Forum Geometricorum
Volume 13 (2013) 209–218.

FORUM GEOM
ISSN 1534-1178

Three Natural Homoteties of The Nine-Point Circle

Mehmet Efe Akengin, Zeyd Yusuf Köroğlu, and Yiğit Yargiç

Abstract. Given a triangle with the reflections of its vertices in the opposite sides, we prove that the pedal circles of these reflections are the images of nine-point circle under specific homoteties, and that their centers form the anticevian triangle of the nine-point center. We also construct two concentric circles associated with the pedals of these reflections on the sidelines, and study the triangle bounded by the radical axes of these pedal circles with the nine-point circle.

1. Three pedal circles

Given a triangle ABC with angles α, β, γ, circumcenter O, orthocenter H, and nine-point center N, we let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, H_a, H_b, H_c the pedals of A on BC, B on CA, C on AB respectively. Consider also the reflections A' of A in BC, B' of B in CA, and C' of C in AB. Our first result (Theorem 3 below) is about the pedal circles of A', B', C' with respect to triangle ABC.

Construct the circle through O, B, C, and let O_a be the second intersection of this circle with the line AO.

Proposition 1. *O_a and A' are the isogonal conjugates in triangle ABC.*

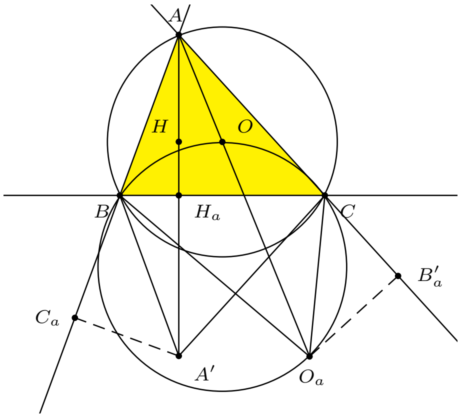

Figure 1

Proof. Clearly the lines AO_a and AH are isogonal with respect to angle A, since O and H are isogonal conjugates. Also,

$$\angle A'BC_a = 2\angle A'AC_a = 2\angle HAB = 2\angle OAC = \angle O^aOC = \angle O^aBC.$$

Publication Date: November 12, 2013. Communicating Editor: Paul Yiu.

Therefore, the lines $A'B$ and O^aB are symmetric in the external bisector of angle B, and so are isogonal with respect to angle B. Similarly, $A'C$ and O^aB are isogonal with respect to angle C. This shows that A' and O^a are isogonal conjugates. □

The points A' and O_a have a common pedal circle, with center at the midpoint N^a of $A'O_a$.

Proposition 2. *O_aA' is parallel to OH.*

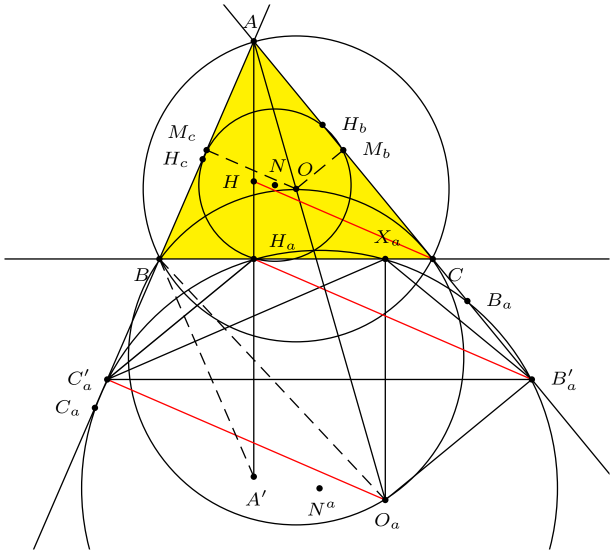

Figure 2

Proof. Let X_a, B'_a, C'_a be the pedals of O_a on BC, CA, AB respectively. From

$$\frac{AM_b}{AB'_a} = \frac{AO}{AO_a} = \frac{AM_c}{AC'_a},$$

we have $B'_aC'_a//M_bM_c//BC$. Therefore the cyclic quadrilateral $B'_aC'_aH_aX_a$, having a pair of parallel sides, must be a symmetric trapezoid. Now,

$$\angle C'_aX_aO_a = \angle C'_aBO_a = \angle CBA' = \beta = \angle C'_aO_aX_a.$$

