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On the Fermat Geometric Problem

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Abstract. We consider the Fermat configuration generated by a point on a circle over the side of a rectangle. When the ratio of the rectangle's sides is $\sqrt{2}$, then many properties do not depend on a position of the point. Some properties hold for all ratios and other ratios can also have interesting geometric consequences. Our proofs use analytic geometry but some parts include also synthetic arguments.

1. Introduction

Among the numerous questions that Pierre Fermat formulated, the following geometric problem is our main concern (see Figure 1).

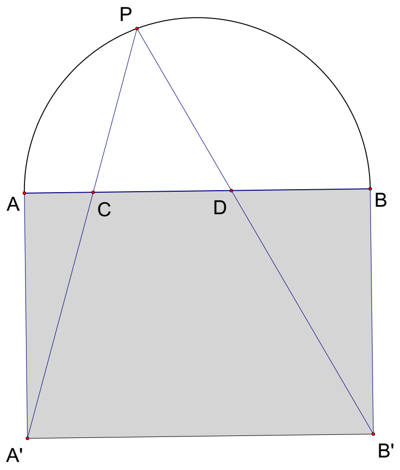

Figure 1. The configuration of the Fermat problem.

Fermat Problem. Let P be a point on the semicircle that has the top side AB of the rectangle $ABB'A'$ as a diameter. Let $\frac{|AB|}{|AA'|} = \sqrt{2}$. Let the segments PA' and PB' intersect the side AB in the points C and D, respectively. Then $|AD|^2 + |BC|^2 = |AB|^2$.

The great Leonard Euler in [6] has provided the first rather long proof, which is old fashioned (for his time), and avoids the analytic geometry (which offers rather simple proofs as we shall see later). Several more concise synthetic proofs are now known (see [10], [7, pp. 602, 603], [1, pp. 168, 169] and [8, pp. 181, 264]). A very nice description of Euler's proof is available on the Internet (see [11]).

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The analytic proof was recently recalled in [9] where it was observed that the above relation holds for all points on the circle with the segment AB as a diameter.

For a circle, we shall consider a slightly more general situation where the quotient $\frac{|AB|}{|AA'|}$ is a positive real number m (see Figure 2.).

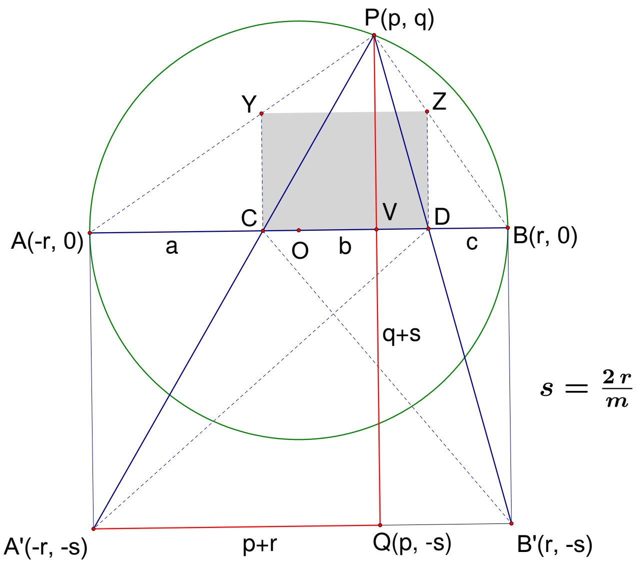

Figure 2. The extended Fermat configuration for a circle.

For given different points A and B and any points P_1, P_2, P_3, P_4 in the plane, let

$$\varphi(P_1P_2, P_3P_4) := \frac{|P_1P_2|^2 + |P_3P_4|^2}{|AB|^2}.$$

In this notation, the above Fermat Problem for the circle is the implication $(a) \Rightarrow (b)$ in Theorem 1 below.

Theorem 1. *Let $U_b = \varphi(AD, BC)$. The following statements are equivalent.*
(a) $m = \sqrt{2}$, *and*
(b) $U_b = 1$.

Proof. We shall use analytic geometry, which offers a simple proof. Let the origin of the rectangular coordinate system be the midpoint O of the side AB so that the points A and B have coordinates $(-r, 0)$ and $(r, 0)$ for some positive real number r (the radius of the circle). The equation of the circle is a standard $x^2 + y^2 = r^2$.

The coordinates of the points A' and B' are $\left(-r, -\frac{2r}{m}\right)$ and $\left(r, -\frac{2r}{m}\right)$. For any real number t, let $u = 1 - t^2$, $v = 1 + t^2$, $z = m\,t$, $\eta = v - z$ and $\vartheta = v + z$. An arbitrary point P on the circle has coordinates $\left(\frac{r\,u}{v}, \frac{2\,r\,t}{v}\right)$. From similar right-angled triangles PVC and PQA' and $s = \frac{2\,r}{m}$, we easily find that $C\left(\frac{r(u-z)}{\vartheta}, 0\right)$

and $D\left(\frac{r(u+z)}{\vartheta}, 0\right)$. The equivalence of the statements (a) and (b) follows from the identity $U_b - 1 = \frac{t^2\left(m^2-2\right)}{\vartheta^2}$. $\square$

The fact that $(a) \implies (b)$ can be proved more simply by synthetic methods. Here is an adaptation of Lionnet's proof from [7, p. 602], which works for any point on the circle (see Figure 2).

