Forum Geometricorum
Volume 14 (2014) 63–86.

FORUM GEOM
ISSN 1534-1178

The Touchpoints Triangles and the Feuerbach Hyperbolas

Sándor N. Kiss and Paul Yiu

Abstract. In this paper we generalize the famous Kariya theorem on the perspectivity of a given triangle with the homothetic images of the intouch triangle from the incenter to the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. We also study in some details the triangle formed by the orthocenters of the touchpoints triangles. An elegant construction is given for the asymptotes of the Feuerbach hyperbolas.

1. Introduction

Consider a triangle ABC with incircle $I(r)$ tangent to the sides BC, CA, AB at X, Y, Z respectively. These form the intouch triangle $\mathbf{T}_{\rm i}$ of ABC. For a real number t, let $\mathbf{T}_{\rm i}(t)$ be the image of the $\mathbf{T}_{\rm i}$ under the homothety $\mathsf{h}\left(I, \frac{t}{r}\right)$. Its vertices are the points $X(t)$, $Y(t)$, $Z(t)$ on the lines IX, IY, IZ respectively, such that

$$IX(t) = IY(t) = IZ(t) = t.$$

The famous Kariya's theorem asserts that the lines $AX(t)$, $BY(t)$, $CY(t)$ are concurrent, i.e., the triangles ABC and $\mathbf{T}_{\rm i}(t)$ are perspective, and that the perspector is a point $Q(t)$ on the Feuerbach hyperbola $\mathscr{F}$, the rectangular circum-hyperbola which is the isogonal conjugate of the line OI joining the circumcenter and the incenter of ABC. This fact was actually known earlier to J. Neuberg and H. Mandart; see [5] and the interesting note in [2, §1242]. We revisit in §3 this theorem with a proof leading to simple relations of the perspectors $Q(t)$ and $Q(-t)$ (Proposition 4 below), and their isogonal conjugates on the line OI. In §5 we obtain analogous results by replacing $\mathbf{T}_{\rm i}(t)$ by homothetic images of the touchpoints triangles of the excircles, leading to the triad of ex-Feuerbach hyperbolas. Some properties of the triad of ex-Feuerbach hyperbolas are established in §§7–10. Specifically, we give an elegant construction of the asymptotes of the Feuerbach hyperbolas in §10. The final section §11 is devoted to further properties of the touchpoints triangles, in particular, the loci of perspectors of the triangle $H^aH^bH^c$ formed by their orthocenters.

Publication Date: March 11, 2014. Communicating Editor: Nikolaos Dergiades.
We thank Nikolas Dergiades for suggestions leading to improvements over an earlier version of this paper.

2. Generalization of Kariya's theorem

Let a, b, c be the sidelengths of triangle ABC, $s = \frac{1}{2}(a+b+c)$ the semiperimeter, and Δ the area of the triangle. It is well known that
(i) $\Delta = rs$,
(ii) $abc = 4R\Delta$ for the circumradius R, and
(iii) $16\Delta^2 = a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)$.

We shall work with homogeneous barycentric coordinates with reference to triangle ABC, and refer to [7] for basic results and formulas.

Let $\mathbf{T}$ be the pedal triangle of a point $P = (u : v : w)$ in homogeneous barycentric coordinates. The vertices of $\mathbf{T}$ are the points

$$\begin{aligned} X =& \ (0 : \ (a^2+b^2-c^2)u + 2a^2v : \ (c^2+a^2-b^2)u + 2a^2w), \\ Y =& \ ((a^2+b^2-c^2)v + 2b^2u : \ 0 : \ (b^2+c^2-a^2)v + 2b^2w), \\ Z =& \ ((c^2+a^2-b^2)w + 2c^2u : \ (b^2+c^2-a^2)w + 2c^2v : \ 0). \end{aligned}$$

For a real number k, let $\mathbf{T}_k$ be the image of $\mathbf{T}$ under the homothety $\mathsf{h}(P, k)$.

