$

This means that W_1W_2 is equal in length to ZZ', thus lines ZW_1 and $Z'W_2$ are parallel, as claimed. The other claims are consequences of this property, since then $W_1W_2W_X$ and $Z'ZW$ are equal right angled triangles and projecting W_X on IW to T, we get $|QT| = |WI|$ (See Figure 12).

Next theorem states a property of the tangents to k_1, k_2 at points K, K', resulting by intersecting these circles respectively with lines WZ and WZ'. These tangents are the reflections of line h on lines W_XW_1 and W_XW_2 respectively. The theorem rests upon a simple criterion recognizing the passing of a variable line through a fixed point. Its proof is a simple calculation which I omit.

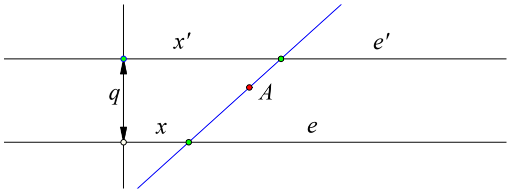

Figure 13. Lines passing through a fixed point A

Lemma 12. *If a variable line intersects the $x-$axis e and a parallel to it e' at distance q, so that the coordinates x, x' of the intersection points satisfy an equation of the form*

$$ax + bx' + c = 0,$$

then the variable line passes through the fixed point with coordinates

$$A = \frac{1}{a+b}(-c, qb).$$

Theorem 13. *The tangents to circles k_1, k_2, which are reflections of h, respectively, on W_XW_1, W_XW_2 are parallel and equidistant to the median W_XM of the right angled triangle $W_XW_1W_2$. This median passes through a fixed point A, independent of the position of X on the circle c.*

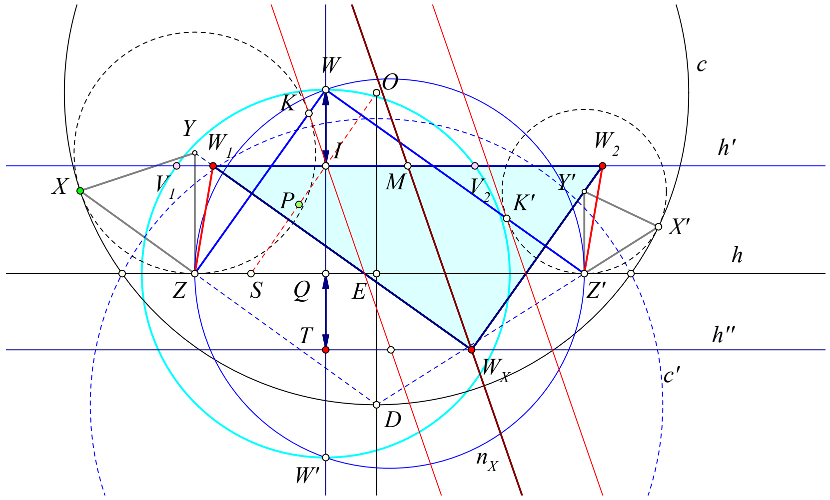

Figure 14. The pivoting line $W_X M$ and the parallel tangents at K, K'

That the tangents are parallel follows easily by measuring the angles at their intersection points with h. Similarly follows their parallelity to the median $W_X M$ (See Figure 14). ?

To show that line $W_X M$ passes through a fixed point it suffices, according to the previous lemma, to show that the x-coordinates of points M and W_X satisfy a linear relation. In fact, a somewhat extended calculation shows that $(W_X)_x$ and M_x satisfy the equation

$$(w-d)(W_X)_x + (2d-w)M_x - ew = 0.$$

From this and the previous lemma follows that line $W_X M$ passes through the fixed point

$$A = \frac{1}{d}\left(ew, i(2d-w)\right).$$

In the sequel line $W_X M$ is called the *pivoting line* of the circle pairing.

6. Sawayama pairings

I call the pairing of circles k_1, k_2 defined by the basic configuration (c, h, P) a *Sawayama pairing*, when the corresponding point I of the configuration is on the arc of circle c' lying inside c (See Figure 15). The formulas of the preceding section imply then that points W and I coincide ($i = w$) and consequently that W_X lies on line h. This implies in turn that the three parallels coincide with the pivoting line of the pairing, which is also the median of triangle $W_X W_1 W_2$ and which becomes simultaneously tangent to the two circles k_1, k_2 while passing also through the fixed point A. A short calculation shows also that in these circumstances point A lies on circle c, thus producing the figure of Sawayama's theorem. In fact, the last ingredient of the figure is the collinearity of points A, I and D, which, in view of the previous conventions for the coordinates reduces to an easy check of the vanishing of a determinant. By a well known property of the incenter ([2, p.76]),

