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A Generalization of the Conway Circle

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Abstract. For any point in the plane of the triangle we show a conic that becomes the Conway circle in the case of the incenter. We give some properties of the conic and of the configuration of the six points that define it.

Let ABC be a triangle and I its incenter. Call B_a the point on line CA in the opposite direction to AC such that $AB_a = BC = a$ and C_a the point on line BA in the opposite direction to AB such that $AC_a = a$. Define C_b, A_b and A_c, B_c cyclically. The six points A_b, A_c, B_c, B_a, C_a, C_b lie in a circle called the *Conway circle* with I as center and squared radius $r^2 + s^2$ as indicated in Figure 1. This configuration also appeared in Problem 6 in the 1992 Iberoamerican Mathematical Olympiad. The problem asks to establish that the area of the hexagon $C_aB_aA_bC_bB_cA_c$ is at least $13\Delta(ABC)$ (see [4]).

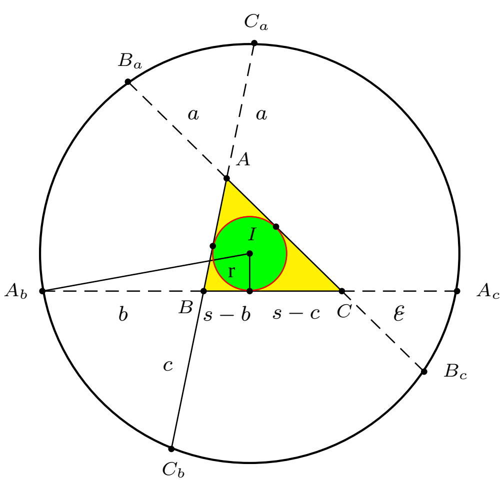

Figure 1. The Conway circle

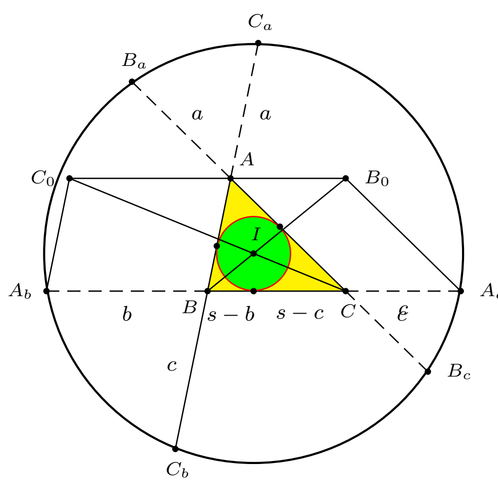

Figure 2

Figure 2 shows a construction of these points which can be readily generalized. The lines BI and CI intersect the parallel of BC through A at B_0 and C_0. The points A_b and A_c are obtained by completing the parallelograms BAC_0A_b and CAB_0A_c. If we take an arbitrary point $P = (u : v : w)$ instead of I, then $B_0 = (u : -w : w)$ and $C_0 = (u : v : -v)$. From these, we determine the points

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A_b, A_c, and analogously the other four points (see Figure 3). In homogeneous barycentric coordinates, these are

$$\begin{aligned} A_b &= (0 : u+v : -v), \quad A_c = (0 : -w : w+u); \\ B_c &= (-w : 0 : v+w), \quad B_a = (u+v : 0 : -u); \\ C_a &= (w+u : -u : 0), \quad C_b = (-v : v+w : 0). \end{aligned}$$

Proposition 1. *The area of ABC is the geometric mean of the areas of the triangles AB_aC_a, BC_bA_b, CA_cB_c.*

Proof. From the coordinates of these points, we have

$$\frac{\Delta(AB_aC_a)}{\Delta(ABC)} = \frac{u^2}{vw}, \quad \frac{\Delta(BC_bA_b)}{\Delta(ABC)} = \frac{u^2}{vw}, \quad \frac{\Delta(CA_cB_c)}{\Delta(ABC)} = \frac{u^2}{vw}.$$

Therefore, $\Delta(ABC)^3 = \Delta(AB_aC_a) \cdot \Delta(BC_bA_b) \cdot \Delta(CA_cB_c)$. □

Theorem 2. *For any point P, the six points A_b, A_c, B_c, B_a, C_a, C_b always lie on a conic.*