Lemma 1. *The vertices of* $\mathbf{T}_k$ *are the points*

$$X_k = (2a^2(1-k)u : \ (a^2+b^2-c^2)ku + 2a^2v : \ (c^2+a^2-b^2)ku + 2a^2w), \quad (1)$$

$$Y_k = ((a^2+b^2-c^2)kv + 2b^2u : \ 2b^2(1-k)v : \ (b^2+c^2-a^2)kv + 2b^2w), \quad (2)$$

$$Z_k = ((c^2+a^2-b^2)kw + 2c^2u : \ (b^2+c^2-a^2)kw + 2c^2v : \ 2c^2(1-k)w). \quad (3)$$

Proof. The point X_k divides the segment PX in the ratio $PX_k : X_kX = k : 1-k$. In absolute barycentric coordinates,

$$\begin{aligned} X_k =& \ (1-k)P + kX \\ =& \ \frac{(1-k)(u,v,w)}{u+v+w} + \frac{k(0, (a^2+b^2-c^2)u + 2a^2v, (c^2+a^2-b^2)u + 2a^2w)}{2a^2(u+v+w)} \\ =& \ \frac{(2a^2(1-k)u, (a^2+b^2-c^2)ku + 2a^2v, (c^2+a^2-b^2)ku + 2a^2w)}{2a^2(u+v+w)}. \end{aligned}$$

Ignoring the denominator, we obtain the homogeneous barycentric coordinates of X_k given above; similarly for Y_k and Z_k. □

Proposition 2. *The only points that satisfy Kariya's theorem, as the incenter I does, are the orthocenter H, the circumcenter O, and the three excenters I_a, I_b, I_c.*

Proof. The equations of the lines AX_k, BY_k, CZ_k are

$$\begin{aligned} ((c^2+a^2-b^2)ku + 2a^2w)y - ((a^2+b^2-c^2)ku + 2a^2v)z =& \ 0, \\ -((b^2+c^2-a^2)kv + 2b^2w)x + ((a^2+b^2-c^2)kv + 2b^2u)z =& \ 0, \\ ((b^2+c^2-a^2)kw + 2c^2v)x - ((c^2+a^2-b^2)kw + 2c^2u)y =& \ 0. \end{aligned}$$

They are concurrent if and only if

$$\begin{vmatrix} 0 & (c^2+a^2-b^2)ku + 2a^2w & -((a^2+b^2-c^2)ku + 2a^2v) \\ -(b^2+c^2-a^2)kv + 2b^2w) & 0 & (a^2+b^2-c^2)kv + 2b^2u \\ (b^2+c^2-a^2)kw + 2c^2v & -((c^2+a^2-b^2)kw + 2c^2u) & 0 \end{vmatrix} = 0.$$

Equivalently, $2F_1(P) \cdot k - F_2(P) = 0$ for every k, where

$$F_1(P) = \sum_{\text{cyclic}} a^2(b^2 + c^2 - a^2)u(c^2v^2 - b^2w^2),$$
$$F_2(P) = \sum_{\text{cyclic}} (c^2 + a^2 - b^2)(a^2 + b^2 - c^2)u(c^2v^2 - b^2w^2).$$

This means that $F_1(P) = F_2(P) = 0$. The point P is common to the McCay cubic p$K(X(6), X(3))$, and the orthocubics p$K(X(6), X(4))$. These appear in [3] as **K003** and **K006** respectively. It is known that the common points of these circumcubics are the vertices of ABC and the points H, O, I, I_a, I_b, I_c, as can be readily verified. □

The case $P = H$ is trivial because the perspector is H for every k.

The case $P = O$ is also trivial because the triangle $\mathbf{T}_k$ is homothetic to ABC, and the perspector is obviously the point Q_k on the Euler line dividing OH in the ratio $k : 2$. This follows from

$$OQ_k : Q_kH = OX(k) : HA = k : 2.$$

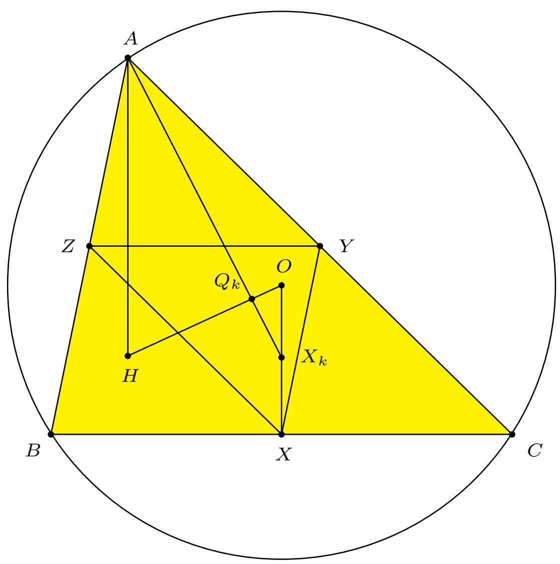

Figure 1

In the remainder of this paper, we study the cases when P is the incenter or an excenter.

