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On Polygons Admitting a Simson Line as Discrete Analogs of Parabolas

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Abstract. We call a polygon which admits a Simson line a *Simson polygon*. In this paper, we show that there is a strong connection between Simson polygons and the seemingly unrelated parabola. We begin by proving a few general facts about Simson polygons. We use an inductive argument to show that no convex n-gon, $n \geq 5$, admits a Simson line. We then determine a property which characterizes Simson n-gons and show that one can be constructed for every $n \geq 3$. We proceed to show that a parabola can be viewed as a limit of special Simson polygons, which we call *equidistant Simson polygons*, and that these polygons provide the best piecewise linear continuous approximation to the parabola. Finally, we show that equidistant Simson polygons can be viewed as discrete analogs of parabolas and that they satisfy a number of results analogous to the pedal property, optical property, properties of Archimedes triangles and Lambert's Theorem of parabolas. The corresponding results for parabolas are easily obtained by applying a limit process to the equidistant Simson polygons.

1. Introduction

The Simson-Wallace Theorem (see, e.g., [5]) is a classical result in plane geometry. It states that

Theorem 1. *(Simson-Wallace Theorem*[1]*). Given a triangle ABC and a point P in the plane, the pedal points of P (That is, the feet of the perpendiculars dropped from P to the sides of the triangle) are collinear if and only if P is on the circumcircle of triangle ABC.*

Such a line is called a Simson line of P with respect to triangle ABC.

A natural question is whether an n-gon with $n \geq 4$ can admit a Simson line. In [3, pp.137–144] and [4], it is shown that every quadrilateral possesses a unique

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[1]One remark concerning the theorem is that the Simson-Wallace Theorem is most commonly known as "Simson's Theorem", even though "Wallace is known to have published the theorem in 1799 while no evidence exists to support Simson's having studied or discovered the lines that now bear his name" [5]. This is perhaps one of the many examples of Stigler's law of eponymy.

Simson Line, called "the Simson Line of a complete[2] quadrilateral". We call a polygon which admits a Simson line a *Simson polygon*. In this paper, we show that there is a strong connection between Simson polygons and the seemingly unrelated parabola.

We begin by proving a few general facts about Simson polygons. We use an inductive argument to show that no convex n-gon, $n \geq 5$, admits a Simson Line. We then determine a property which characterizes Simson n-gons and show that one can be constructed for every $n \geq 3$. We proceed to show that a parabola can be viewed as a limit of special Simson polygons, called *equidistant Simson polygons*, and that these polygons provide the best piecewise linear continuous approximation to the parabola. Finally, we show that equidistant Simson polygons can be viewed as discrete analogs of parabolas and that they satisfy a number of results analogous to the pedal property, optical property, properties of Archimedes triangles and Lambert's Theorem of parabolas. The corresponding results for parabolas are easily obtained by applying a limit process to the equidistant Simson polygons.

2. General Properties of Simson Polygons

We begin with an easy Lemma. Throughout, we will use the notation that (XYZ) is the circle through points X, Y, Z.

Lemma 2. *Let S be a point in the interior of two rays AB and AC. Suppose that $ABSC$ is cyclic, and let X be a point on ray AB such that $|AX| < |AB|$. Let $Y = (AXS) \cap AC$. Then $|AY| > |AC|$.*

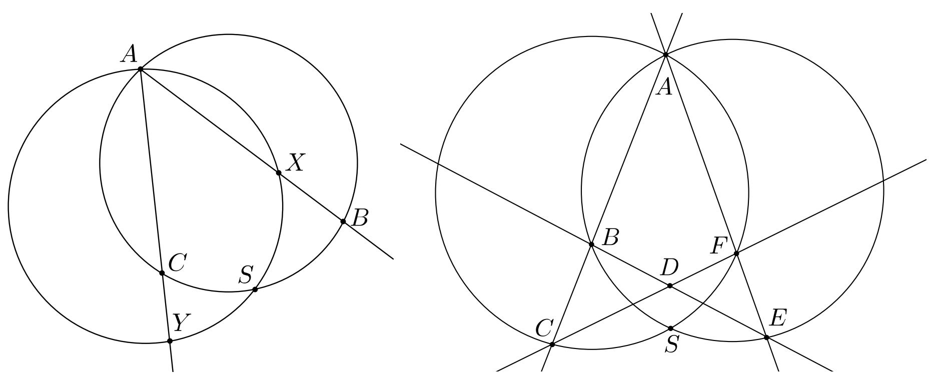

Figure 1. Lemma 2 and Lemma 3

Proof. Since $|AX| < |AB|$, $\angle AXS > \angle ABS$. Since $ABSC$ and $AXSY$ are cyclic, $\angle ACS = \pi - \angle ABS$ and $\angle AYS = \pi - \angle AXS$. Therefore $\angle AYS < \angle ABS$ so that $|AY| > |AC|$. $\square$