Let the directed lengths $\overrightarrow{AC}$, $\overrightarrow{CD}$, $\overrightarrow{DB}$ be a, b, c. Then

$$\begin{aligned} AD^2 + BC^2 - AB^2 &= (a+b)^2 + (b+c)^2 - (a+b+c)^2 \\ &= b^2 - 2\,a\,c, \end{aligned}$$

so that

$$AD^2 + BC^2 = AB^2 \qquad \iff \qquad b^2 = 2\,a\,c. \tag{1}$$

Now draw CY and DZ perpendicular to AB, with Y on PA and Z on PB. Using pairs of similar triangles, we have

$$\frac{YC}{AA'} = \frac{PC}{PA'} = \frac{CD}{A'B'} = \frac{PD}{PB'} = \frac{ZD}{BB'}.$$

Hence $YCDZ$ is a rectangle similar to $AA'B'B$. The triangles YCA, BDZ are equiangular, so $\frac{YC}{a} = \frac{c}{DZ}$. But $YC = DZ = \frac{CD}{\sqrt{2}} = \frac{b}{\sqrt{2}}$. Thus $b^2 - 2\,a\,c$ vanishes, hence $AD^2 + BC^2 = AB^2$.

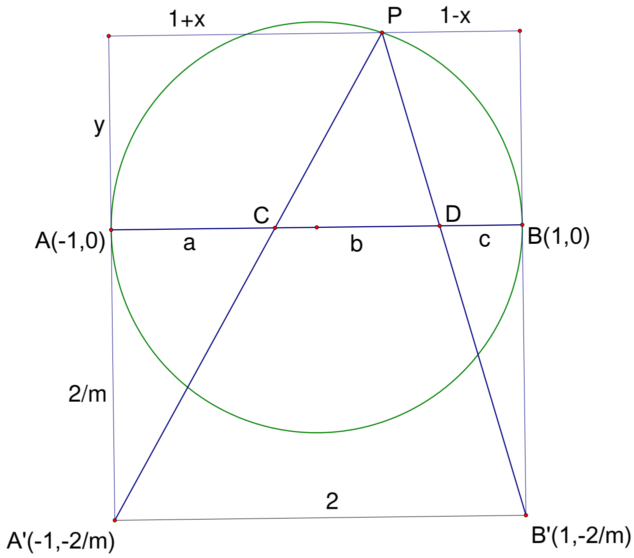

Figure 3. Simple analytic proof.

We can also use the equivalence (1) to obtain a simple analytic proof (see Figure 3). There is no loss of generality in taking the radius of the circle as the unit of length. By similar triangles,

$$\frac{a}{\frac{2}{m}} = \frac{1+x}{\frac{2}{m}+y}, \quad \frac{b}{y} = \frac{2}{\frac{2}{m}+y} \quad \text{and} \quad \frac{c}{\frac{2}{m}} = \frac{1-x}{\frac{2}{m}+y}.$$

Simplifying $b^2 - 2\,a\,c$ and putting $x^2 = 1 - y^2$ we find that this difference is $\frac{4\,y^2(m^2-2)}{(2+m\,y)^2}$.

Let A'', B'', P' be the reflections of the points A', B', P in the line AB. We close this introduction with a remark that most of our results come in related pairs. The second version, which requires no extra proof, comes (for example in Theorem 1) by replacing the points C and D with the points C' and D', which are the intersections of the line AB with the lines PA'' (or $P'A'$) and PB'' (or $P'B'$). In other words, if $V_b = \varphi(AD', BC')$, then (a) and
(b$'$) $V_b = 1$
are also equivalent. Moreover, (a) and
(b$''$) $U_b = V_b$
are equivalent as well.

2. Invariants of the Fermat configuration

Our primary goal is to present several statements similar to (b), (b$'$) and (b$''$) that could replace it in Theorem 1. In other words, we explore what other relationships in the Fermat configuration remain invariant as the point P changes position on the circle.

We begin with the diagonals of the trapezium $A'B'DC$ (see Figure 2).

Theorem 2. *Let $U_c = \varphi(A'D, B'C)$ and $V_c = \varphi(A'D', B'C')$.*
Consider the statements
(c) $U_c = 2$, (c$'$) $V_c = 2$, (c$''$) $U_c = V_c$.
The following are true.
(a) $\Leftrightarrow$ (c$''$),
(a) *implies each of the statements* (c) *and* (c$'$).

Proof. With straightforward computations one can easily check that

$$\begin{aligned} V_c - U_c &= \frac{2(m^2-2)v\,t^3}{\vartheta^2\,\eta^2}, \\ 2 - U_c &= \frac{(m^2-2)v(2\,z+v)}{m^2\,\vartheta^2}, \\ V_c - 2 &= \frac{(m^2-2)v(2\,z-v)}{m^2\,\eta^2}. \end{aligned}$$

Of course, Pythagoras' theorem might be useful in computing the function φ. For instance, for U_c we have

$$A'D^2 + B'C^2 = AA'^2 + BB'^2 + AD^2 + BC^2.$$

But $AA'^2 + BB'^2 = AB^2$ and $AD^2 + BC^2 = AB^2$. Hence $U_c = 2$.

