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Kitta's Double-Locked Problem

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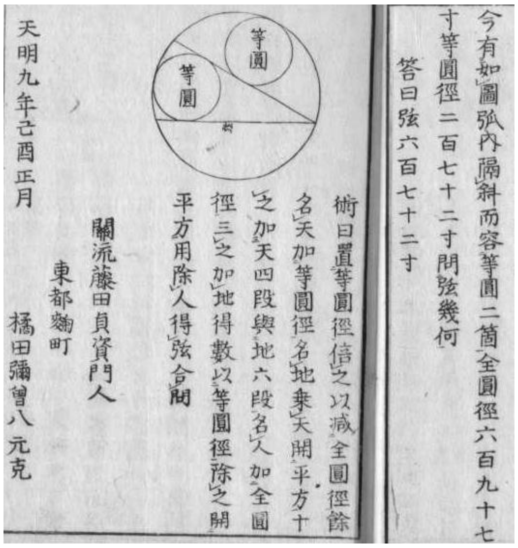
Abstract. I present strongly contrasting solutions of a remarkable *sangaku*. The general solution requires Fujita's celebrated mixtilinear circle theorem. The data-specific solution shows the premodern Japanese preoccupation with Pythagorean triples.

1. The Problem

Shinpeki sanpō [2] is a collection of problems that FUJITA Sadasuke ordered his son Yoshitoki to compile to show off the prowess of his disciples, although problems of enthusiasts from other schools were also included. The problem in question ([2, 1.42-3]), which was not discussed in [3], [4] or [5], is reproduced in Figure 1 followed by my translation.


今有如圖弧內隔斜而容等圓二箇全圓徑六百九十七
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平方用除人得弦合問

關流藤田貞資門人
東都麴町
橘田彌曾八元克

天明九年己酉正月

Figure 1.

> A circular segment is split by a line such that its resulting parts contain two congruent circles as shown in the figure. The diameter of the large circle is 697 inches; the diameters of the congruent circles are 272 inches each. What is the length of the chord of the segment?

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> The answer is 672 inches.
> The method is as follows: Substract twice the small diameter (d) from the large diameter (D). Call the difference heaven (h). Add this to d. Call this earth (e). Add 10 times the square root of eh to $4h + 6e$. Call this man (m). Now divide $3(m + D) + e$ by d, take the square root, and divide it into m. You get the chord asked for.
>
> By KITTA Yasohachi Motokatsu, a disciple of Fujita Sadasuke of the Seki School, at Kōjimachi in the Eastern capital, in the 9th year of Tenmei (1789), first (lunar) month.

As we shall see, one can analyze the figure and verify that the numerical solution is correct for the given data. Yet the analysis that emerges does not lead to the stated solution procedure, which may be paraphrased in modern notation as follows:

$$h = D-2d,\ e = h+d,\ m = 10\sqrt{eh}+4h+6e,\ w = \sqrt{\frac{3(m+D)+e}{d}},\ x = \frac{m}{w}.$$

2. A data-driven solution

Given $D = 697$ and $d = 272$, we notice that all but the chord length are divisible by 17 and that $(\sqrt{h},\ \sqrt{d},\ \sqrt{e})$ is a Pythagorean triple.

	D	d	h	e	m	x
n	697	272	153	425	5712	672
$n/17$	41	16	9	25	336	39.52941

Since $w = \frac{17}{2}$ whether one uses the first or second row of numbers, Kitta evidently scaled up the data to ensure that x would be an integer.

Because segment-inscribed circles in many but not all other *sangaku* problems touch the segment's chord at its midpoint, and since that is (approximately) what one sees in Figure 1, Kitta's readers were likely to take this for granted even though it was not included in the statement of the problem. In fact, this is a necessary condition for Kitta's solution, but it would have led readers acquainted with other *sangaku* results into a less than general line of reasoning such as the following.