3. Kariya's theorem and the Feuerbach hyperbola

The pedal triangle of the incenter $I = (a : b : c)$ has vertices

$$\begin{aligned} X =& \ (0 : a+b-c : c+a-b), \\ Y =& \ (a+b-c : 0 : b+c-a), \\ Z =& \ (c+a-b : b+c-a : 0). \end{aligned}$$

The coordinates of $X(t)$, $Y(t)$, $Z(t)$ can be determined from equations (1), (2), (3) by putting $k = \frac{t}{r}$.

Proposition 3. *The lines $AX(t)$, $BY(t)$, $CZ(t)$ are concurrent at the point*

$$Q(t) = \left(\frac{1}{2rbc+t(b^2+c^2-a^2)} : \frac{1}{2rca+t(c^2+a^2-b^2)} : \frac{1}{2rab+t(a^2+b^2-c^2)}\right), \tag{4}$$

which is the isogonal conjugate of the point $P(t)$ dividing OI in the ratio

$$OP(t) : P(t)I = R : t.$$

Proof. Writing the homogeneous barycentric coordinates of $X(t)$, $Y(t)$, $Z(t)$ as

$$\begin{aligned} X(t) =& \ \left(*****: \frac{1}{2rca+t(c^2+a^2-b^2)} : \frac{1}{2rab+t(a^2+b^2-c^2)}\right), \\ Y(t) =& \ \left(\frac{1}{2rbc+t(b^2+c^2-a^2)} : *****: \frac{1}{2rab+t(a^2+b^2-c^2)}\right), \\ Z(t) =& \ \left(\frac{1}{2rbc+t(b^2+c^2-a^2)} : \frac{1}{2rca+t(c^2+a^2-b^2)} : *****\right), \end{aligned}$$

we note that the lines $AX(t)$, $BY(t)$, $CZ(t)$ are concurrent at a point $Q(t)$ with coordinates given in (4) above. This is clearly the isogonal conjugate of the point

$$\begin{aligned} P(t) =& \ (2rabc\cdot a + ta^2(b^2+c^2-a^2), 2rabc\cdot b + tb^2(c^2+a^2-b^2), \\ & \quad 2rabc\cdot c + tc^2(a^2+b^2-c^2)) \\ =& \ 2rabc(a,b,c) + t((a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)) \\ =& \ 2r\cdot 4Rrs(a+b+c)\cdot I + t\cdot 16\Delta^2\cdot O \\ =& \ 16r^2s^2\cdot R\cdot I + 16\Delta^2\cdot t\cdot O \\ =& \ 16r^2s^2(R\cdot I + t\cdot O). \end{aligned}$$

In absolute barycentric coordinates, $P(t) = \frac{R\cdot I+t\cdot O}{R+t}$. This is the point dividing OI in the ratio $OP(t) : P(t)I = R : t$. □

It follows that the locus of the point $Q(t)$ is the Feuerbach hyperbola $\mathscr{F}$. The center is the Feuerbach point F_e, the point of tangency of the incircle and the nine-point circle (see Figure 2). Note that for each t, the points $P(t)$ and $P(-t)$ divide OI harmonically.

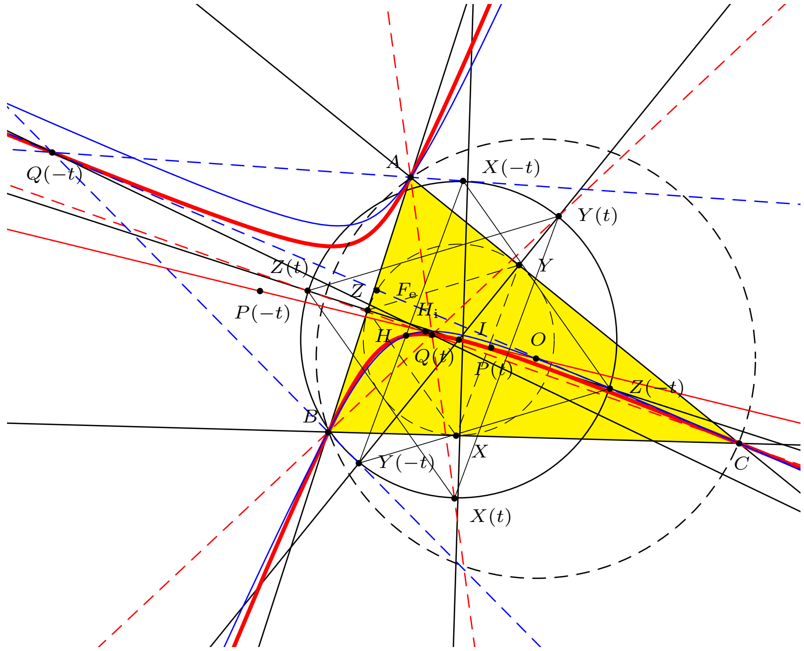

Figure 2.