The second equality is valid because O_aB and $A'B$ are isogonal with respect to B, and the last one because B, X_a, O_a, C'_a are concyclic. It follows that $C'_aO_a = C'_aX_a = B'_aH_a$. Similarly, $B'_aO_a = C'_aH_a$. Therefore, $C'_aO_aB'_aH_a$ is a parallelogram, and B'_aH_a is parallel to $O_aC'_a$, and also to CH, being all perpendicular to AB.

Since M_b and M_c are the midpoints of AC and AB, we have

$$\frac{AO}{AO_a} = \frac{AM_b}{AB'_a} = \frac{AC}{2 \cdot AB'_a} = \frac{AH}{2 \cdot AH_a} = \frac{AH}{AA'}.$$

Therefore, O_aA' is parallel to OH. □

Theorem 3. *The pedal circle of A' (and O_a) is the image of the nine-point circle of ABC under the homothety $\mathsf{h}(A, t_a)$, where $t_a = \frac{2\sin\beta\sin\gamma}{\cos\alpha}$.*

Proof. The circle $B_aB'_aC'_a$ is homothetic to the nine-point circle $H_bM_bM_c$ at A since

$$\frac{AB_a}{AH_b} = \frac{AA'}{AH} = \frac{AO_a}{AO} = \frac{AB'_a}{AM_b} = \frac{AC'_a}{AM_c}.$$

The ratio of homothety is

$$t_a = \frac{AA'}{AH} = \frac{2\cdot AH_a}{2\cdot OM_a} = \frac{2R\sin\beta\sin\gamma}{2R\cos\alpha} = \frac{\sin\beta\sin\gamma}{\cos\alpha}.$$

.

Since the center N^a of the pedal circle of A' and O_a is the midpoint of O_aA', the line AN^a intersects OH at its midpoint N, the nine-point center of ABC. □

Analogously let O_b, O_c be the second intersections of the circles OCA, OAB with the lines BO, CO respectively. The common pedal circle of B' and O_b has center N^b the midpoint of O_bB' and that of C' and O_c has center N^c the midpoint of O_cC'. These pedal circles are images of the nine-point circle under the homotheties $\mathsf{h}(B, t_b)$ and $\mathsf{h}(C, t_c)$ with $t_b = \frac{\sin\gamma\sin\alpha}{\cos\beta}$ and $t_c = \frac{\sin\alpha\sin\beta}{\cos\gamma}$ respectively.