This method will also yield generalizations. Take points A_*, B_*, C_*, D_* with $\mathbf{AA}_* = \lambda\mathbf{AA'}$, $\mathbf{BB}_* = \lambda\mathbf{BB'}$, $\mathbf{BC}_* = \mu\mathbf{BC}$, $\mathbf{AD}_* = \mu\mathbf{AD}$. Then it follows easily that $\varphi(A_*D_*, B_*C_*) = \lambda^2 + \mu^2$. □

For points X and Y, let $X \oplus Y$ be the center of the square built on the segment XY such that the triangle $X(X \oplus Y)Y$ has the positive orientation (counterclockwise). When the point $X \oplus Y$ is shortened to M, then M^* denotes $Y \oplus X$.

The midpoints G, H, G', H' of the segments AC, BD, AC', BD' and the top N of the semicircle over AB are used in the next theorem. In other words, $N = B \oplus A$. The center O of the circle (i. e., the midpoint of the segment AB; the origin of the rectangular coordinate system) appears also.

Theorem 3. *Let* $U_d = \varphi(NG, NH)$, $V_d = \varphi(NG', NH')$,
$U_e = \varphi(OG, OH)$ *and* $V_e = \varphi(OG', OH')$ (see Figure 4).
The following statements are equivalent.
(a) $m = \sqrt{2}$,
(d) $U_d = \frac{3}{4}$, (d′) $V_d = \frac{3}{4}$, (d″) $U_d = V_d$, (d*) $U_d = 3\,U_e$,
(e) $U_e = \frac{1}{4}$, (e′) $V_e = \frac{1}{4}$, (e″) $U_e = V_e$, (e*) $V_d = 3\,V_e$.

Proof. This time the differences $U_d - \frac{3}{4}$ and $U_e - \frac{1}{4}$ both simplify to $\frac{t^2(m^2-2)}{4\,\vartheta^2}$, which has the factor $m^2 - 2$ again. Similarly, $V_d - U_d$ and $V_e - U_e$ both are equal $\frac{m\,v\,t^3(m^2-2)}{\eta^2\,\vartheta^2}$. Finally, $3\,U_e - U_d$ is $\frac{t^2(m^2-2)}{2\,\vartheta^2}$. The other differences are analogous quotients that have η instead of ϑ.

The following synthetic proof shows that the statements (a) and (b) together imply (d) and (e).

A dilatation with center A and scale factor 2 maps GO on to CB, thus $GO = \frac{CB}{2}$; similarly $OH = \frac{AD}{2}$. Consequently for U_e we have

$$OG^2 + OH^2 = \tfrac{BC^2}{4} + \tfrac{AD^2}{4} = \tfrac{1}{4}AB^2.$$

Similarly, for U_d we have $NG^2 = NO^2 + OG^2$ and $NH^2 = NO^2 + OH^2$ so that

$$NG^2 + NH^2 = 2\,NO^2 + \frac{BC^2 + AD^2}{4} = \frac{AB^2}{2} + \frac{AB^2}{4} = \frac{3}{4}AB^2.$$

□

Let G_s, H_s, G'_s, H'_s be the points that divide the segments NG, NH, NG', NH' in the same ratio $s \neq -1$ (i. e., $NG_s : G_sG = s : 1$, etc.).

Theorem 4. *Let* $m_s = \frac{s^2+2}{4(s+1)^2}$, $n_s = \frac{3\,s^2}{4(s+1)^2}$,
$U_f = \varphi(OG_s, OH_s)$, $V_f = \varphi(OG'_s, OH'_s)$,
$U_g = \varphi(NG_s, NH_s)$, $V_g = \varphi(NG'_s, NH'_s)$.
If $s \neq 0$, *then the following statements are equivalent.*
(a) $m = \sqrt{2}$,
(f) $U_f = m_s$, (f′) $V_f = m_s$, (f″) $U_f = V_f$,
(g) $U_g = n_s$, (g′) $V_g = n_s$, (g″) $U_g = V_g$.

Proof. Since $G_s = \left(\frac{-r\,s\,t(m+t)}{(s+1)\vartheta}, \frac{r}{s+1}\right)$ and $H_s = \left(\frac{r\,s(1+z)}{(s+1)\vartheta}, \frac{r}{s+1}\right)$, the difference $U_f - m_s$ is $\frac{s^2 t^2 (m^2-2)}{4(s+1)^2\,\vartheta^2}$. The other parts are proved with analogous arguments. □

Theorem 5. *Let* $\lambda = \frac{3\,s^2}{s^2+2}$. *If* $s \neq 0,\ 1$, *then the following statements are equivalent.*
(a) $m = \sqrt{2}$,
(f*) $U_g = \lambda\, U_f$,
(g*) $V_g = \lambda\, V_f$.