[2]A complete quadrilateral is the configuration formed by 4 lines in general position and their 6 intersections. When it comes to pedals, we are only concerned with the sides making up the polygon. Since we extend these, the pedal of a quadrilateral is equivalent to that of its complete counterpart. For this reason, we will refer to a polygon simply by the number of sides it has.

As mentioned in the introduction, in the case of a quadrilateral there is always a unique Simson point defined as a point from which the projections into the sides are collinear. Let A, B, C, D, E, F denote the vertices of the complete quadrilateral, as in fig. 1. It is shown in [3] that the Simson point is the unique intersection of $(AFC) \cap (ABE) \cap (BCD) \cap (DEF)$, also known as the *Miquel point of a complete quadrilateral*. Using Lemma 2, we can conclude the following:

Lemma 3. *Let $ABCDEF$ be a complete quadrilateral where points in each of the triples A, B, C; B, D, E, etc. as in fig. 1 are collinear and angle $\angle CDE$ is obtuse. Denote the Miquel point of $ABCDEF$ by S. There exist no two points X and Y on rays AF, AB respectively with $|AX| < |AF|$, $|AY| < |AB|$ such that (AXY) passes through S.*

Proof. The Miquel point S lies on (AFC) and (ABE). By Lemma 2, no such X and Y exist. □

We call a polygon for which no three vertices lie on a line nondegenerate. In Lemma 4 and Theorems 5 and 6 we will assume that the polygon is nondegenerate.

Lemma 4. *If $\Pi = V_1 \cdots V_n$, $n \geq 5$ is a convex Simson polygon, then Π has no pair of parallel sides.*

Proof. By the nondegeneracy assumption, it is clear that no two consecutive sides can be parallel. So suppose that $V_1V_2 \parallel V_iV_{i+1}$, $i \notin \{1, 2, n\}$. Then S lies on the Simson line L orthogonal to V_1V_2 and V_iV_{i+1}. The projection of S into each other side V_jV_{j+1} must also lie on L, so that either V_jV_{j+1} is parallel to V_1V_2 or it passes through S. By the nondegeneracy assumption, no two consecutive sides can pass through S. Therefore the sides of Π must alternate between being parallel to V_1V_2 and passing through S. It is easy to see that no such polygon can be convex. □

It is worth noting that both the convexity hypothesis and the restriction to $n \geq 5$ in the last result are necessary, for one can construct a non-convex n-gon, $n \geq 5$ having pairs of parallel sides and the trapezoid (if not a parallelogram) is a convex Simson polygon with $n = 4$ having a pair of parallel sides. Using the above result, we can prove:

Theorem 5. *A convex pentagon does not admit a Simson point.*

Proof. Let $\Pi = ABCDE$ be a nondegenerate convex pentagon. Suppose that S is a point for which the pedal in Π is a line. Then in particular the pedal is a line for every 4 sides of the pentagon. Therefore if $BC \cap DE = F$, then S must be a Simson point for $ABFE$, so that S is the Miquel point of $ABFE$. This implies that

$$S = (GAB) \cap (GFE) \cap (HAE) \cap (HBF),$$

where $BC \cap AE = G$ and $AB \cap DE = H$. By the same reasoning applied to quadrilateral $CGED$, S must be the Miquel point of $CGED$. Therefore S lies on (FCD). Because Π is convex, $|FC| < |FB|$ and $|FD| < |FE|$. We can now apply corollary 3 with C and D playing the role of points X and Y to conclude that S cannot lie on (FCD) - a contradiction. □

Consider a convex polygon Π as the boundary of the intersection of half planes $H_1, H_2, \ldots, H_n$. Then the polygon formed from the boundary of $\bigcap\limits_{\substack{i \neq k \\ i=1}}^{n} H_i$ for $k \in \{1, 2, \ldots, n\}$ is also convex.