Lemma 1 ([3, 2.2.7])**.** *The radius of the mixtilinear incircle touching the legs of a right triangle with sides a, b, c* (hypotenuse) *and its circumcircle is $\rho = a + b - c$.*

Proof. In Figure 2, triangle ABC has a right angle at C, and the circle (J) touches BC, AC at T_a, T_b respectively. P is the orthogonal projection of the circumcenter O on JT_A. In triangle POJ, we have

$$\left(\frac{a}{2}-\rho\right)^2 + \left(\rho-\frac{b}{2}\right)^2 = \left(\frac{c}{2}-\rho\right)^2$$

or

$$\frac{a^2}{4} + \frac{b^2}{4} - a\rho - b\rho + 2\rho^2 = \frac{c^2}{4} - c\rho + \rho^2.$$

Since $a^2 + b^2 = c^2$, this quickly reduces to the result. □

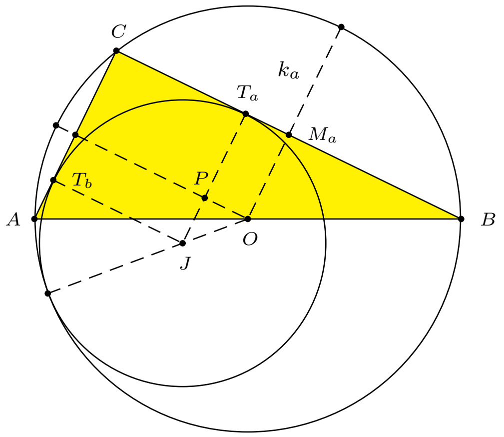

Figure 2.

Corollary 2. *$\rho = 2r$, where r is the inradius of the right triangle ABC.*

Corollary 3 ([5, 254, 275-76])**.** *The difference between each leg and the radius of the mixtilinear circle is twice the sagitta on that leg,* i.e.,

$$2k_a = a - \rho \quad \textit{and} \quad 2k_b = b - \rho.$$

Proof. $a - \rho = a - (a + b - c) = c - b = 2\left(\frac{c}{2} - \frac{b}{2}\right) = 2(R - OM_a) = 2k_a$. The proof for b is analogous. $\square$

If $2\rho = k_a$, then $2(a + b - c) = \frac{1}{2}(c - b)$, $4a + 4b - 4c = c - b$, $4a + 5b = 5c$. From this, $(4a + 5b)^2 = (5c)^2 = 25(a^2 + b^2)$, $16a^2 + 40ab + 25b^2 = 25a^2 + 25b^2$, $9a^2 = 40ab$, and $a : b = 40 : 9$. It follows that $a : b : c : \rho = 40 : 9 : 41 : 8$.

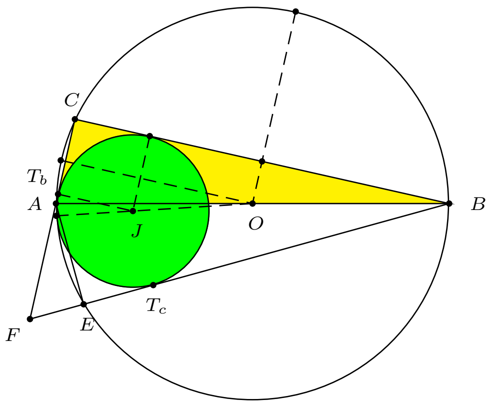

Figure 3.

Assume $BC = 40$, $CA = 9$, and $AB = 41$. Now add E such that BE touches (OJ) at T_c. We seek the length of BE (see Figure 3). Let F be the intersection of CA and BE extended. By equal tangents on (J), let $FT_b = FT_c = x$. Since $BT_c = BT_a = 40 - 8 = 32$, by the Pythagorean relation for the triangle FBC,

$$(8+x)^2 + 40^2 = (32+x)^2 \implies x = \frac{40}{3}.$$

This means that $CF = 8 + \frac{40}{3} = \frac{64}{3}$ and $BF = 32 + \frac{40}{3} = \frac{136}{3}$.

By the intersecting chords theorem, $EF \cdot BF = AF \cdot CF$, and $EF = \frac{AF \cdot CF}{BF} = \frac{37}{3} \cdot \frac{64}{3} \cdot \frac{3}{136} = \frac{296}{51}$. This gives $BE = \frac{136}{3} - \frac{296}{51} = \frac{672}{17}$. Scaling up by 17 gives the answer.