Proof. For $s \neq 0,\ 1$, the equivalence of (a) and (f^*) is a consequence of the equality $\lambda\, U_f - U_g = \frac{s^2\,t^2 (s-1)(m^2-2)}{2\,(s+1)(s^2+2)\,\vartheta^2}$. □

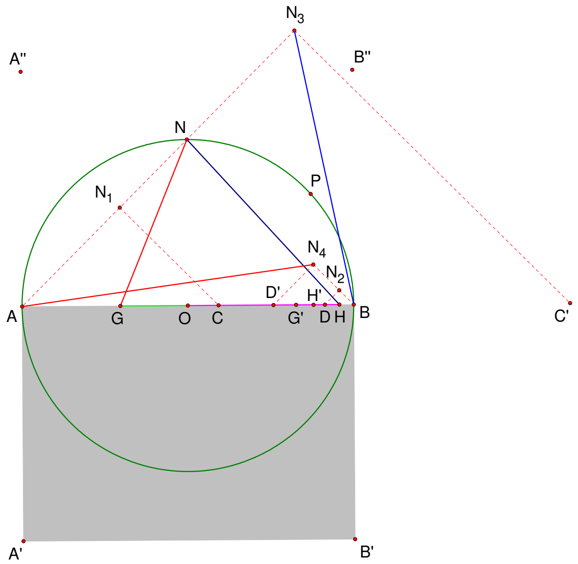

Figure 4. Points O, N, $N_1, \ldots\ N_4$ in Theorems 3 and 6.

Let N_1, N_2, N_3, N_4 denote the highest points on the semicircles built on the segments AC, BD, AC', BD' above the line AB. In other words, $N_1 = C \oplus A$, $N_2 = B \oplus D$, $N_3 = C' \oplus A$, $N_4 = B \oplus D'$.

Theorem 6. *Let* $U_h = \varphi(BN_1, AN_2)$, $V_h = \varphi(BN_3, AN_4)$, $U_i = \varphi(NN_1, NN_2)$, $V_i = \varphi(NN_3, NN_4)$ (see Figure 4).
The following statements are equivalent.
(a) $m = \sqrt{2}$,
(h) $U_h = \frac{3}{2}$, (h′) $V_h = \frac{3}{2}$, (h″) $U_h = V_h$, (h*) $3\,U_i = U_h$,
(i) $U_i = \frac{1}{2}$, (i′) $V_i = \frac{1}{2}$, (i″) $U_i = V_i$, (i*) $3\,V_i = V_h$.

Proof. Since $N_1 = \left(\frac{-r\,t(m+t)}{\vartheta}, \frac{r}{\vartheta}\right)$ and $N_2 = \left(\frac{r(1+z)}{\vartheta}, \frac{r\,t^2}{\vartheta}\right)$, we get

$$U_h - \tfrac{3}{2} = U_i - \tfrac{1}{2} = \frac{t^2\left(m^2-2\right)}{2\,\vartheta^2}.$$

This proves the equivalence of (a) with (h) and (i). The proofs of the other equivalences are similar.

The synthetic proof of the implication $(a) \Rightarrow (h)$ uses the right-angled triangles AHN_2 and BGN_1 to get $AN_2^2 = (a+b+\frac{c}{2})^2 + \frac{c^2}{4}$ and $BN_1^2 = (\frac{a}{2}+b+c)^2 + \frac{a^2}{4}$. But $b^2 = 2\,a\,c$ implies $AN_2^2 + BN_1^2 = \frac{3}{2}AB^2$.

Let $a' = \frac{a}{\sqrt{2}}$, etc. For the implication $(a) \Rightarrow (i)$, from the isosceles right-angled triangles ABN, ACN_1, BDN_2 we get $NN_1^2 = (b'+c')^2$ and $NN_2^2 = (a'+b')^2$. But $b^2 = 2\,a\,c$ implies $NN_1^2 + NN_2^2 = \frac{1}{2}AB^2$. □

In the following theorem we also use the points N_1, N_2, N_3 and N_4. However, we do not use the function φ.

Theorem 7. *The following statements are equivalent.*
(a) $m = \sqrt{2}$,
(j) $|N_1N_2| = |AN|$, (j′) $|N_3N_4| = |AN|$, (j″) $|N_1N_2| = |N_3N_4|$,
(k) $|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 = 2\,|AB|^2$.

Proof. For $(a) \Leftrightarrow (j)$, we easily get $|N_1N_2|^2 - |AN|^2 = \frac{2\,r^2t^2\left(m^2-2\right)}{\vartheta^2}$.

Since the coordinates of N_3 and N_4 are $\left(\frac{r\,t(m-t)}{\eta}, \frac{r}{\eta}\right)$ and $\left(\frac{r(1-z)}{\eta}, \frac{r\,t^2}{\eta}\right)$,

$$|N_1N_2|^2 + |N_2N_3|^2 + |N_3N_4|^2 + |N_4N_1|^2 - 2\,|AB|^2 = \frac{8\,r^2\,t^2(m^2-2)(v^2+z^2)}{\eta^2\,\vartheta^2}.$$

This shows that $(a) \Leftrightarrow (k)$.