We are now ready to prove the following result by induction:

Theorem 6. *A convex n-gon with $n \geq 5$ does not admit a Simson point.*

Proof. The base case has been established. Assume the hypothesis for $n \geq 5$, and consider the case for an $(n+1)$-gon Π with vertices $V_1, \ldots, V_{n+1}$. Suppose that Π admits a Simson point. Let $V_{n-1}V_n \cap V_{n+1}V_1 = V'$. This intersection exists by Lemma 4. Since Π admits a Simson point, $\Pi' = V_1 \ldots V_{n-1}V'$ must also admit one. By the preceding remark, Π' is convex, and since it has n sides, the hypothesis is contradicted. Therefore Π cannot admit a Simson line, completing the induction. □

Now that we have established that no convex n-gon (with $n \geq 5$) admits a Simson line, we will proceed to find a necessary and sufficient condition for an n-gon $\Pi = V_1V_2 \ldots V_n$ to have a Simson point. Let $W_i = V_{i-1}V_i \cap V_{i+1}V_{i+2}$ for each i, with $V_{n+k} = V_k$. In case that $V_{i-1}V_i$ and $V_{i+1}V_{i+2}$ are parallel, view W_i as a point at infinity and $(V_iW_iV_{i+1})$ as the line V_iV_{i+1}. For example, in a right-angled trapezoid with $AB \perp BC$ and $AB \perp AD$, S will necessarily lie on the line AB (in fact $S = AB \cap CD$).

Theorem 7. *An n-gon $\Pi = V_1 \cdots V_n$ admits a Simson point S if and only if all circles $(V_iW_iV_{i+1})$ have a common intersection.*

Proof. Assume first that S is a Simson point for Π. The projections of S into $V_{i-1}V_i$, V_iV_{i+1} and $V_{i+1}V_{i+2}$ are collinear. By the Simson-Wallace Theorem (Theorem 1), S is on the circumcircle of $V_iW_iV_{i+1}$.

Conversely, let $S = \bigcap\limits_i (V_iW_iV_{i+1})$. For each i, this implies that the projections of S into $V_{i-1}V_i$ and $V_{i+1}V_{i+2}$ are collinear. As i ranges from 1 to n we see that all projections of S into the sides are collinear. □

To construct an n-gon with a given Simson point S and Simson line L, let X_1, $X_2, \ldots, X_n$ be n points on L. The n lines the ith of which is perpendicular to SX_i and passing through X_i, $i = 1, \ldots, n$ are the sides of an n-gon with Simson point S and Simson line L. The X_i are the projections of S into the sides of the n-gon and the vertices V_i are the intersections of consecutive pairs of sides.

3. Simson Polygons and Parabolas

In this section we will show that there is a strong connection between Simson polygons and parabolas. In particular, we may view a special type of Simson polygons, which we call equidistant Simson polygons, as discrete analogs of the parabola.

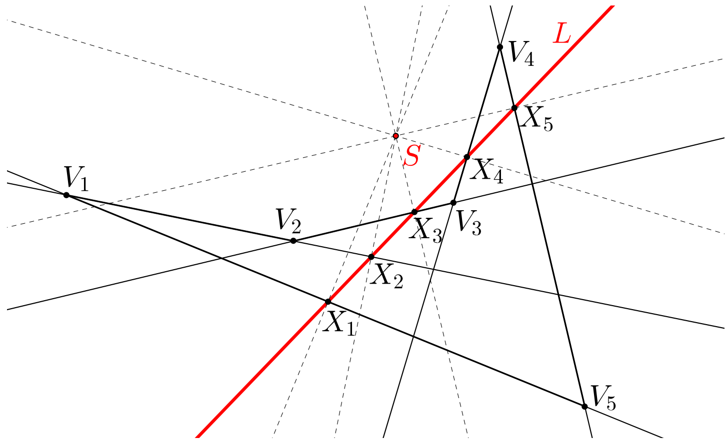

Figure 2. A construction of a pentagon $V_1V_2V_3V_4V_5$ with Simson point S and Simson line L. The points $X_1, \ldots, X_5$ on L are the projections of S into the sides of the pentagon.