Theorem 4. *$N^aN^bN^c$ is the anticevian triangle of the nine-point N.*

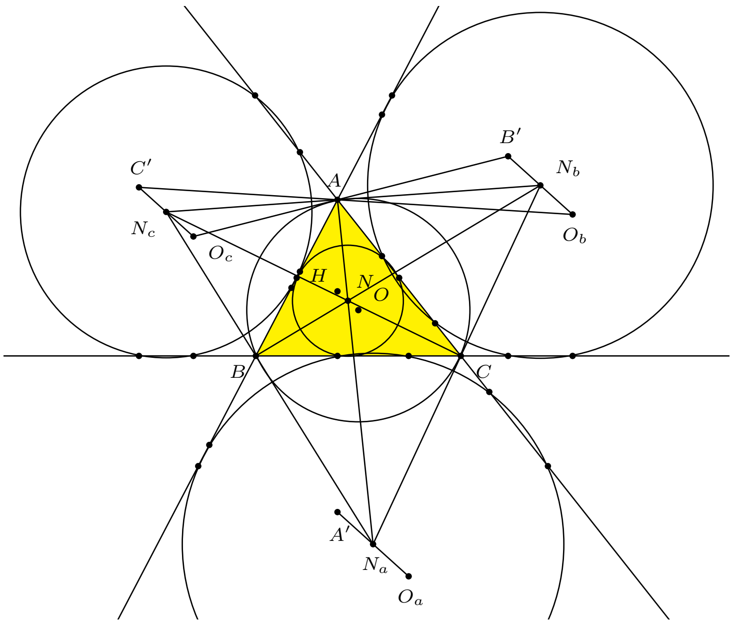

Figure 3

Proof. Since N^a is the midpoint of O_aA' and the nine-point center N is the midpoint of OH, by Proposition 2, A, N, N_a are collinear. Similarly, N^b and N^c are on the cevians BN and CN respectively. We show that the line N^bN^c, N^cN^a,

N^aN^b contain A, B, C respectively. From this the result follows. It is enough to show that N^bN^c contains A. For this, note that B', A, O_c are collinear because

$$\begin{aligned}\angle B'AB + \angle BAO_c &= 2\angle B'AC + \angle BOO_c \\ &= 2\alpha + (180^\circ - \angle BOC) \\ &= 2\alpha + (180^\circ - 2\alpha) \\ &= 180^\circ.\end{aligned}$$

Similarly, O_b, A, and C' are collinear. Therefore, the midpoints of O_bB' and O_cC', namely, N^b and N^c, are collinear with A. □

2. Two concentric circles associated with six pedals

Let $A''B''C''$ be the triangle bounded by the lines B_aC_a, C_bA_b, and A_cB_c.

Theorem 5. *The incenter of triangle $A''B''C''$ is the orthocenter of the orthic triangle $H_aH_bH_c$, and the incircle touches the sides at the midpoints P_a, P_b, P_c of the segments B_aC_a, C_bA_b, A_cB_c respectively.*

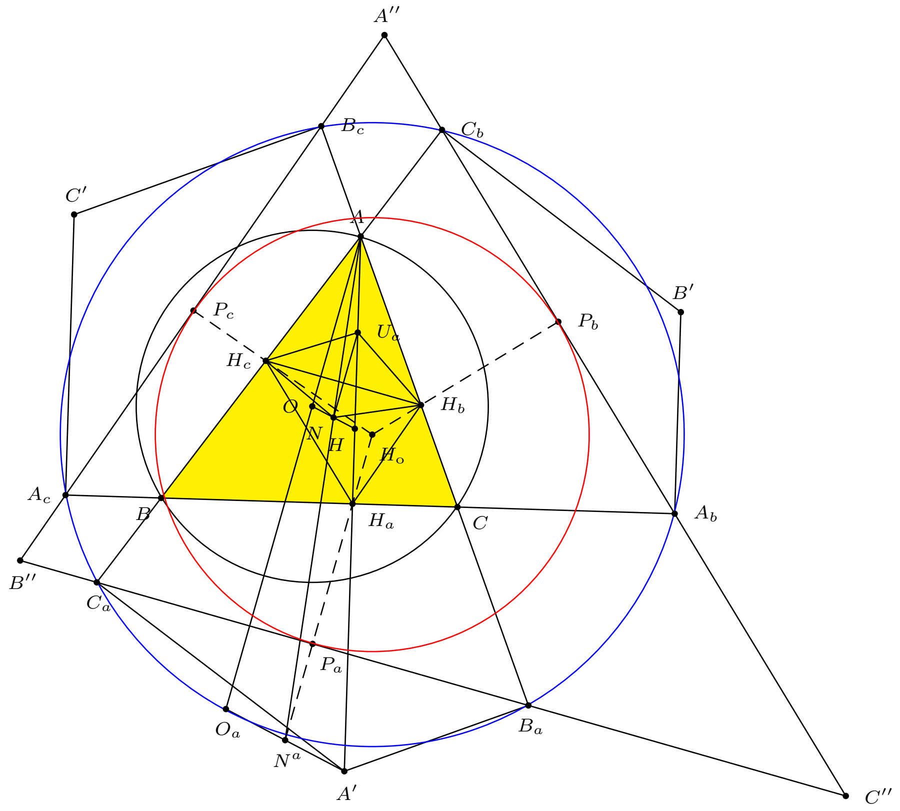

Figure 4.