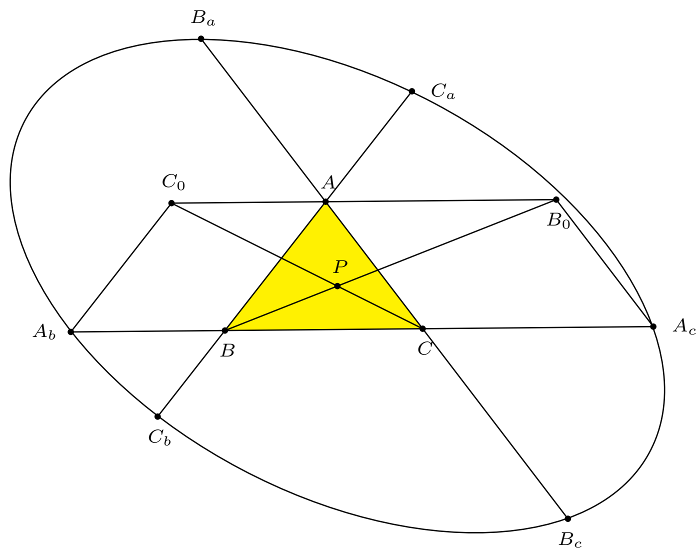

Figure 3

Proof. They all lie on the conic $\Gamma(P)$ with equation

$$\begin{aligned} &u(v+w)x^2 + v(w+u)y^2 + w(u+v)z^2 + (vw + (u+v)(u+w))yz \\ &+(wu + (v+w)(v+u))zx + (uv + (w+u)(w+v))xy \\ &= 0. \end{aligned}$$

□

From the matrix form $XMX^t = 0$ for $\Gamma(P)$ where

$$M = \begin{pmatrix} 2u(v+w) & (u+w)(v+w)+uv & (u+v)(v+w)+uw \\ (u+w)(v+w)+uv & 2v(u+w) & (u+v)(u+w)+vw \\ (u+v)(v+w)+uw & (u+v)(u+w)+vw & 2w(u+v) \end{pmatrix}$$

we have

$$|M| = 2(v+w-u)(u+v-w)(u-v+w)(uvw+u^2v+uv^2+u^2w+v^2w+uw^2+vw^2),$$

and $\Gamma(P)$ has discriminant

$$\delta(P) = -\frac{1}{4}(v+w-u)(u+v-w)(u-v+w)(u+v+w).$$

Therefore we can get the following results:

Proposition 3. (a) *The conic $\Gamma(P)$ is degenerate when P lies on the sides of the medial triangle or in the cubic K327.*

(b) *If the conic $\Gamma(P)$ is nondegenerate, it is homothetic to the circumconic with perspector $(u^2 : v^2 : w^2)$ and has center P.*

Proof. (b) follows from rewriting the equation of the conic $\Gamma(P)$ in the form

$$(u^2yz+v^2zx+w^2xy)+(x+y+z)(u(v+w)x+v(w+u)y+w(u+v)z) = 0.$$

□

Proposition 4. *The conic $\Gamma(P)$ and the circumconic with perspector P^2 have the same infinite points.*

If $P = (u : v : w)$ and $u = v + w$, then $\Gamma(P)$ factors as

$$(vx+wx+vy+2wy+wz)(vx+wx+vy+2vz+wz) = 0,$$

two parallel lines to the cevian AP. If P lies on $K327$, then $\Gamma(P)$ factors as two lines through P in the directions of the asymptotes of the circumconic with perspector P^2.

If P lies outside the sidelines of the medial triangle and the cubic $K327$, the conic $\Gamma(P)$ is an ellipse when P is interior to the medial triangle and a hyperbola otherwise.

Corollary 5. *The conic $\Gamma(P)$ is*
(a) *a circle if and only if P is the incenter or an excenter of the triangle,*
(b) *a rectangular hyperbola if and only if P lies on the polar circle.*

Remarks. (1) If $P = I_a$, the center of the excircle on the side BC, then $\Gamma(I_a)$ is a circle with squared radius $r_a^2 + (s-a)^2$

(2) The polar circle $S_Ax^2 + S_By^2 + S_Cz^2 = 0$ contains real points only if triangle ABC has an obtuse angle.

Proposition 6. (a) *If A', B', C' are the midpoints of B_aC_a, C_bA_b, A_cB_c respectively, the lines AA', BB', CC' concur at P.*

(b) *If A'', B'', C'' are the midpoints of B_cC_b, C_aA_c, A_bB_a respectively, the lines AA'', BB'', CC'' concur at P.*