Definition. Let $\Pi = V_1 \cdots V_n$ be a Simson polygon with Simson point S and projections $X_1, \ldots, X_n$ of S into its sides. In the special case that $|X_iX_{i+1}| = \Delta$ for each $i = 1, \ldots, n-1$, we call such a polygon Π an *equidistant Simson polygon*.

The following result shows that all but one of the vertices of an equidistant Simson polygon lie on a parabola. Moreover, the parabola is independent of the position of X_1 (but depends on Δ).

Theorem 8. *Let S be a point and L a line not passing through S. Suppose that $X_1, \ldots, X_n$ are points on L such that $|X_iX_{i+1}| = \Delta$ for all $i = 1, \ldots, n-1$ and let $\Pi = V_1 \cdots V_n$ be the equidistant Simson polygon with Simson point S and projections $X_1, \ldots, X_n$ of S into its sides. Then $V_1, \ldots, V_{n-1}$ lie on a parabola C. Moreover, C is independent of the position of X_1 on L.*

Proof. Without loss of generality, let $S = (0, s)$, L be the x-axis, $X_i = (X + (i-1)\Delta, 0)$ and $X_{i+1} = (X + i\Delta, 0)$. A calculation shows that the perpendiculars at X_i and X_{i+1} to the segments SX_i and SX_{i+1}, respectively, intersect at the point $(2X + (2i-1)\Delta, \frac{(X+(i-1)\Delta)(X+i\Delta)}{s})$. Therefore the coordinates of the intersection satisfy $y = \frac{x^2-\Delta^2}{4s}$ independently of X. It follows that $V_1, \ldots, V_{n-1}$ lie on the parabola $y = \frac{x^2-\Delta^2}{4s}$. □

The fact that C is independent of the position of X_1 on L can be illustrated on figure 3 by supposing that $X_1, X_2, \ldots, X_8$ are being translated on L as a rigid body. Then the independence of C from X_1 implies that C remains fixed and $V_1, \ldots, V_7$ slide together about C.

Corollary 9. *Let C be a parabola with focus F. The locus of projections of F into the lines tangent to C is the tangent to C at its vertex.*

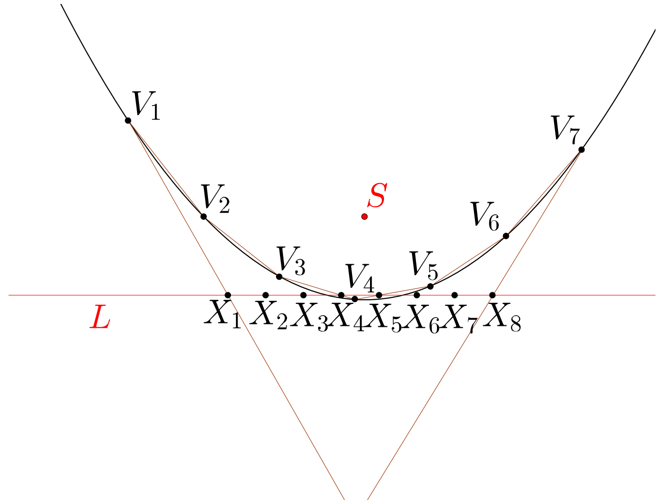

Figure 3. Points $V_1, \dots, V_8$ are the vertices of an equidistant Simson octagon with a Simson point S, Simson line L and projections $X_1, \dots, X_8$. By Theorem 8, $V_1, \dots, V_7$ lie on a parabola.

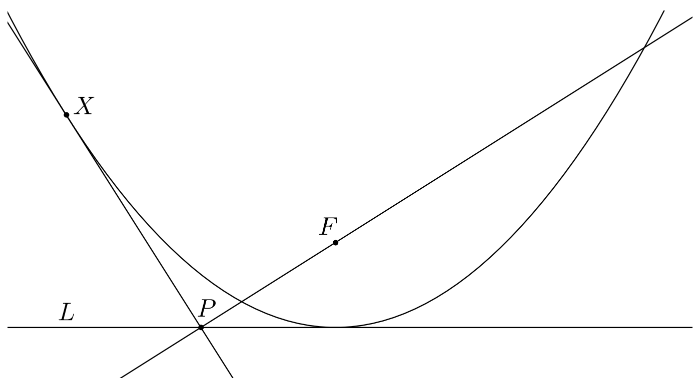

Figure 4. Corollary 9: X is a variable point of C, F is the focus, P is the projection of F into the tangent at X and L is the tangent to C at its vertex.