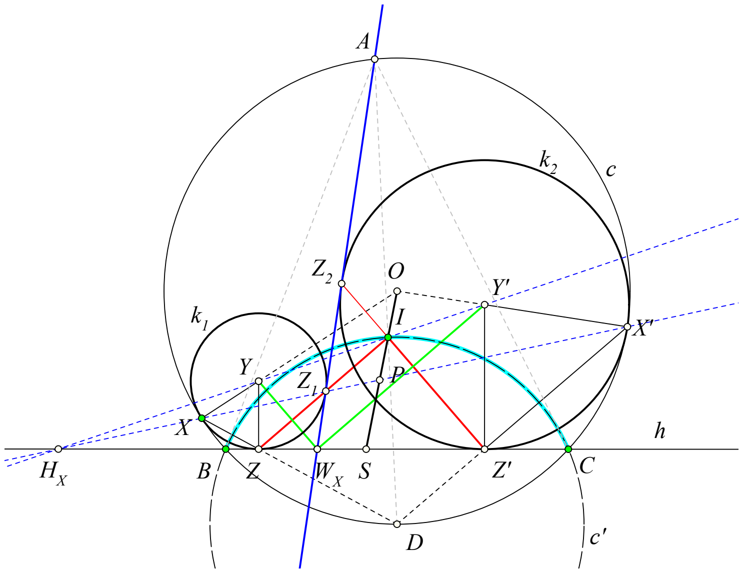

Figure 15. Sawayama pairing characterized by $Q \in c'$

point I coincides with the incenter of triangle ABC, thereby proving the next theorem.

Theorem 14. *Each Sawayama pairing k_1, k_2 corresponds to a triangle ABC with incenter I and cevians AW_X, such that the two circles are tangent to the circumcircle c of the triangle and also tangent to the cevian AW_X and side BC. The centers Y, Y' of the circles and the incenter are collinear.*

Fixing the basic configuration (c, h, P), so that the corresponding point I lies on the arc of c' contained in c, we obtain another proof of Sawayama's theorem, once we can show, that point I can obtain every possible position on the aforementioned arc of c' and, for each position of I, the corresponding pivoting lines AW_X can obtain all positions of the cevians through A of triangle ABC. That I can take every position on the claimed arc of c', is a simple consequence of the homographic relation of I to P described in §3. Next lemma makes this point clear and shows that the whole arc BC on c is obtained from a proper arc of an ellipse via the homography f, defined in §2. The somewhat more interesting verification that the pivoting line AW_X obtains all positions of cevians from A is handled in the next section.

Lemma 15. *For I varying on circle c' the corresponding point P varies on an ellipse, which is tangent to circle c at points B and C.*

In fact, since $I = f(P)$ is a homology, the claimed relation is given through the inverse homology $P = f^{-1}(I)$, which has the same center and axis and ratio $k' = \frac{DE}{DO}$. Hence the curve $c'' = f^{-1}(c')$ is a conic and the other claims result from simple geometric considerations.

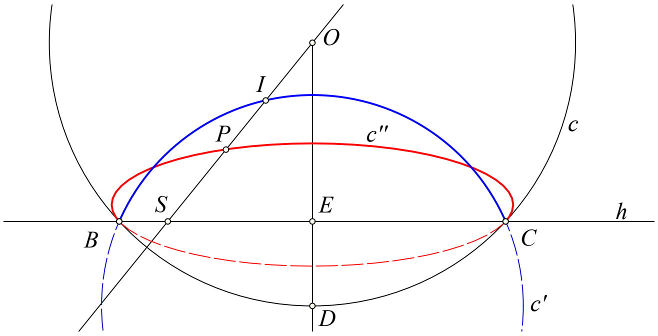

Figure 16. The locus c'' of P as Q varies on c'

7. The cevians

Using the notation of the previous sections, we deduce here various properties of the Sawayama pairings, appearing to be of interest and leading to the proof that, for a fixed point I on the arc of c' lying inside c, the corresponding pivoting lines AW_X, for variable X, obtain all positions of the cevians through A.

Lemma 16. *The circle c'' with diameter Z_1Z_2 is orthogonal to circles k_1, k_2 and the diametral point I' defines line II' which is parallel to BC. Also the triples of points (Z_1, I', V) and (Z_2, I', V') are collinear.*

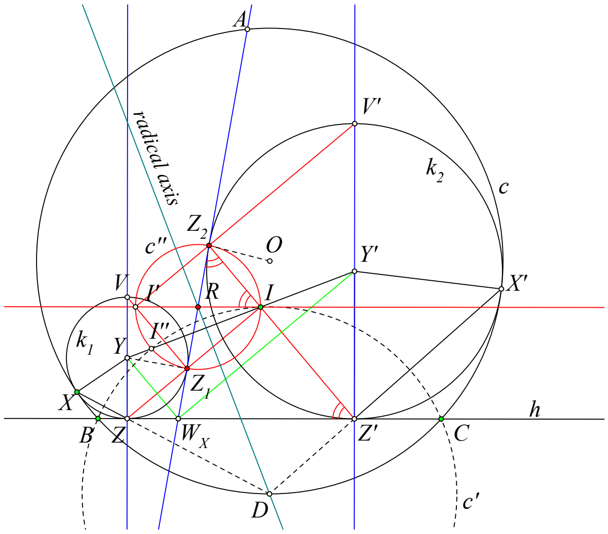

Figure 17. The parallel to h line II' and the radical axis of $c_x, c_{X'}$

The proof of parallelity follows from the equality of angles $W_XZ'Z_2$, W_XZ_2Z', TIZ_2 (See Figure 17). The collinearity of Z_1, I', V follows from the equality of angles $W_XZ'I$, ZW_XY and the parallelity of VZ_1 to YW_X.