Proof. We first claim that the segments B_aC_a, C_bA_b, A_cB_c have equal lengths. Note that the homothety $\mathsf{h}(A,\ t_a)$ maps H_b, H_c to B_a, C_a respectively. Hence,

$$B_aC_a = t_a \cdot H_bH_c = \frac{\sin\beta\sin\gamma}{\cos\alpha} \cdot 2R\sin\alpha\cos\alpha = 4R\sin\alpha\sin\beta\sin\gamma.$$

Since this expression is symmetric in α, β, γ, it also gives the lengths of C_bA_b and A_cB_c.

Note that the corresponding sidelines of triangles $A''B''C''$ and the orthic triangle $H_aH_bH_c$ are parallel. The two triangles are homothetic. By parallelism,

$$\angle A''A_bA_c = \angle H_cH_aA = \angle CH_aH_b = \angle A_bA_cC''.$$

Therefore, $A''A_cA_b$ is an isosceles triangle with $A''A_b = A''A_c$. Since $A_bC_b = A_cB_c$, we deduce that $A''P_b = A''P_c$. Similarly, $B''P_c = B''P_a$ and $C''P_a = C''P_b$. Hence, P_a, P_b, P_c are the points of tangency of the incircle of triangle $A''B''C''$ with its sides.

Next we claim that H_a, N_a and P_a all lie on a line perpendicular to B_aC_a. Let U_a be the midpoint of AH. Since NU_a is parallel to OA, it is perpendicular to B_bH_c. As $NH_b = NH_c$, the line NU_a is the perpendicular bisector of H_bH_c. The homothety $\mathsf{h}(A,\ t_a)$ maps $U_aH_bNH_c$ into $H_aB_aN^aC_a$, and H_aN^a is the perpendicular bisector of B_aC_a. Therefore, it passes through the midpoint P_a of B_aC_a. Since H_aN^a is perpendicular to H_bH_c, it passes through the orthocenter of the orthic triangle $H_aH_bH_c$. The same is true for the other two lines H_bN^b and H_cN^c, which are the perpendiculars to the sides $C''A''$ and $A''B''$ at the points P_b and P_c respectively. Therefore, the incenter of $A''B''C''$ is the orthocenter of the orthic triangle. □

Remarks. (1) The common length of B_aC_a, C_bA_b, A_cB_c is also the perimeter of the orthic triangle, being $4R\sin\alpha\sin\beta\sin\gamma = R(\sin 2\alpha + \sin 2\beta + \sin 2\gamma)$.

(2) The orthocenter of the orthic triangle is the triangle center $X(52)$ in [3].

Corollary 6. *The lines N^aH_a, N^bH_b, N^cH_c are concurrent at H_o.*

Theorem 7. *The six pedals A_b, A_c, B_c, B_a, C_a, C_b lie on a circle with center H_o.*

Proof. From Theorem 5, we have $H_\text{o}P_a = H_\text{o}P_b = H_\text{o}P_c$. Also recall from the proof of the same theorem, the segments B_aC_a, C_bA_b, A_cB_c have equal lengths. Therefore, $H_\text{o}B_aC_a$, $H_\text{o}C_bA_b$, and $H_\text{o}A_cB_c$ are congruent isosceles triangles, and H_o is the center of a circle containing these six pedals (see Figure 4). □

Theorem 8. *The triangles ABC, $A''B''C''$, and $P_aP_bP_c$ are perspective at the symmedian point of triangle ABC*