Proposition 4. *The line joining $Q(t)$ and $Q(-t)$ contains the triangle center*

$$H_{\rm i} = \left(\frac{a(b+c)}{b+c-a} : \frac{b(c+a)}{c+a-b} : \frac{c(a+b)}{a+b-c} \right). \tag{5}$$

Proof. The line joining $Q(t)$ to $Q(-t)$ has equation

$$\sum_{\text{cyclic}} a(b-c)(b+c-a)(4b^2c^2r^2 - t^2(b^2+c^2-a^2)^2)x = 0.$$

With $(x:y:z)$ given in (5) we have

$$\begin{aligned} & \sum_{\text{cyclic}} a^2(b^2-c^2)(4b^2c^2r^2 - t^2(b^2+c^2-a^2)^2) \\ = \ & 4a^2b^2c^2r^2 \left(\sum_{\text{cyclic}} (b^2-c^2) \right) - t^2 \left(\sum_{\text{cyclic}} a^2(b^2-c^2)(b^2+c^2-a^2)^2 \right) \\ = \ & 0. \end{aligned}$$

While the first sum obviously is zero, the second sum vanishes because the Euler line

$$\sum_{\text{cyclic}} (b^2-c^2)(b^2+c^2-a^2)x = 0 \tag{6}$$

contains the circumcenter. □

Remark. The triangle center $H_{\rm i}$ is the orthocenter of the intouch triangle $\mathbf{T}_{\rm i}$; it appears as $X(65)$ in [4]. It divides OI in the ratio $R+r : -r$. Its isogonal conjugate is the Schiffler point

$$S_{\rm c} = \left(\frac{a(b+c-a)}{b+c} : \frac{b(c+a-b)}{c+a} : \frac{c(a+b-c)}{a+b}\right), \tag{7}$$

which is the point of concurrency of the Euler lines of the four triangles ABC, IBC, ICA, and IAB. Therefore, $H_{\rm i}$ is a point on the Jerabek hyperbola $\mathscr{J}$:

$$\frac{a^2(b^2-c^2)(b^2+c^2-a^2)}{x}+\frac{b^2(c^2-a^2)(c^2+a^2-b^2)}{y}+\frac{c^2(a^2-b^2)(a^2+b^2-c^2)}{z}=0,$$

the isogonal conjugate of the Euler line.

4. The touchpoints triangles

Consider the A-excircle $I_a(r_a)$ of triangle ABC, tangent to the sidelines BC at X_a, CA at Y_a, and AB at Z_a respectively. We call triangle $\mathbf{T}_a := X_aY_aZ_a$ the A-touchpoints triangle. Clearly,

$$I_aX_a = I_aY_a = I_aZ_a = r_a = \frac{\Delta}{s-a}.$$

In homogeneous barycentric coordinates these are the points

$$\begin{aligned} I_a =&\ (-a : b : c),\\ X_a =&\ (0 : c+a-b : a+b-c),\\ Y_a =&\ (-(c+a-b) : 0 : a+b+c),\\ Z_a =&\ (-(a+b-c) : a+b+c : 0). \end{aligned}$$

Similarly, we also have the B-touchpoints triangle $\mathbf{T}_b := X_bY_bZ_b$ from the B-excircle $I_b(r_b)$ and the C-touchpoints triangles $\mathbf{T}_c := X_cY_cZ_c$ from the C-excircle $I_c(r_c)$ (see Figure 3).

Consider the reflection I'_a of I_a in the line Y_aZ_a. Since Y_aZ_a is perpendicular to the line AI_a, I'_a lies on AI_a, and $I_aI'_a = 2r_a\sin\frac{A}{2}$. On the other hand, I_aI, being a diameter of the circle through I, B, I_a, C, has length $\frac{a}{\sin\left(\frac{B}{2}+\frac{C}{2}\right)} = \frac{a}{\cos\frac{A}{2}}$. It follows that

$$I_aI'_a : I_aI = 2r_a\sin\frac{A}{2} : \frac{a}{\cos\frac{A}{2}} = r_a : \frac{a}{\sin A} = r_a : 2R = I_aX_a : I_aI'.$$

Therefore, $X_aI'_a$ and $I'I$ are parallel. Note that the midpoint of $X_aI'_a$ is the nine-point center of $\mathbf{T}_a$.