In summary, if one focuses on the data given and applies other well-known *sangaku* results, one can verify the value of the given answer. Kitta does not state that AC touches (J) in Figure 3, although this is true for the given data. But neither does Kitta state that the circle inscribed in the segment touches BC at its midpoint, which is in fact a necessary condition, so it would not be unreasonable for the reader to think that both pieces of information were to be inferred from the data. Yet a little experimentation shows that Kitta's method is valid even when AC does not touch (J).

3. The general case

Let us start afresh with the Figure 4, in which $YJ = \frac{d}{2}$. By an elementary *sangaku* theorem ([3, 1.1]), $EY = 2\sqrt{\frac{d}{2}\left(\frac{D}{2} - d\right)} = \sqrt{d(D-2d)} = \sqrt{dh}$.

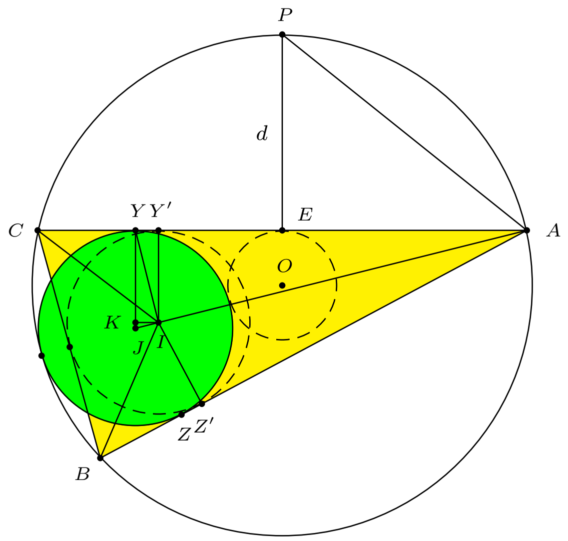

Figure 4.

By the intersecting chords theorem, $EA = \sqrt{d(D-d)} = \sqrt{de}$. Hence $b = 2EA = 2\sqrt{de}$. Let r be the inradius of ABC, $p = \sqrt{de} + \sqrt{dh} = YA$, and $q = Y'A$. Notice that all seven right triangles

$$IJK,\ AIY',\ YIK,\ AJY,\ IYY',\ YJI,\ AYI$$

are similar. In particular, $\frac{KJ}{IK} = \frac{Y'I}{Y'A}$ and $\frac{IK}{KY} = \frac{IY'}{Y'A}$. That is, $\frac{d}{2} - r = \frac{r \cdot IK}{q}$ and $IK = \frac{r^2}{q}$. Hence, $\frac{d}{2} - r = r \cdot \frac{r^2}{q^2}$. But by applying Fujita's mixtilinear circle theorem ([3, 2.2.8], also [7, 8] [1]) with A on (O), $\frac{r}{q} = \frac{d}{2p}$. Therefore, $\frac{d}{2} - r = r \cdot \frac{d^2}{4p^2}$. Solving for r, $r = \frac{2dp^2}{d^2+4p^2}$.

Likewise, since the triangles AEP and $BZ'I$ are similar, $\frac{b}{2d} = \frac{c-q}{r}$. Using $\frac{q}{r} = \frac{2p}{d}$ again, this becomes $\frac{b}{2d} = \frac{c}{r} - \frac{2p}{d}$. Solving for c, $c = \frac{r(b+4p)}{2d}$, or, replacing r,

$$c = \frac{2dp^2}{d^2+4p^2} \cdot \frac{b+4p}{2d} = \frac{p^2(b+4p)}{d^2+4p^2}.$$