Proof. (1) Since AA_cA_b and AB_cC_b are isosceles triangles, B_cC_b and A_cA_b are parallel, and the triangles AC_bB_c and ABC are homothetic (see Figure 5). Now,

$$\angle A''B_cC_b = \angle A''A_cA_b = \angle H_bH_aC = \alpha = \angle B_cAC_b.$$

Similarly, $\angle A''C_bB_c = \angle B_cAC_b$. Therefore, $A''B_c$ and $A''C_b$ are tangents from A'' to the circumcircle of triangle AC_bB_c. The line $A''A$ is a symmedian of triangle AC_bB_c. Since ABC and AC_bB_c are homothetic at A, the same line $A''A$ is a

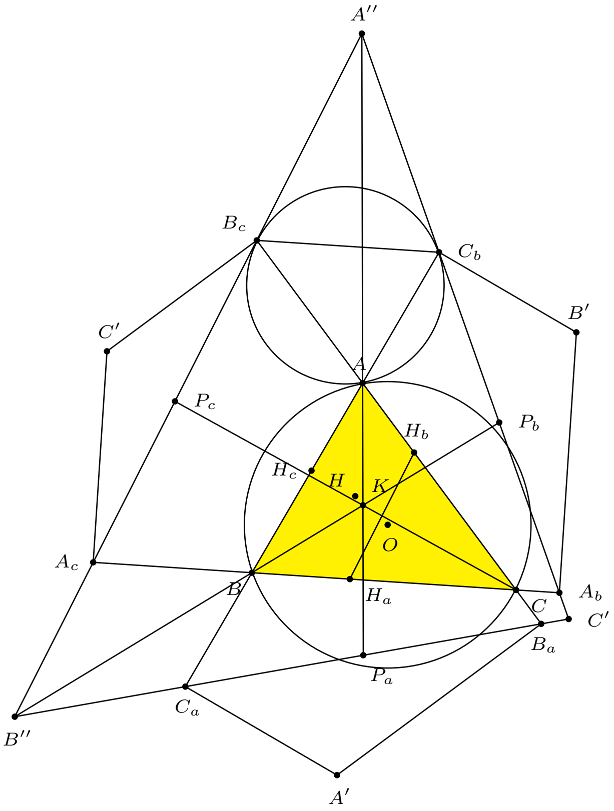

Figure 5.

symmedian of triangle ABC, and it contains the symmedian point K of triangle ABC. The same reasoning shows that $B''B$ and $C''C$ also contain K. Therefore, triangles $A''B''C''$ and ABC are perspective at K.

(2) In triangle ABC, B_aC_a is antiparallel to BC since

$$\angle C_aB_aA = \angle C_aC_bC = \angle BH_aH_b$$

The the reflection of triangles AC_aB_a in the bisector of angle A is homothetic to ABC. Therefore, the median AP_a of triangle AC_aB_a is the same as the symmedian AK; similarly for BP_b and CP_c. The three lines are concurrent at the symmedian point K. □

3. A triangle bounded by three radical axes

Let $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ be the radical axes of the nine-point circle with the pedal circles of A', B', C' respectively. These lines bound a triangle $Q_aQ_bQ_c$. The vertex Q_a is the radical center of the nine-point circle and the pedal circles of B' and C'; similarly for the vertices Q_b and Q_c.