The same conclusions apply to the other two touchpoints triangles $\mathbf{T}_b := X_bY_bZ_b$ and $\mathbf{T}_c := X_cY_cZ_c$ associated with the B- and C-excircles $I_b(r_b)$ and $I_c(r_c)$.

Corollary 5. (a) *The Euler lines of the touchpoints triangles of the excircles are concurrent at* O.
(b) *The nine-point centers of the touchpoints triangles form a triangle perspective with the extouch triangle* $X_aY_bZ_c$ *at the infinite point of the* OI *line.*

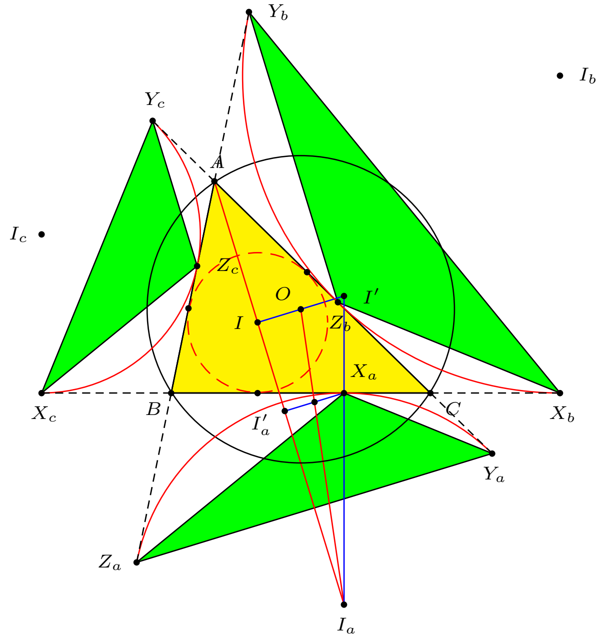

Figure 3.

Let H^a be the orthocenter of $\mathbf{T}_a$. Since the nine-point center N_a is also the midpoint of its circumcenter I_a and H^a, we obtain, from the parallelogram $H^aX_aI_aI'_a$,

$$\begin{aligned} H^a =&\ X_a + I'_a - I_a \\ =&\ X_a + \frac{r_a}{2R}(I - I_a) \\ =&\ X_a + \frac{2s(s-b)(s-c)}{abc}(I - I_a) \\ =&\ \frac{(s-b)B + (s-c)C}{a} + \frac{2s(s-b)(s-c)}{abc}\left(\frac{aA + bB + cC}{2s} - \frac{-aA + bB + cC}{2(s-a)}\right) \\ =&\ \frac{(b+c)(s-b)(s-c)}{bc(s-a)}A + \frac{s(c-a)(s-b)}{ac(s-a)}B - \frac{s(a-b)(s-c)}{ab(s-a)}C. \end{aligned}$$

Proposition 6. *In homogeneous barycentric coordinates, the orthocenters of the touchpoints triangles are the points*

$$\begin{aligned} H^a =& \left(\frac{a(b+c)}{a+b+c} : \frac{b(c-a)}{a+b-c} : \frac{-c(a-b)}{c+a-b}\right), \\ H^b =& \left(\frac{-a(b-c)}{a+b-c} : \frac{b(c+a)}{a+b+c} : \frac{c(a-b)}{b+c-a}\right), \\ H^c =& \left(\frac{a(b-c)}{c+a-b} : \frac{-b(c-a)}{b+c-a} : \frac{c(a+b)}{a+b+c}\right). \end{aligned}$$

Remark. The orthocenter H^a of $\mathbf{T}_a$ divides OI_a in the ratio $R - r_a : r_a$; similarly for H^b and H^c. These orthocenters lie on the Jerabek hyperbola $\mathscr{J}$ (see Figure 4).