Proof. As seen in the proof of Theorem 8, the coordinates of the V_i, $i = 1, \dots, n$ are continuous functions of Δ. Therefore as $n \to \infty$ and $\Delta \to 0$ in Theorem 8, the limit of the polygon is a parabola with focus S and tangent line at the vertex equal to L. □

This property can be equivalently stated as: "the pedal curve of the focus of a parabola with respect to the parabola is the line tangent to it at its vertex". This property is by no means new, but its derivation does give a nice connection between the pedal of a polygon and the pedal of the parabola. Specifically, we can view the focus F as the Simson point of a parabola (considered as a polygon with infinitely many points) and the tangent at the vertex as the Simson line of the parabola.

Let $V_1 \cdots V_{n+2}$ be an equidistant Simson polygon. We will now prove that the sides connecting the vertices $V_1, V_2, \ldots, V_{n+1}$ form an optimal piecewise linear continuous approximation of the parabola. To be precise, we show that it is a solution to the following problem:

Problem. Consider a continuous piecewise linear approximation $l(x)$ of a parabola $f(x)$, $x \in [a, b]$ obtained by connecting several points on the parabola. That is, let

$$l(x) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_{i+1}) + f(x_{i+1}) \text{ for } x \in [x_i, x_{i+1}]$$

where $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Find $x_1, x_2, \ldots, x_{n-1} \in (a, b)$ such that the error

$$\int_a^b |f(x) - l(x)|dx$$

is minimal.

The points $(x_i, f(x_i))$, $i = 0, \ldots, n$ are called knot points and a continuous piecewise linear approximation which solves the problem is called optimal. Since all parabolas are similar, it suffices to consider $f(x) = \frac{x^2 - \Delta^2}{4s}$.

Theorem 10. *The optimal piecewise-continuous linear approximation to $f(x)$ with the setup above is given by the sides V_1V_2, $V_2V_3, \ldots, V_nV_{n+1}$ of an equidistant Simson $(n+2)$-gon with $X_1 = \frac{a}{2}$, $\Delta = \frac{b-a}{n}$ and $V_i = (a+(i-1)\Delta, f(a+(i-1)\Delta))$. The knot points $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$ are the vertices $V_1, V_2, \ldots, V_{n+1}$.*

Proof. The equation of the ith line segment simplifies to

$$l(x) = \frac{x(x_{i+1} + x_i) - x_i x_{i+1} - \Delta^2}{4s}, \text{ for } x \in [x_i, x_{i+1}].$$

Therefore $f(x) - l(x) = \frac{(x - x_{i+1})(x - x_i)}{4s}$ for $x \in [x_i, x_{i+1}]$. Integrating $|f(x) - l(x)|$ from x_i to x_{i+1} we get

$$\int_{x_i}^{x_{i+1}} |f(x) - l_i(x)|dx = \frac{(x_{i+1} - x_i)^3}{24|s|}.$$

It is enough to minimize

$$S(x_1, \ldots, x_{n-1}) = 24|s| \int_a^b |f(x) - l(x)|dx = \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3.$$

Taking the partial derivative with respect to x_i for $1 \leq i \leq n-1$ and setting to zero, we get

$$\frac{\partial}{\partial x_i} S(x_1, \ldots, x_{n-1}) = 3(x_i - x_{i-1})^2 - 3(x_{i+1} - x_i)^2 = 0$$
$$\iff (x_i - x_{i-1})^2 = (x_{i+1} - x_i)^2.$$

Since the points are ordered and distinct, $x_i = \frac{x_{i+1}+x_{i-1}}{2}$, so that the x_i's form an arithmetic progression. The x-coordinates of the vertices V_i satisfy this relation, and by uniqueness, the theorem is proved.

□

By similar reasoning, one can see that the same sides of the $(n+2)$-gon are also optimal if the problem is modified to solving the least-squares problem

$$\min_{x_1,\dots,x_{n-1}} \int_a^b (f(x)-l(x))^2 dx.$$

From the proof of Theorem 10, we have the following interesting result about parabolas.