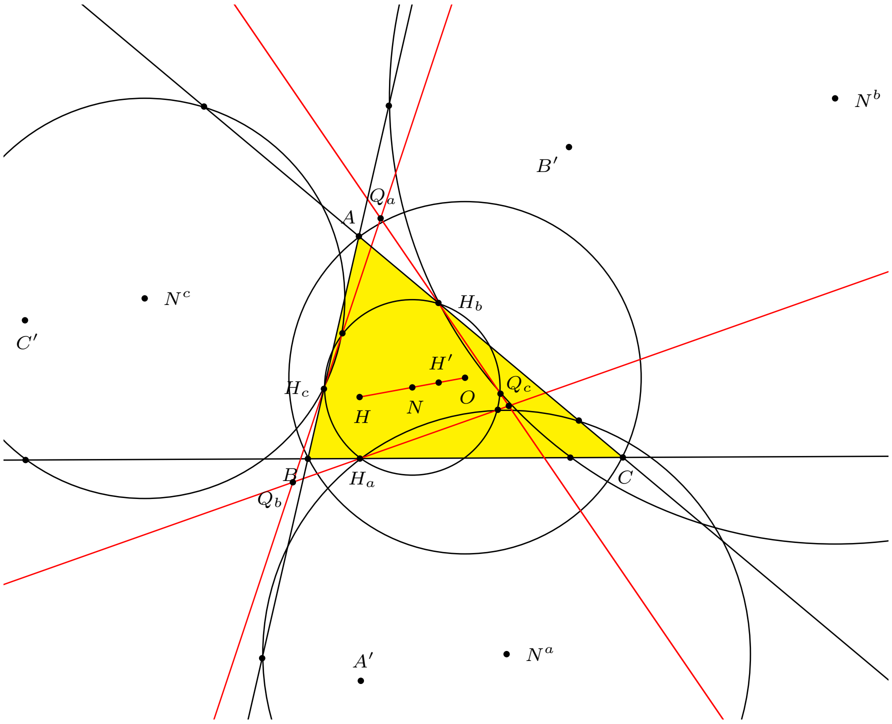

Figure 6

Lemma 9. *Let J_a be the midpoint of OA. The line J_aM_a is perpendicular to Q_bQ_c and contains the midpoint of ON.*

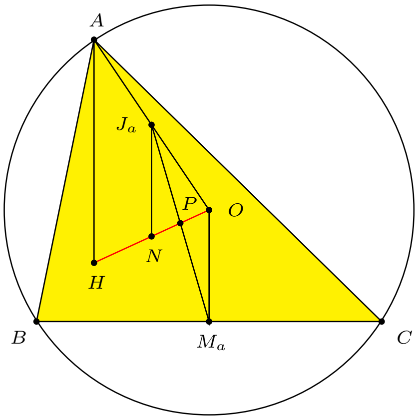

Figure 7.

Proof. Since N is the midpoint of OH, the segment J_aN is parallel to AH and therefore to OM_a. Furthermore, $J_aN = \frac{1}{2}AH = OM_a$. It follows that J_aM_a intersects ON at its midpoint. $\square$

Proposition 10. *Given triangle ABC with incentral triangle DEF, extend AB and AC to P and Q such that $BP = BC = CQ$. Let T be the midpoint PQ, and M the midpoint of the arc BAC of the circumcircle.*

(a) *The line TM is perpendicular EF.*

(b) *BT and CT are parallel to DF and DE respectively.*

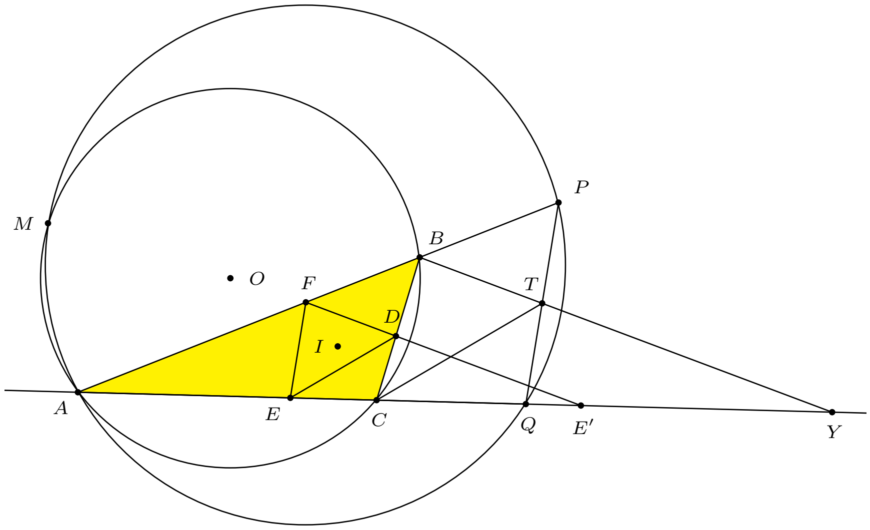

Figure 8.