Corollary 17. *Line YY' passes through the intersection points I, I'' of the circles c' and c''.*

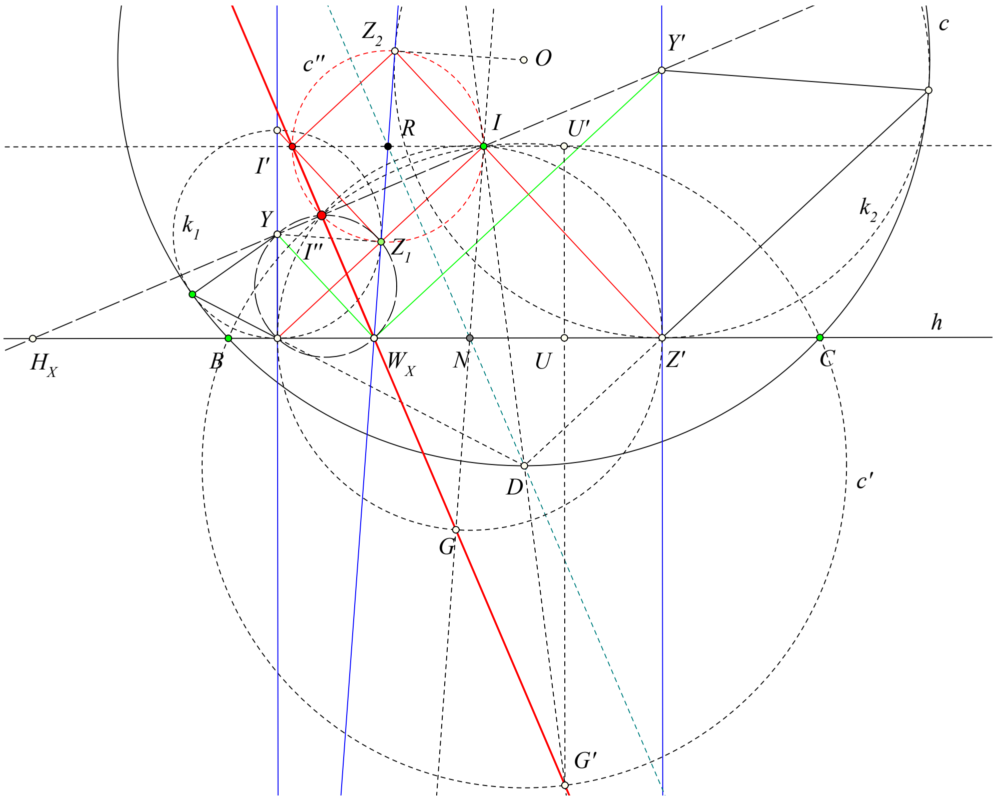

Figure 18. Line $I'W_X$

Lemma 18. *Line $I'I''$ is orthogonal to II'' and passes through point W_X and also through points G, G', which are correspondingly diametrals of I with respect to the circle with diameter ZZ' and the circle c'.*

In fact, by the previous lemma, angle $I'I''I$ is a right one (See Figure 18). Also I'' is on the circle with diameter YW_X, which passes through Z, Z_1. Thus I', I'', W_X are collinear. The center N of the circle with diameter ZZ' is on line RD, which is the radical axis of circles k_1, k_2 and is orthogonal to II' at its middle. Hence $I'W_X$ is parallel to RD. This implies the two other statements of the lemma.

Corollary 19. *The cross ratio $\kappa = (BCW_XH_X)$ is constant and equal to $\frac{UB}{UC}$. Here U is the projection of G' on BC.*

This follows by considering the pencil $I''(B, C, W, H_X)$ of lines through I''. The lines of the pencil pass through fixed points of the circle c'. Thus their cross ratio is independent of the position of I'' on this circle. Its value is easily calculated by letting I'' take the place of U', which is the projection of G' on II'. Then H_X goes to infinity and κ becomes equal to the stated value.

Corollary 20. *Fixing point I, which becomes the incenter of triangle ABC, the pivoting lines AW_X of the Sawayama pairing take the positions of all cevians through A and Sawayama's theorem is true.*

In fact, by the previous corollary, the constancy of cross ratio implies that for variable X on c points H_X obtain all positions on line BC and consequently W_X, being related to H_X by a line-homography, obtains also all possible positions on this line.

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