Corollary 11. *Let $f(x)$ be the equation of parabola, Δ be a real number and let $l(x)$ be the line segment with end points $(y, f(y)), (y+\Delta, f(y+\Delta))$. Then the area*

$$\int_y^{y+\Delta} |f(x)-l(x)| dx$$

bounded by $f(x)$ and $l(x)$ is independent of y.

This property also explains why the x-coordinates of the knot points of the optimal piecewise linear continuous approximation of the parabola are at equal intervals.

We now list some of the properties of equidistant Simson polygons:

Theorem 12. *An equidistant Simson polygon $V_1V_2\dots V_n$ with projections $X_1, X_2, \dots, X_n$ has the following properties:*

(1) If $j-i>0$ is odd, the segments V_iV_j, $V_{i+1}V_{j-1}$, …,$V_{\frac{j+i+1}{2}}V_{\frac{j+i-1}{2}}$ are parallel for every $i, j \in \{1, 2, \dots, n-1\}$.
(2) If $j-i>0$ is even, the segments V_iV_j, $V_{i+1}V_{j-1}$, …,$V_{\frac{j+i}{2}-1}V_{\frac{j+i}{2}+1}$ and the tangent to the parabola at $V_{\frac{j+i}{2}}$ are parallel for every $i, j \in \{1, 2, \dots, n-1\}$.
(3) The midpoints of the parallel segments in (1) (respectively (2)) lie on a line orthogonal to the Simson line L.

Proof. (1). The slope between V_i and V_j is easily calculated to be $\frac{2X+(i+j-1)\Delta}{2s}$.

(2). Recall that the parabola is given by $y = \frac{x^2-\Delta^2}{4s}$ so that its slope at $V_{\frac{j+i}{2}}$ is $\frac{2X+(2(\frac{j+i}{2})-1)\Delta}{2} = \frac{2X+(j+i-1)\Delta}{2}$.

(3). The x-coordinate of the midpoint of V_iV_j is $2X+(i+j-1)\Delta$. □

The following property of Simson polygons can be viewed as a discrete analog of the isogonal property of the parabola.

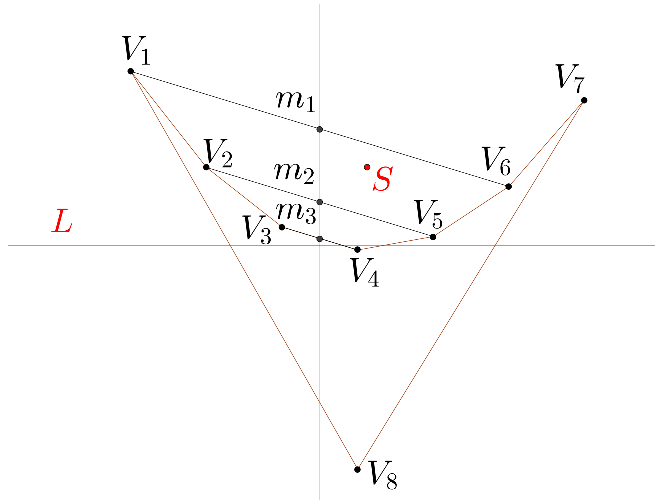

Figure 5. Points $V_1, \dots, V_8$ are the vertices of an equidistant Simson octagon with Simson point S and Simson line L. By Theorem 12, the segments V_1V_6, V_2V_5 and V_3V_4 are parallel, and their midpoints m_1, m_2 and m_3 all lie on a line perpendicular to L.

Property 1. *Let S and L be the Simson point and Simon line of a Simson polygon (not necessarily equidistant) with vertices $V_1, \dots, V_n$ and define $X_1, \dots, X_n$ as before. Let V_i' be the reflection of V_i in L. Then the lines V_iX_i and V_iX_{i+1} are isogonal with respect to the lines V_iV_i' and V_iS (i.e. $\angle V_i'V_iX_i = \angle X_{i+1}V_iS$) for $i = 1, \dots, n$.*

Proof. The proof is by a straightforward angle count. □

In the case when the Simson polygon in Proposition 1 is equidistant, we can take limits to obtain the isogonal property of the parabola:

Corollary 13. *Let C be a parabola with focus F and tangent line L at its vertex. Let X be any point on C and K the tangent at X. Furthermore, let X' be the reflection of X in L. Then K forms equal angles with $X'X$ and FX.*

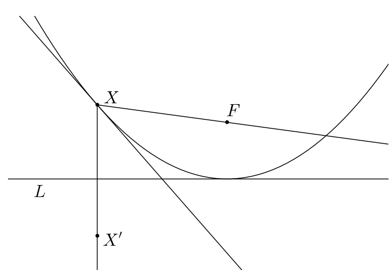

Figure 6. Corollary 13: X is a variable point of the parabola C, F is the focus, L is the tangent to C at its vertex and X' is the reflection of X in L. The lines XF and XX' form equal angles with the tangent at X.