Note that by projecting BC orthogonally on to AN, we obtain NN_1, and its length is $\frac{BC}{\sqrt{2}}$. Similarly, $|NN_2| = \frac{AD}{\sqrt{2}}$. The fourth vertex of the rectangle with sides NN_1, NN_2 is the point M_1 used later in the paper. Hence $|NM_1| = |N_1N_2| = |AN|$, proving $(a) \Rightarrow (o)$ in Theorem 9 below. □

Notice that $|N_1N_2^*|^2 + |N_2^*N_3^*|^2 + |N_3^*N_4|^2 + |N_4N_1|^2 = 2\,|AB|^2$ if and only if $m = 1$.

Let $N_5 = A \oplus D$, $N_6 = C \oplus B$, $N_7 = A \oplus D'$ and $N_8 = C' \oplus B$.

Theorem 8. *Let* $U_\ell = \varphi(AN_5, BN_6)$, $V_\ell = \varphi(AN_7, BN_8)$,
$U_m = \varphi(GN_6, HN_5)$, $V_m = \varphi(G'N_8, H'N_7)$ *and*
$U_n = \varphi(NN_5, NN_6)$, $V_n = \varphi(NN_7, NN_8)$.
The following ten statements are equivalent:
(a) $m = \sqrt{2}$,
(ℓ) $U_\ell = \frac{1}{2}$, (ℓ') $V_\ell = \frac{1}{2}$, (ℓ'') $U_\ell = V_\ell$,
(m) $U_m = \frac{3}{4}$, (m′) $V_m = \frac{3}{4}$, (m″) $U_m = V_m$,
(n) $U_n = \frac{3}{2}$, (n′) $V_n = \frac{3}{2}$, (n″) $U_n = V_n$.

Proof. Since $N_5 = \left(\frac{-r\,t^2}{\vartheta}, \frac{-r(1+z)}{\vartheta}\right)$ and $N_6 = \left(\frac{r}{\vartheta}, \frac{-r\,t(m+t)}{\vartheta}\right)$, we obtain $U_\ell - \frac{1}{2} = \frac{t^2\left(m^2-2\right)}{2\,\vartheta^2}$. This shows the equivalence of (a) and (ℓ). The other equivalences have similar proofs.

In order to prove the implication $(a) \Rightarrow (\ell)$ in synthetic fashion, from the isosceles right-angled triangles ADN_5 and BCN_6, we get $AN_5^2 + BN_6^2 = \frac{AD^2}{2} + \frac{BC^2}{2} = \frac{1}{2}AB^2$. □

Let $\lambda_s = \frac{3((s+1)^2+1)}{4(s+1)^2}$. The equality $\varphi(G_sN_6, H_sN_5) = \lambda_s$ is true if and only if $m = \sqrt{2}$. In fact, this is the first equivalence from another similar group that involve points G_s and H_s.

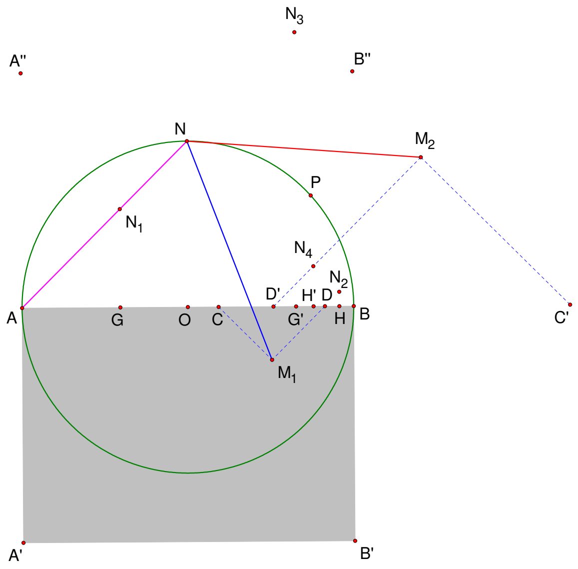

Figure 5. Points M_1 and M_2 in Theorem 9.

The next theorem uses the centers of squares on the segments CD and $C'D'$. Let $M_1 = C \oplus D$ and $M_2 = C' \oplus D'$.

Theorem 9. *Let* $U_o = |N\,M_1| - |A\,N|$, $V_o = |N\,M_2| - |A\,N|$, $U_p = \varphi(M_1N_1, M_1N_2)$ *and* $V_p = \varphi(M_2N_3, M_2N_4)$ (see Figure 5).
The following statements are equivalent:
(a) $m = \sqrt{2}$,
(o) $U_o = 0$, (o′) $V_o = 0$, (o″) $U_o = V_o$,
(p) $U_p = \frac{1}{2}$, (p′) $V_p = \frac{1}{2}$, (p″) $U_p = V_p$.

Proof. Since $M_1 = \left(\frac{r\,u}{\vartheta}, -\frac{r\,z}{\vartheta}\right)$, the difference $|M_1N|^2 - |AN|^2$ is $\frac{2\,r^2\,t^2(m^2-2)}{\vartheta^2}$. Similarly, $|M_2N|^2 - |M_1N|^2 = \frac{8\,r^2\,t^2\,v\,z(m^2-2)}{\eta^2\vartheta^2}$. This shows that $(a) \Leftrightarrow (o)$ and $(a) \Leftrightarrow (o'')$. The $(a) \Rightarrow (o)$ is proved also as follows.