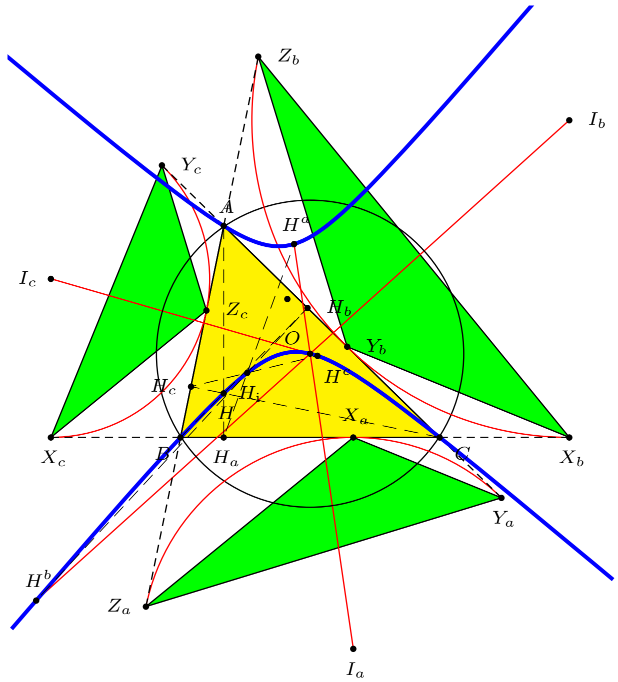

Figure 4.

Proposition 7. *The triangle $H^aH^bH^c$ is perspective with the orthic triangle $H_aH_bH_c$ (of ABC) at H_{i}.*

5. Kariya's theorem for the A-touchpoints triangle

Consider the image of the A-touchpoints triangle $\mathbf{T}_a$ under the homothety $\mathsf{h}\left(I_a, \frac{t}{r_a}\right)$ for a real number t. This is the triangle $\mathbf{T}_a(t)$ with vertices $X_a(t)$, $Y_a(t)$, $Z_a(t)$ on the lines I_aX_a, I_aY_a, I_aZ_a respectively, such that

$$I_aX_a(t) = I_aY_a(t) = I_aZ_a(t) = t.$$

These points can be determined from Lemma 1 by putting $k = \frac{t}{r_a}$. In homogeneous barycentric coordinates, they are

$$X_a(t) = (2(r_a - t)a^2 : -2r_aab + t(a^2 + b^2 - c^2) : -2r_aca + t(c^2 + a^2 - b^2)), \quad (8)$$
$$Y_a(t) = (-2r_aab + t(a^2 + b^2 - c^2) : 2(r_a - t)b^2 : 2r_abc + t(b^2 + c^2 - a^2)), \quad (9)$$
$$Z_a(t) = (-2r_aca + t(c^2 + a^2 - b^2) : 2r_abc + t(b^2 + c^2 - a^2) : 2(r_a - t)c^2). \quad (10)$$

Proposition 8. *The lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at the point*

$$Q_a(t) = \left(\frac{1}{2r_abc + t(b^2+c^2-a^2)} : \frac{1}{-2r_aca + t(c^2+a^2-b^2)} : \frac{1}{-2r_aab + t(a^2+b^2-c^2)}\right), \tag{11}$$

which is the isogonal conjugate of the point $P_a(t)$ dividing OI_a in the ratio

$$OP_a(t) : P_a(t)I_a = R : -t.$$

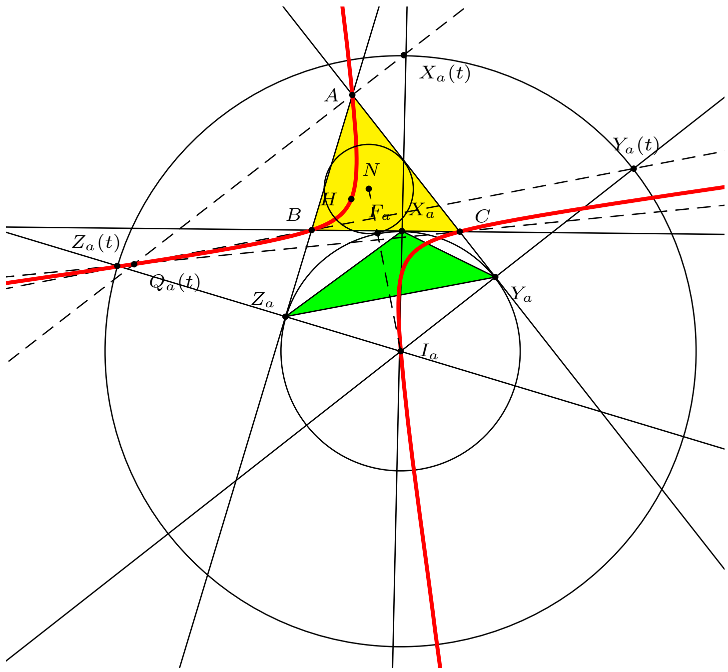

Figure 5.