Using the same setup as in Theorem 8 for an equidistant Simson polygon,

Theorem 14. *Let M_i be the midpoint of V_iV_{i+1}, $i = 1, \ldots, n-2$. Then the midpoints M_i lie on a parabola C' with focus S and tangent line at its vertex L.*

Proof. Since $V_i = (2X + (2i-1)\Delta, \frac{(x+(i-1)\Delta)(x+i\Delta)}{s})$,

$$M_i = (2(X + i\Delta), \frac{(X + i\Delta)^2}{s}).$$

Therefore the M_i lie on the parabola $p(x) = \frac{x^2}{4s}$ with focus S. The slope of V_iV_{i+1} is $\frac{X+i\Delta}{s}$, which is the same as that of $p(x)$ at M_i.

□

In a coordinate system where S lies above L, the parabolas C and C' form sharp upper and lower bounds to the piecewise linear curve $f(x)$ formed by the sides connecting $V_1, \ldots, V_{n-1}$ (discussed in Theorem 10). Informally, one can think of C and C' as “sandwiching” $f(x)$, and in the limit $n \to \infty$ and $\Delta \to 0$, the two curves coincide and equal the limit of the polygon.

The following result is a discrete analog of the famous optical reflection property of the parabola.

Corollary 15. *Let M_i be the midpoints of V_iV_{i+1} as in Theorem 14 and p_i be the line passing through M_i orthogonal to L for $i = 1, 2, \ldots, n-2$. Then the reflection p_i' of p_i in V_iV_{i+1} passes through S for each $i = 1, 2, \ldots, n-2$.*

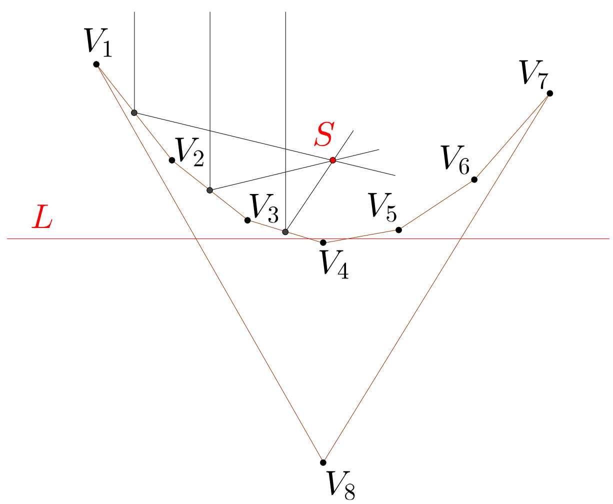

Figure 7. Corollary 15: $\Pi = V_1 \cdots V_8$ is an equidistant Simson polygon. The reflections at the midpoints of the sides of Π of rays orthogonal to L pass through S.

Let X and Y be two points on a parabola C. The triangle formed by the two tangents at X and Y and the chord connecting X and Y is called an *Archimedes Triangle* [2]. The chord of the parabola is called the triangle's base. One of the results stated in Archimedes' Lemma is that if Z is the vertex opposite to the base of an Archimedes triangle and M is the midpoint of the base, then the median MZ is parallel to the axis of the parabola. The following result yields a discrete analog to Archimedes' Lemma. Let $V_1 \cdots V_n$ be an equidistant Simson polygon.