From the right-angled triangle NN_1M_1, since $\frac{b^2}{2} = a\,c$, we get $NM_1^2 = (a'+b')^2 + (b'+c')^2 = a'^2 + b^2 + c'^2 + a\,b + b\,c = (a'+b'+c')^2 = NA^2$. $\square$

For any point X in the plane, let G_1, G_2, G_3, G_4, G_5 and G_6 denote the centroids of the triangles ACX, CDX, DBX, $AC'X$, $C'D'X$ and $BD'X$.

Theorem 10. *Let $U_q = \varphi(G_2G_1, G_2G_3)$ and $V_q = \varphi(G_5G_4, G_5G_6)$.*
The following statements are equivalent:
(a) $m = \sqrt{2}$,
(q) $U_q = \frac{1}{9}$, (q′) $V_q = \frac{1}{9}$, (q″) $U_q = V_q$.

Proof. If $X = (x, y)$, then the points G_1, G_2 and G_3 have the same ordinate $\frac{y}{3}$ while their abscissae are $\frac{x}{3} - \frac{2\,r\,t(m+t)}{3\,\vartheta}$, $\frac{x}{3} + \frac{2\,r\,u}{3\,\vartheta}$ and $\frac{x}{3} + \frac{2\,r(z+1)}{3\,\vartheta}$. It follows that $U_q - \frac{1}{9} = \frac{t^2\left(m^2-2\right)}{9\,\vartheta^2}$ that proves $(a) \Leftrightarrow (q)$.

The implication $(b) \Rightarrow (q)$ could be proved as follows. Let I denote the midpoint of the segment CD. A dilatation with center C and scale factor 2 maps GI on to AD, thus $GI = \frac{AD}{2}$; similarly $HI = \frac{BC}{2}$. Hence $\varphi(GI, HI) = \frac{1}{4}$. On the other hand, a dilatation with center X and scale factor $\frac{3}{2}$ maps G_1G_2 on to GI, thus $G_1G_2 = \frac{2}{3}GI$; similarly $G_3G_2 = \frac{2}{3}HI$. Hence $U_q = \frac{1}{9}$. $\square$

Let U and V be the midpoints of the segments CC' and DD'.

Theorem 11. *Let $U_s = \varphi(NU, NV)$, $U_t = \varphi(OU, OV)$ and $V_t = \varphi(N_6U, N_5V)$.*
The following statements are equivalent:
(a) $m = \sqrt{2}$,
(s) $U_s = 1$,
(t) $U_t = \frac{1}{2}$, (t′) $V_t = \frac{1}{2}$, (t″) $U_t = V_t$.

Proof. Since abscissae of U and V are $\frac{r\left(u\,v+z^2\right)}{\eta\,\vartheta}$ and $\frac{r\left(u\,v-z^2\right)}{\eta\,\vartheta}$, we get $U_s - 1 = U_t - \frac{1}{2} = \frac{t^2\,v^2\left(m^2-2\right)}{\eta^2\vartheta^2}$. This proves $(a) \Leftrightarrow (s)$ and $(a) \Leftrightarrow (t)$. $\square$

Theorem 12. *Let $W = U \oplus V$, $U_u = |WO| - \frac{|AB|}{2}$,*
$U_{v(i,j)} = \varphi(WN_i, WN_j)$ for $i \in \{1,3\}$ and $j \in \{2,4\}$ and $U_w = \varphi(W^*N, WN)$.
The following statements are equivalent:
(a) $m = \sqrt{2}$,
(u) $U_u = 0$,
(w) $U_w = 1$,
(x) *the lines WN_1 and WN_2 are perpendicular,*
(y) *the lines WN_3 and WN_4 are perpendicular,*
($\mathrm{v}_{i,j}$) $U_{v(i,j)} = \frac{1}{2}$ *for $i \in \{1,3\}$ and $j \in \{2,4\}$.*

Proof. Since the coordinates of the point W is the pair $\left(\frac{r\,u\,v}{\eta\,\vartheta}, \frac{r\,z^2}{\eta\,\vartheta}\right)$, we get that $|WO|^2 - r^2 = \frac{2\,r^2\,t^2\,v^2\left(m^2-2\right)}{\eta^2\,\vartheta^2}$. This proves $(a) \Leftrightarrow (u)$.

For the equivalence $(a) \Leftrightarrow (x)$, note that the lines WN_1 and WN_2 have equations

$$(z^2 - \eta)x + (z^2 - \vartheta)y = \lambda \qquad \text{and} \qquad (m^2 - \eta)x - (m^2 - \vartheta)y = \mu,$$

where λ and μ are real numbers. These lines are perpendicular if and only if $2\,v\,z(m^2-2)$ is zero. $\square$

Let $K_1 = B \oplus N_1$, $K_2 = N_2 \oplus A$, $K_3 = B \oplus N_3$, $K_4 = N_4 \oplus A$. These points can be defined more simply. They all are at the same hight as N and vertically above the points N_6, N_5, N_8, N_7, respectively.