Proof. (a) By the angle bisector theorem, $AE = \frac{bc}{a+b}$, $AF = \frac{bc}{a+c}$. Therefore, $\frac{AE}{AF} = \frac{a+c}{a+b} = \frac{AQ}{AP}$, showing that PQ is parallel to EF (see Figure 8). On the other hand, the circumcircles of ABC and APQ intersect at A and M, which is the center of the rotation taking the oriented segments BP and CQ into each other (see [4, p.5]). Since $MB = MC$, M is the center of this rotation. Hence, MT is the perpendicular bisector of PQ. We conclude that MT and EF are perpendicular to each other.

(b) We show that BT is parallel to DF.

Let Y be the intersection of the lines BT and AC. Applying Menelaus' theorem to triangle APQ with transversal BTY, we have

$$\frac{AY}{YQ} \cdot \frac{QT}{TP} \cdot \frac{PB}{BA} = -1 \implies \frac{AY}{YQ} = -\frac{AB}{BP} = -\frac{c}{a} \implies \frac{AY}{AQ} = \frac{c}{c-a}.$$

Therefore, $AY = \frac{c(a+b)}{c-a}$. Now, DF intersects AC at E' such that BE' is the external bisector of angle E. $\frac{AE'}{E'C} = -\frac{c}{a} \implies \frac{AE'}{AC} = \frac{c}{c-a}$. It follows that $AE' = \frac{c}{c-a} \cdot b$. From these, $\frac{AE'}{AY} = \frac{b}{a+b} = \frac{AF}{AB}$. Therefore, BT is parallel to DF.

The same reasoning shows that CT is parallel to DE. □

Remark. Proposition 10 remains valid if P and Q are chosen on the rays BA and CA instead, and BE, BF are external bisectors.