Theorem 16. *Let $W_{i,j} = V_iV_{i+1} \cap V_jV_{j+1}$ for each $i, j \in \{1, 2, \ldots, n-2\}$ and $i \neq j$. Let $M_{i,j+1}$ and $M_{i+1,j}$ be the respective midpoints of chords V_iV_{j+1} and $V_{i+1}V_j$. Then $W_{i,j}M_{i,j+1}$ and $W_{i,j}M_{i+1,j}$ are orthogonal to L.*

Proof. As shown in the proof of Theorem 12, the x-coordinate of $M_{i,j+1}$ is $2X + (i+j)\Delta$ and that of $M_{i+1,j}$ is the same. The point $W_{i,j}$ is the intersection of the line V_iV_{i+1} given by $y = \frac{X+i\Delta}{s}x - \frac{(X+i\Delta)^2}{s}$ and the line V_jV_{j+1} given by $y = \frac{X+j\Delta}{s}x - \frac{(X+j\Delta)^2}{s}$, so that $W_{i,j} = (2X + (i+j)\Delta, \frac{(X+i\Delta)(X+j\Delta)}{s})$. □

Corollary 17. *The points $W_{i,j+1}, W_{i+1,j}$, $W_{i+2,j-1}$, etc. and the points $M_{i,j+1}$, $M_{i+1,j}$, $M_{i+2,j-1}$, are collinear. The line on which they lie is orthogonal to L.*

Taking limits, we get the following Corollary which includes the part of Archimedes' Lemma stated previously:

Corollary 18. *The vertices opposite to the bases of all Archimedes triangles with parallel bases lie on a single line parallel to the axis of the parabola and passing through the midpoints of the bases.*

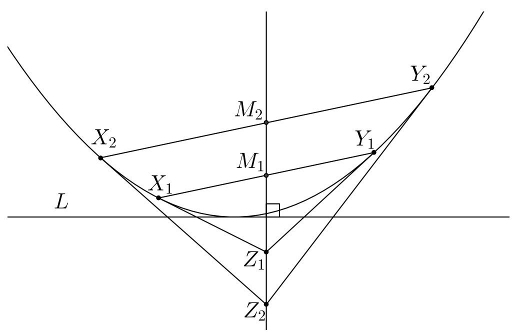

Figure 8. Corollary 18: Triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are two Archimedes triangles with parallel bases X_1Y_1, X_2Y_2. Points Z_1,Z_2 and the midpoints of the bases M_1, M_2 all lie on a line parallel to the axis of the parabola.

The final theorem to which we give generalization is *Lambert's Theorem*, which states that the circumcircle of a triangle formed by three tangents to a parabola passes through the focus of the parabola [2]. We can prove it using the Simson-Wallace Theorem.

Theorem 19. *Let $V_1 \cdots V_n$ be a Simson polygon (not necessarily equidistant) with Simson point S. Let $i, j, k \in \{1, 2, \ldots, n\}$, be distinct. Then the circumcircle of the triangle T formed from lines V_iV_{i+1}, V_jV_{j+1} and V_kV_{k+1} passes through S.*

Proof. Since the projections of S into V_iV_{i+1}, V_jV_{j+1} and V_kV_{k+1} are collinear, S is a Simson point of the triangle T. Therefore by the Simson-Wallace Theorem (Theorem 1), S lies on the circumcircle of T. □

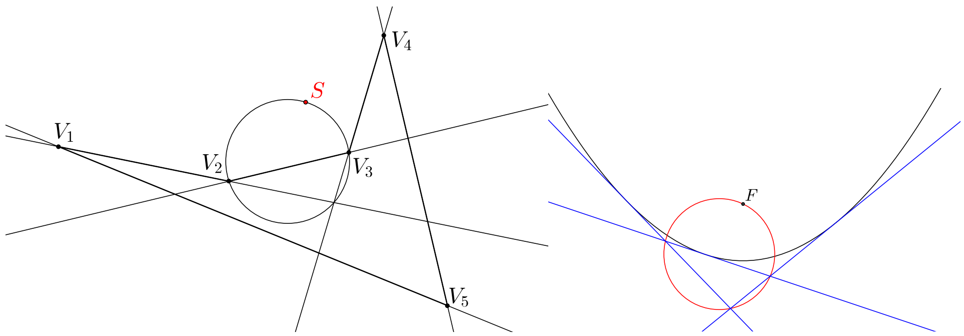

Figure 9. Theorem 19 and Corollary 20.

Corollary 20. *(Lambert's Theorem). The focus of a parabola lies on the circumcircle of a triangle formed by any three tangents to the parabola.*

Proof. Taking the limit of a sequence of equidistant Simson polygons gives Lambert's Theorem for a parabola, since the lines V_iV_{i+1}, V_jV_{j+1}, V_kV_{k+1} become tangents in the limit. □

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