Theorem 13. *Let* $\lambda = \frac{7}{4} + \sqrt{2}$, $U_z = \varphi(A'K_2, B'K_1)$ *and* $V_z = \varphi(A'K_4, B'K_3)$. *For the statements*
(a) $m = \sqrt{2}$,
(z) $U_z = \lambda$, (z′) $V_z = \lambda$,
we have (a)⇒(z) *and* (a)⇒(z′).

Proof. Since K_2 and K_1 have abscissae $-\frac{r\,t^2}{\vartheta}$ and $\frac{r}{\vartheta}$, we conclude that $\lambda - U_z = \frac{[(4(m+1)\vartheta^2 - z^2)\sqrt{2} + m(z+2\,v)(3\,z+2\,v)](m-\sqrt{2})}{4\,m^2\,\vartheta^2}$. When (a) is true, then this difference is zero. The proof of $(a) \Rightarrow (z')$ is similar. $\square$

Let $K = P \oplus A'$, $L = P \oplus B'$, $S = P \oplus A''$, $T = P \oplus B''$.

Theorem 14. *Let* $\lambda = 1 + \frac{\sqrt{2}}{2}$, $U_\alpha = \varphi(AK^*, BL)$ *and* $V_\alpha = \varphi(AS, BT^*)$. *The following statements are equivalent:*
(a) $m = \sqrt{2}$,
(α) $U_\alpha = \lambda$, (α') $V_\alpha = \lambda$.

Proof. Since $\left(\frac{r(\vartheta - t\,z)}{m\,v}, -\frac{r(m+\eta)}{m\,v}\right)$ and $\left(\frac{r(m-\vartheta)}{m\,v}, -\frac{r(t\,z+\eta)}{m\,v}\right)$ are the coordinates of the points K^* and L, we get that $\lambda - U_\alpha$ is equal to $\frac{(1+\sqrt{2})(m-\sqrt{2})(m+2-\sqrt{2})}{2\,m^2}$. It follows from Theorem 18 that the same holds for V_α. $\square$

Let S_1, T_1, S_2 and T_2 denote the midpoints of the segments $A'C$, $B'D$, $A'C'$ and $B'D'$. Note that (a) implies that

$$\varphi(G_sS_1, H_sT_1) = \varphi(G'_sS_2, H'_sT_2) = \frac{(s+1+\sqrt{2})^2+1}{4(s+1)^2}.$$

Theorem 15. *Let* $\lambda_\pm = 1 \pm \frac{\sqrt{2}}{2}$,
$U_\beta = \varphi(NS_1, NT_1)$, $V_\beta = \varphi(NS_2, NT_2)$,
$U_\gamma = \varphi(N^*S_1, N^*T_1)$, $V_\gamma = \varphi(N^*S_2, N^*T_2)$,
$U_\delta = \varphi(OS_1, OT_1)$ *and* $V_\delta = \varphi(OS_2, OT_2)$.
Consider the statements
(a) $m = \sqrt{2}$,
(β) $U_\beta = \lambda_+$, (β') $V_\beta = \lambda_+$, (β'') $U_\beta = V_\beta$,
(γ) $U_\gamma = \lambda_-$, (γ') $V_\gamma = \lambda_-$, (γ'') $U_\gamma = V_\gamma$,
(δ) $U_\delta = \frac{1}{2}$, (δ') $V_\delta = \frac{1}{2}$, (δ'') $U_\delta = V_\delta$.
The following are true:
(i) *The statements* (a), (β''), (γ'') *and* (δ'') *are equivalent.*
(ii) (a) *implies each of the statements* (β), (β'), (γ), (γ'), (δ) *and* (δ').

Proof. For the implication $(a) \Rightarrow (\delta)$, from the right-angled triangles OGS_1 and OHT_1 and Theorem 3, we get that the sum $OS_1^2+OT_1^2$ is equal to $(OG^2+GS_1^2)+(OH^2+HT_1^2)$, i. e., to $(OG^2+OH^2)+\frac{AB^2}{4}=\frac{1}{2}AB^2$. □

For points X and Y, let ϱ_X^Y be the reflection of the point X in the point Y. Let $Q=\varrho_A^D$, $R=\varrho_B^C$, $Q'=\varrho_A^{D'}$, $R'=\varrho_B^{C'}$.

Theorem 16. *Let* $U_\varepsilon=\varphi(A'Q, B'R)$ *and* $V_\varepsilon=\varphi(A'Q', B'R')$.
Consider the statements
(a) $\quad m=\sqrt{2}$,
$(\varepsilon) \quad U_\varepsilon=5, \qquad (\varepsilon') \quad V_\varepsilon=5, \qquad (\varepsilon'') \quad U_\varepsilon=V_\varepsilon$.
The following are true:
(i) *The statements* (a), (ε'') *are equivalent.*
(ii) (a) *implies each of the statements* (ε) *and* (ε').