Theorem 11. *The orthocenter of triangle $Q_aQ_bQ_c$ is the midpoint of ON.*

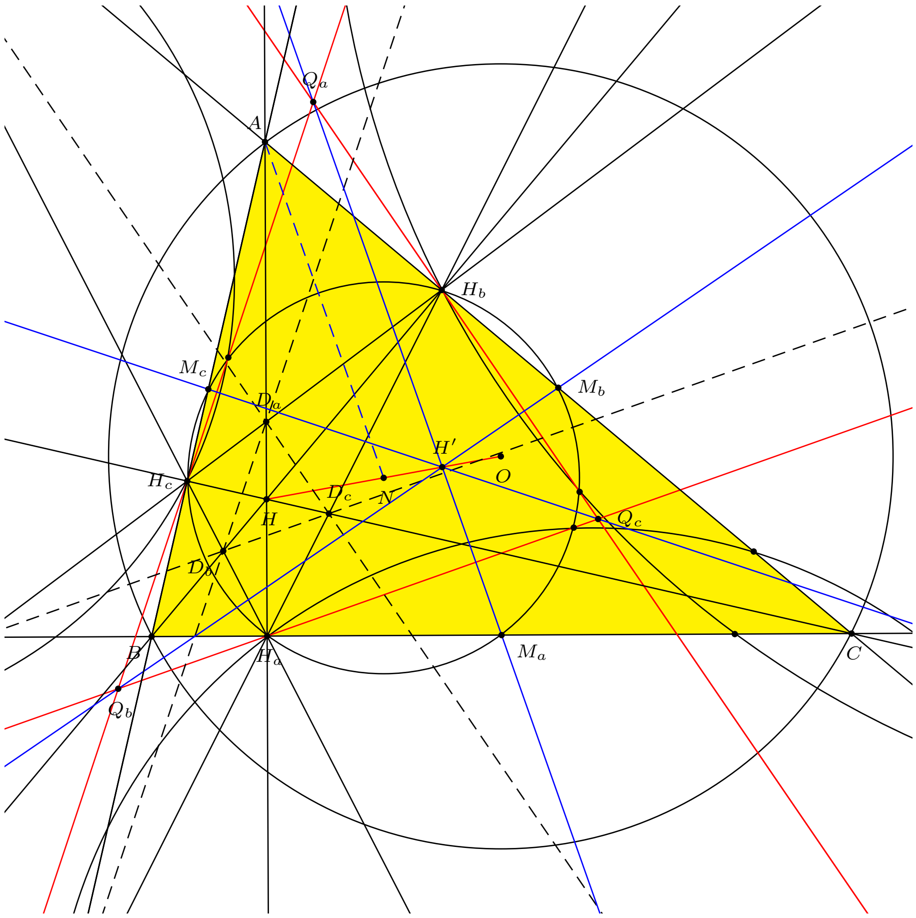

Figure 9

Proof. It is enough to prove that Q_aM_a is parallel to AN.

Let $D_a = AH \cap H_bH_c$, $D_b = BH \cap H_cH_a$, $D_c = CH \cap H_aH_b$. We claim that D_bD_c is perpendicular to AN. The points D_b and D_c have equal powers with respect to the nine-point circle of ABC and the circumcircle of HBC. Therefore, the line D_bD_c is the radical axis of the these two circles. The circumcenter of HBC is the reflection of O in BC, and form a parallelogram with O, A, H, with N as the common midpoint of the diagonals. Therefore AN is the line joining the centers of the nine-point circle and the center of the circle HBC, and is perpendicular to the radical axis D_bD_c.

The line AN also contains the center N^a of the circle Γ_a. Therefore the radical axes Q_bQ_c and D_bD_c are parallel, and AN is perpendicular to Q_bQ_c.

Now we show that Q_aM_a is parallel to AN.

It is easy to see that $D_aD_bD_c$ is the incentral triangle of $H_aH_bH_c$. (If triangle ABC is obtuse, then the two bisectors not corresponding to obtuse angle have to be replaced by external bisectors; see Remark following Proposition 10). Applying Proposition 10 to the orthic triangle $H_aH_bH_c$, the lines Q_aQ_b and Q_aQ_c are parallel to D_aD_b and D_aD_c respectively, and the midpoint of the arc $H_bH_aH_c$ is M_a, the midpoint of BC. Therefore, Q_aM_a is perpendicular to D_bD_c, which is parallel to Q_bQ_c.

The lines Q_aM_a, Q_bM_b, Q_cM_c are the altitudes of the triangle $Q_aQ_bQ_c$. But these lines are parallel to AN, BN, CN respectively. They are concurrent at the midpoint of ON. □

Remark. The midpoint of ON is the triangle center $X(140)$ in [3].

References

[1] http://www.artofproblemsolving.com/Forum/viewtopic.php?f=46&t=384694
[2] R. Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Math. Assoc. America, 1995.
[3] C. Kimberling, *Encyclopedia of Triangle Centers*, available at `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.
[4] Y. Zhao, Three Lemmas in Geometry, available at: `web.mit.edu/yufeiz/www/olympiad/three_geometry_lemmas.pdf`.

Mehmet Efe Akengin: Istanbul Lisesi (Istanbul High School), Türkocaği Caddesi, No:4, Fatih 34440 Istanbul, Turkey
E-mail address: mehmetefeakengin@hotmail.com

Zeyd Yusuf Köroğlu: Istanbul Lisesi (Istanbul High School), Türkocaği Caddesi, No:4, Fatih 34440 Istanbul, Turkey
E-mail address: zeyd.yusuf@gmail.com

Yiğit Yargiç: Ludwig Maximillian University of Munich, Geschwister-Scholl-Platz 1, 80539 München, Germany
E-mail address: yigityargic@hotmail.com