Proof. Rewrite the homogeneous barycentric coordinates of $X_a(t)$, $Y_a(t)$, $Z_a(t)$ as follows:

$$X_a(t) = \left(*****: \frac{1}{-2r_aca + t(c^2+a^2-b^2)} : \frac{1}{-2r_aab + t(a^2+b^2-c^2)}\right),$$

$$Y_a(t) = \left(\frac{1}{2r_abc + t(b^2+c^2-a^2)} : *****: \frac{1}{-2r_aab + t(a^2+b^2-c^2)}\right),$$

$$Z_a(t) = \left(\frac{1}{2r_abc + t(b^2+c^2-a^2)} : \frac{1}{-2r_aca + t(c^2+a^2-b^2)} : *****\right).$$

It follows easily that the lines $AX_a(t)$, $BY_a(t)$, $CZ_a(t)$ are concurrent at a point with coordinates given in (11) above (see Figure 5). This is clearly the isogonal conjugate of the point

$$\begin{aligned} P_a(t) = & \ (a^2(2r_abc+t(b^2+c^2-a^2)), b^2(-2r_aca+t(c^2+a^2-b^2)), \\ & \qquad c^2(-2r_aab+t(a^2+b^2-c^2))) \\ = & \ -2r_aabc(-a,b,c)+t((a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2)) \\ = & \ -2r_a \cdot 4Rrs(b+c-a)\cdot I_a + t\cdot 16\Delta^2 \cdot O \\ = & \ -16rr_as(s-a)\cdot R\cdot I_a + 16\Delta^2\cdot t\cdot O \\ = & \ -16\Delta^2(R\cdot I_a - t\cdot O). \end{aligned}$$

In absolute barycentric coordinates, $P_a(t) = \frac{R\cdot I_a - t\cdot O}{R-t}$. This is the point dividing OI_a in the ratio $OP_a(t) : P_a(t)I_a = R : -t$. □

Proposition 9. *The locus of the point $Q_a(t)$ is the rectangular circum-hyperbola*

$$\mathscr{F}_a:\ a(b-c)(a+b+c)yz+b(c+a)(a+b-c)zx-c(a+b)(c+a-b)xy=0$$

with center

$$F_a = (-(b-c)^2(a+b+c) : (c+a)^2(a+b-c) : (a+b)^2(c+a-b)),$$

the point of tangency of the nine-point circle with the A-excircle.

Proof. By Proposition 8, the locus of $Q_a(t)$ is the isogonal conjugate of the line OI_a. It is a rectangular hyperbola since it contains the orthocenter H, the isogonal conjugate of O. The center of the hyperbola is a point on the nine-point circle. The equation of the line OI_a is

$$\begin{vmatrix} x & y & z \\ a^2(b^2+c^2-a^2) & b^2(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \\ -a & b & c \end{vmatrix} = 0.$$

After simplification, this becomes

$$\begin{aligned} & -bc(b-c)(a+b+c)(b+c-a)x \\ & -ca(c+a)(a+b-c)(b+c-a)y \\ & +ab(a+b)(b+c-a)(c+a-b)z = 0. \end{aligned}$$

Replacing (x,y,z) by (a^2yz, b^2zx, c^2xy) we obtain the equation of the hyperbola $\mathscr{F}_a$ given above. Since the center of the circumconic $pyz+qzx+rxy=0$ is the point

$$(p(q+r-p) : q(r+p-q) : r(p+q-r)),$$

with

$$p = bc(b-c)(a+b+c), \quad q = ca(c+a)(a+b-c), \quad r = -ab(a+b)(c+a-b),$$

we obtain the center of the hyperbola as the point

$$\begin{aligned} F_a = & \ p(q+r-p) : q(r+p-q) : r(p+q-r) \\ = & \ -2abc(b-c)^2(a+b+c) : 2abc(c+a)^2(a+b-c) : 2abc(a+b)^2(c+a-b) \\ = & \ -(b-c)^2(a+b+c) : (c+a)^2(a+b-c) : (a+b)^2(c+a-b). \end{aligned}$$

This is indeed a point on the line joining I_a to the nine-point center

$$N = (a^2(b^2+c^2)-(b^2-c^2)^2 : b^2(c^2+a^2)-(c^2-a^2)^2 : c^2(a^2+b^2)-(a^2-b^2)^2). \tag{12}$$