Proof. Since $Q=\left(\frac{r(u+3\,z+2)}{\vartheta},0\right)$ and $R=\left(\frac{r(3\,u-3\,z-2)}{\vartheta},0\right)$, we get that $5-U_\varepsilon$ is equal to $\frac{\eta(m^2-2)(\vartheta+2\,z)}{m^2\,\vartheta^2}$. From this we conclude that $(a) \Rightarrow (\varepsilon)$.

From the right-angled triangles $AA'Q$ and $BB'R$, we get that $A'Q^2$ and $B'R^2$ are $4(a+b)^2+A'A^2$ and $4(b+c)^2+B'B^2$. By adding we conclude from (1) that $U_\varepsilon=5$. Also, we have $\varphi(N_5Q, N_6R)=\frac{5}{2}$ and $\varphi(AQ, BR)=4$. □

3. Common properties for all ratios

Of course, there are many properties that hold for all ratios m. The following are two examples of such properties.

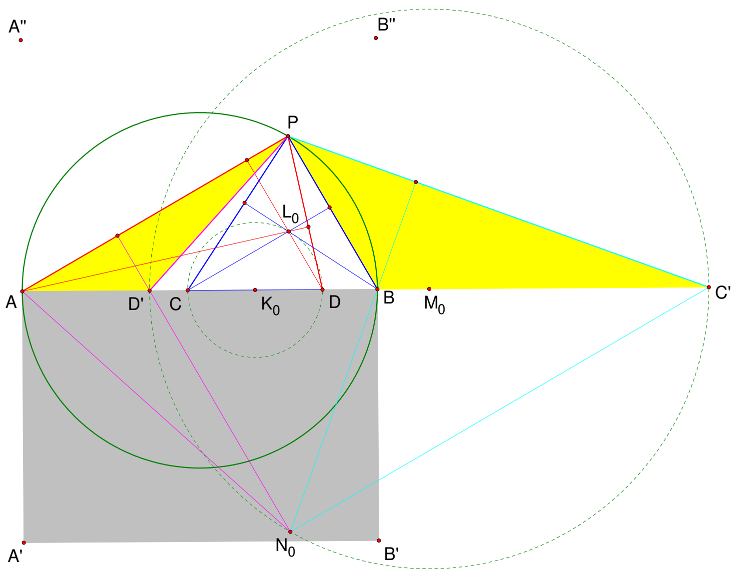

Figure 6. Common orthocenters in Theorem 17.

Theorem 17. (i) *The triangles ADP and BCP have the same orthocenter that lies on a circle with the segment CD as a diameter.*

(ii) *The triangles $AD'P$ and $BC'P$ have a common orthocenter. It lies on a circle with the segment $C'D'$ as a diameter.*

Proof. The orthocenters of the triangles ADP and BCP both have coordinates $\left(\frac{r\,u}{v}, \frac{2\,m\,r\,t^2}{\vartheta\,v}\right)$.

If K_0 is the midpoint of the segment CD and L_0 is the orthocenter of the triangle ADP, then $|CK_0|^2 - |K_0L_0|^2 = 0$ so that L_0 lies on the circle with the segment CD as a diameter. □

Theorem 18. *The equalities*

$$\varphi(AK, BL^*) = \varphi(AS^*, BT) \quad \textit{and} \quad \varphi(AK^*, BL) = \varphi(AS, BT^*)$$

hold for all points P on the circle and every ratio m.

Proof. Both $\varphi(AK, BL^*)$ and $\varphi(AS^*, BT)$ have the value $\frac{(m-1)^2+1}{2\,m^2}$ while both $\varphi(AK^*, BL)$ and $\varphi(AS, BT^*)$ have the value $\frac{(m+1)^2+1}{2\,m^2}$. □

4. Some other interesting ratios

Our last result is different because in it some other ratios besides $\sqrt{2}$ appear. This has happened already in a comment following the proof of Theorem 9. Let $\lambda = 1 - \frac{\sqrt{2}}{2}$.

Theorem 19. *The ratio m is either $\sqrt{2}$ or $2 + \sqrt{2}$ if and only if for every point P on the circle the equalities $\varphi(AK, BL^*) = \lambda$ and/or $\varphi(AS^*, BT) = \lambda$ hold.*

Proof. Since $\left(-\frac{r(\vartheta+t\,z)}{m\,v}, \frac{r(m-\eta)}{m\,v}\right)$ and $\left(\frac{r(m+\vartheta)}{m\,v}, \frac{r(t\,z-\eta)}{m\,v}\right)$ are the coordinates of the points K and L^*, we get that $\lambda - \varphi(AK, BL^*)$ will factor out as the quotient $\frac{\left(1-\sqrt{2}\right)\left(m-\sqrt{2}\right)\left(m-2-\sqrt{2}\right)}{2\,m^2}$. The same is true for $\varphi(AS^*, BT)$ by Theorem 18. □

Conclusion. Fermat conjectured that $(a) \Rightarrow (b)$ and Euler proved this. We added (b'), (b''), $(c), \ldots, (\varepsilon'')$ and discovered here that many are equivalent. Also, in related papers, we consider how to replace the circle with an arbitrary conic and propose the space versions for the sphere and some other surfaces (see [2], [3], [4] and [5]).

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