Now, $b + 4p = 6\sqrt{de} + 4\sqrt{dh}$, and, because $d = (\sqrt{e} + \sqrt{h})(\sqrt{e} - \sqrt{h})$, we see that $p^2 = (\sqrt{e} + \sqrt{h})^3(\sqrt{e} - \sqrt{h})$ and

$$\begin{aligned} d^2 + 4p^2 &= (\sqrt{e}+\sqrt{h})^2(\sqrt{e}-\sqrt{h})^2 + 4(\sqrt{e}+\sqrt{h})(\sqrt{e}-\sqrt{h})(\sqrt{e}+\sqrt{h})^2 \\ &= (\sqrt{e}+\sqrt{h})^2(\sqrt{e}-\sqrt{h})[\sqrt{e}-\sqrt{h}+4(\sqrt{e}+\sqrt{h})] \\ &= (\sqrt{e}+\sqrt{h})^2(\sqrt{e}-\sqrt{h})(5\sqrt{e}+3\sqrt{h}). \end{aligned}$$

Hence, $\frac{p^2}{d^2+4p^2} = \frac{\sqrt{e}+\sqrt{h}}{5\sqrt{e}+3\sqrt{h}}$ and

$$c = \frac{2\sqrt{d}(3\sqrt{e}+2\sqrt{h})(\sqrt{e}+\sqrt{h})}{5\sqrt{e}+3\sqrt{h}}.$$

This is Kitta's formula because $m = 10\sqrt{eh}+4h+6e = 2(3\sqrt{e}+2\sqrt{h})(\sqrt{e}+\sqrt{h})$, and

$$3(m+D)+e = 3(m+2e-h)+e = 3m+7e-3h = 25e+30\sqrt{eh}+9h = (5\sqrt{e}+3\sqrt{h})^2.$$

But we are not quite done since Kitta defined w as $\sqrt{\frac{(5\sqrt{e}+3\sqrt{h})^2}{d}}$ and used m to define the numerator of $c = \frac{m}{w}$. Why not set $w_0 = \frac{5\sqrt{e}+3\sqrt{h}}{\sqrt{d}}$ and $m_0 = \frac{dw_0^2-e}{3} - D$ and have $c = \frac{m_0}{w_0}$? I suspect the reason is that Kitta relied on two other *sangaku* results, which took him on a circuitous path to c.

One result was an expression for the diameter of the circumcircle in terms of two sagittae and the inradius of a circumscribed triangle (see Figure 5), $D = \frac{4d\ell(d+\ell+r)}{4d\ell-r^2}$. This follows from the equivalent of Carnot's Theorem stated in terms of the sagittae on the sides of a triangle rather than the signed perpendicular distances from the circumcenter to the sides. This elegant version of the theorem was cited as if common knowledge, with no mention of Carnot, in a Meiji period paper by Y. Sawayama ([6, 153]), and we can reasonably conclude that it was known in premodern Japan.

Proof. If n is the third sagitta in Figure 5, Sawayama's form of Carnot's Theorem states that $d + \ell + n = D - r$. Hence $2D - 2n = 2(d + \ell + r)$. Since the right triangles CPQ and AIY are similar, $\frac{a}{2n} = \frac{q}{r}$; but by the intersecting chords

[1]The proof in [7] was flawed and is superseded by [8].

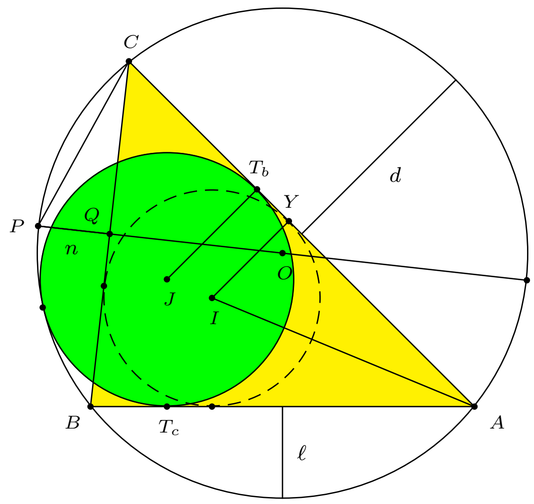

Figure 5.