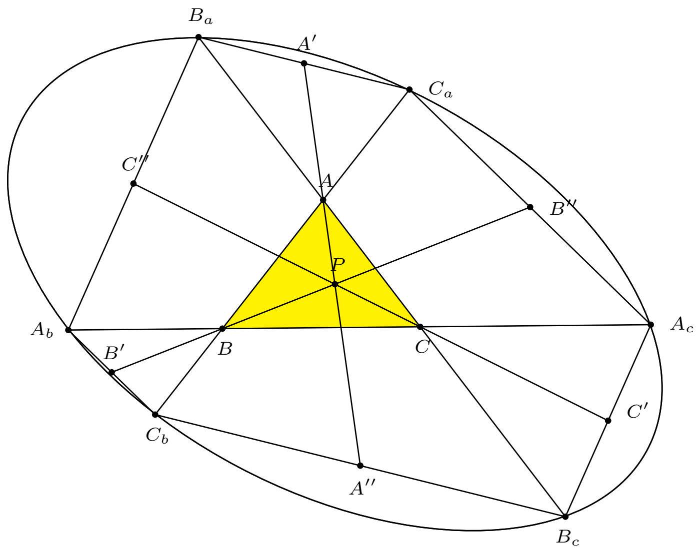

Figure 4

Proof. In homogeneous barycentric coordinates,

$$A' = (2vw + wu + uv : -uv : -uw),$$
$$A'' = (-(v^2 + w^2) : v(v + w) : w(v + w).$$

These points clearly lie on the line AP: $wy - vz = 0$. Similarly, B', B'' lie on BP and C', C'' lie on CP. □

Proposition 7. *If $P = (u : v : w)$, both of the triangles formed by the lines B_aC_a, C_bA_b, A_cB_c and the lines B_cC_b, C_aA_c, A_bB_a are perspective with ABC at the isotomic conjugate of the anticomplement of P,* i.e.,

$$Q = \left(\frac{1}{v + w - u} : \frac{1}{w + u - v} : \frac{1}{u + v - w} \right).$$

Proof. (a) The lines B_aC_a, C_bA_b, A_cB_c have equations

$$\begin{array}{rcrcrcl} ux & + & (w+u)y & + & (u+v)z & = & 0, \\ (v+w)x & + & vy & + & (u+v)z & = & 0, \\ (v+w)x & + & (w+u)y & + & wz & = & 0. \end{array}$$

They bound a triangle with vertices

$$X = (-u(u + v + w) : (v + w)(u + v - w) : (v + w)(w + u - v)),$$
$$Y = ((w + u)(v + w - u) : -v(u + v + w) : (w + u)(v + w - u)),$$
$$Z = ((u + v)(w + u - v) : (u + v)(w + u - v) : -w(u + v + w)).$$

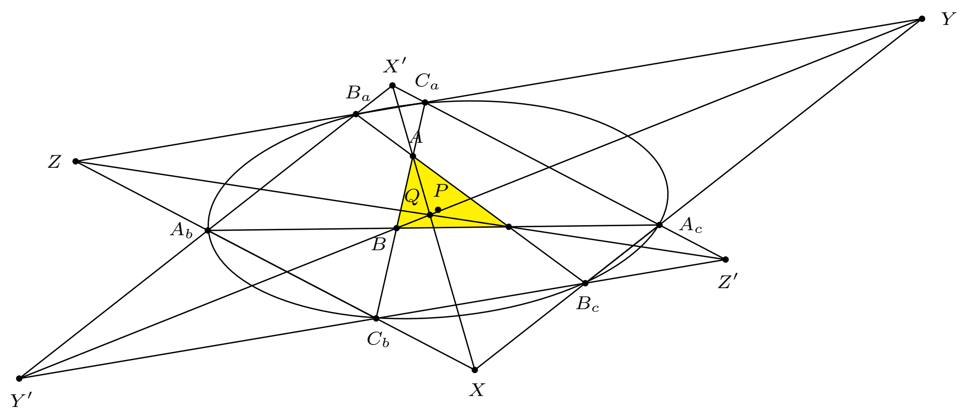

Figure 5

See Figure 5. This is perspective with ABC at

$$Q = \left(\frac{1}{v+w-u} : \frac{1}{w+u-v} : \frac{1}{u+v-w}\right).$$

(b) The lines B_cC_b, C_aA_c, A_bB_a have equations

$$\begin{array}{rcrcrcl} (v+w)x & + & vy & + & wz & = & 0, \\ ux & + & (w+u)y & + & wz & = & 0, \\ ux & + & vy & + & (u+v)z & = & 0. \end{array}$$

They bound a triangle with vertices

$$\begin{aligned} X' &= (-(u+v+w) : u+v-w : w+u-v), \\ Y' &= (v+w-u : -(u+v+w) : v+w-u), \\ Z' &= (w+u-v : w+u-v : -(u+v+w)). \end{aligned}$$

The triangle $X'Y'Z'$ is clearly perspective at the same point Q. □

In the case of the Conway configuration, this is the Gergonne point.

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