theorem, $n(D-n) = \left(\frac{a}{2}\right)^2$. Hence $D - n = \frac{q^2}{r^2} \cdot n$, or, $n = \frac{Dr^2}{q^2+r^2}$. Now, another *sangaku* theorem ([3, 2.2]) states that *the square of the distance from the vertex of a triangle to its incenter is four times the product of the sagittae on the adjacent sides*. In the present case, $4d\ell = x^2$, where $x^2 = q^2 + r^2$. Therefore, $2D - 2n = 2D - \frac{D(4d\ell - r^2)}{2d\ell}$ and so $2(d + \ell + r) = \frac{D(4d\ell-r^2)}{2d\ell}$, or $D = \frac{4d\ell(d+\ell+r)}{4d\ell - r^2}$. □

Armed with $D = \frac{4d\ell(d+\ell+r)}{4d\ell-r^2}$ and $\ell = \frac{x^2}{4d}$, one calculates c by the intersecting chords theorem without much difficulty because $D - \ell$ factors nicely into

$$\frac{4d^2\ell + 4d\ell^2 + 4d\ell r}{4d\ell - r^2} - \ell = \frac{4d^2\ell + 4d\ell r + \ell r^2}{4d\ell - r^2} = \frac{\ell(2d+r)^2}{4d\ell - r^2}.$$

Therefore,

$$\begin{aligned}
\left(\frac{c}{2}\right)^2 &= \ell(D-\ell) = \frac{\ell^2(2d+r)^2}{4d\ell - r^2} = \frac{(2d+r)^2 x^4}{16d^2(x^2-r^2)} = \frac{(2d+r)^2(q^2+r^2)^2}{16d^2q^2} \\
&= \frac{(d^2+4p^2)^2 r^2 (2d+r)^2}{64d^4p^2} = \frac{(\sqrt{d}(5\sqrt{e}+3\sqrt{h})+p)^2(d^2+4p^2)^2}{4d^2(5\sqrt{e}+3\sqrt{h})^4} \\
&= \frac{d(d+4(\sqrt{e}+\sqrt{h})^2)^2(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^4} \\
&= \frac{(\sqrt{e}-\sqrt{h})(\sqrt{e}+\sqrt{h})^3(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^2}.
\end{aligned}$$

Hence,

$$c = 2\sqrt{\frac{(\sqrt{e}-\sqrt{h})(\sqrt{e}+\sqrt{h})^3(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^2}} = 2\sqrt{d}\sqrt{\frac{(\sqrt{e}+\sqrt{h})^2(3\sqrt{e}+2\sqrt{h})^2}{(5\sqrt{e}+3\sqrt{h})^2}}.$$

(We get to the same expression if we eliminate D using $D = 2e - h$, but the algebra is lengthier.) Thus we obtain a quadratic in c with no linear term and therefore with symmetric roots, one negative definite and one positive. From this perpsective, $c = \sqrt{\frac{m^2}{w^2}}$ and Kitta's definitions of m and w make sense.

I have recently discovered that this problem is the first one discussed by AIDA Yasuaki in a manuscript criticizing various solutions in *Shinpeki sanpō* ([1, 1.5-6]). Aida's solution is equivalent to the equation $c = \frac{2\sqrt{d}(3\sqrt{e}+2\sqrt{h})(\sqrt{e}+\sqrt{h})}{5\sqrt{e}+3\sqrt{h}}$ derived above. In terms of the notation we have been using, Aida sets $h_0 = \sqrt{d(D-d)}$ and $e_0 = \sqrt{h_0^2 - d^2} + h_0$ (that is, $e_0 = \sqrt{d(D-2d)} + \sqrt{d(D-d)}$), and writes $c = \frac{e_0^2}{2h_0+3e_0} + e_0$, avoiding Kitta's m and w placeholders altogether.

4. Discussion

Kitta evidently took pains to write up this problem in such a way that even a reader who could verify the numerical result might still be baffled by the general procedure. In this way, it is like a double-locked box. For the sophisticated solver, the unanswered question is why neat Pythagorean relationships emerge in the data.

A connection between the general and special cases of the problem lies in the fact that, if d and p are integers, then h, d, and e are perfect squares and $(\sqrt{h}, \sqrt{d}, \sqrt{e})$ is a Pythagorean triple. To prove this, recall that $dp^2 = (\sqrt{e}+\sqrt{h})^4(\sqrt{e}-\sqrt{h})^2$. Replacing h with $e-d$ and solving for e, $e = \frac{(d^2+p^2)^2}{4dp^2}$; that is, $\frac{e}{d} = \left(\frac{d^2+p^2}{2dp}\right)^2$. Thus, provided d and p are integers, e and d are perfect squares, and, since $h+d = e$ implies $h = (p^2-d^2)^2$, h is a perfect square too.

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Appendix (by the editor): Kitta's configurations with integer diameters

Given a triangle ABC with sidelengths $BC = a$, $CA = b$, $AB = c$, semiperimeter s, and area Δ, the radius of the mixtilinear incircle in angle A is

$$\rho_a = \frac{r}{\cos^2 \frac{A}{2}} = \frac{bc}{s(s-a)} \cdot \frac{\Delta}{s} = \frac{bc\Delta}{s^2(s-a)},$$

where $r = \frac{\Delta}{s}$ is the inradius of the triangle. With reference to Figure 5, $d = R(1 - \cos B) = 2R\sin^2 \frac{B}{2}$. The condition $2\rho_a = d$ becomes $\frac{2bc\Delta}{s^2(s-a)} = \frac{2abc}{4\Delta} \cdot \frac{(s-c)(s-a)}{ca}$. Since $\Delta^2 = s(s-a)(s-b)(s-c)$, this reduces to $4c(s-b) = s(s-a)$, and $a^2 - b^2 + 7c^2 - 10bc + 8ca = 0$. By completing squares, we rewrite this as $(a+4c)^2 = (b+c)(b+9c)$. It follows that

$$a = -4c + \sqrt{(b+c)(b+9c)} = -4c + \sqrt{(b+5c)^2 - (4c)^2}.$$

To obtain integer solutions we put $b + 5c = p^2 + q^2$, $4c = 2pq$ for relatively prime integers p and q. This leads to

$$b = 2p^2 + 2q^2 - 5pq, \quad c = pq, \quad a = 2(p^2 - q^2 - 2pq).$$

These satisfy the triangle inequality if and only if $p > \frac{5q}{2}$. The circumradius of the triangle is rational if and only if the area is an integer. Since $\Delta = 2pq(p-2q)\sqrt{(2p-5q)q}$, we choose p and q such that $2p - 5q : q = u^2 : v^2$ for relatively prime integers u, v. Equivalently, $(p, q) = (\frac{u^2+5v^2}{g}, \frac{2v^2}{g})$ with $g = \gcd(u^2 + 5v^2, 2v^2)$. The following table shows that with small values of u and v, we exhaust all examples in which a, b, c, after reduction by their gcd, are all integers below 1000.

u	v	p	q	a	b	c	D	$d = 2\rho_a$	ℓ	AT_b
1	1	3	1	4	5	3	5	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{5}{2}$
1	2	21	8	41	85	84	$\frac{697}{8}$	34	32	68
2	1	9	2	41	40	9	41	17	$\frac{1}{2}$	8
1	3	23	9	68	185	207	$\frac{629}{3}$	$\frac{111}{2}$	$\frac{243}{2}$	$\frac{333}{2}$
3	1	7	1	68	65	7	$\frac{221}{3}$	$\frac{39}{2}$	$\frac{1}{6}$	$\frac{13}{2}$
2	3	49	18	313	520	441	$\frac{1565}{3}$	240	$\frac{243}{2}$	360
3	2	29	8	313	325	116	$\frac{7825}{24}$	150	$\frac{32}{3}$	100
4	1	21	2	353	340	21	$\frac{1765}{4}$	80	$\frac{1}{4}$	20
5	1	15	1	388	377	15	$\frac{2813}{5}$	$\frac{145}{2}$	$\frac{1}{10}$	$\frac{29}{2}$

Kitta's example is the one obtained from $(u, v) = (1, 2)$, magnified by a factor 8 to make the circumradius an integer. Here are the only configurations with integer values of a, b, c, D, and $2\rho_a = d$, all below 1000:

a	b	c	D	$d = 2\rho_a$	ℓ	AT_b
8	10	6	10	5	1	5
41	40	9	41	17	$\frac{1}{2}$	8
328	680	672	697	272	256	544
408	390	42	442	117	1	39

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