It is routine to verify that

$$\begin{aligned} & 2abc(-a,b,c) \\ & + (a^2(b^2+c^2)-(b^2-c^2)^2, b^2(c^2+a^2)-(c^2-a^2)^2, c^2(a^2+b^2)-(a^2-b^2)^2) \\ = & \ (b+c-a)(-(b-c)^2(a+b+c), (c+a)^2(a+b-c), (a+b)^2(c+a-b)). \end{aligned}$$

Since the coordinate sum in (12) is

$$\begin{aligned} & 2(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4) \\ = & \ 2(a+b+c)(b+c-a)(c+a-b)(a+b-c) \\ = & \ 32\Delta^2, \end{aligned}$$

this is the point dividing NI_a in the ratio

$$2abc(b+c-a) : 32\Delta^2 = 2\cdot 4R\Delta(b+c-a) : 32\Delta^2 = \frac{R}{2} : \frac{2\Delta}{b+c-a} = \frac{R}{2} : r_a,$$

i.e., the point of tangency of the nine-point circle and the A-excircle. □

Proposition 10. *The line joining $Q_a(t)$ and $Q_a(-t)$ contains the orthocenter H^a of the A-touchpoints triangle $\mathbf{T}_a$.*

Proof. The equation of the line $Q_a(t)Q_a(-t)$ is

$$\begin{aligned} & a(b-c)(a+b+c)(-4b^2c^2r_a^2+t^2(b^2+c^2-a^2)^2)x \\ & + b(c+a)(a+b-c)(-4c^2a^2r_a^2+t^2(c^2+a^2-b^2)^2)y \\ & - c(a+b)(c+a-b)(-4a^2b^2r_a^2+t^2(a^2+b^2-c^2)^2)z \\ & = 0. \end{aligned}$$

Substituting the coordinates of the point H^a given in Proposition 6, we obtain

$$\begin{aligned} & -4a^2b^2c^2r_a^2\left((b^2-c^2)+(c^2-a^2)+(a^2-b^2)\right) \\ & + t^2\left(a^2(b^2-c^2)(b^2+c^2-a^2)^2+b^2(c^2-a^2)(c^2+a^2-b^2)^2\right. \\ & \qquad \left. + c^2(a^2-b^2)(a^2+b^2-c^2)^2\right) \\ = & \ 0, \end{aligned}$$

as in the proof of Proposition 4. □

6. The triad of ex-Feuerbach hyperbolas

We call the hyperbola $\mathscr{F}_a$ in Proposition 9 the A-ex-Feuerbach hyperbola. We also consider the triangles $\mathbf{T}_b(t) := X_b(t)Y_b(t)Z_b(t)$ and $\mathbf{T}_c(t) := X_c(t)Y_c(t)Z_c(t)$. These vertices are the points on the lines I_bX_b, I_bY_b, I_bZ_b satisfying

$$I_bX_b(t) = I_bY_b(t) = I_bZ_b(t) = t,$$

and $X_c(t)$, $Y_c(t)$, $Z_c(t)$ on I_cX_c, I_cY_c, I_cZ_c satisfying

$$I_cX_c(t) = I_cY_c(t) = I_cZ_c(t) = t.$$

In homogeneous barycentric coordinates,

$$\begin{aligned}
X_b(t) =&\ (2(r_b - t)a^2 : -2r_bab + t(a^2 + b^2 - c^2) : 2r_bca + t(c^2 + a^2 - b^2)),\\
Y_b(t) =&\ (-2r_bab + t(a^2 + b^2 - c^2) : 2(r_b - t)b^2 : -2r_bbc + t(b^2 + c^2 - a^2)),\\
Z_b(t) =&\ (2r_bca + t(c^2 + a^2 - b^2) : -2r_bbc + t(b^2 + c^2 - a^2) : 2(r_b - t)c^2);\\
X_c(t) =&\ (2(r_c - t)a^2 : 2r_cab + t(a^2 + b^2 - c^2) : -2r_cca + t(c^2 + a^2 - b^2)),\\
Y_c(t) =&\ (2r_cab + t(a^2 + b^2 - c^2) : 2(r_c - t)b^2 : -2r_cbc + t(b^2 + c^2 - a^2)),\\
Z_c(t) =&\ (-2r_cca + t(c^2 + a^2 - b^2) : -2r_cbc + t(b^2 + c^2 - a^2) : 2(r_c - t)c^2).
\end{aligned}$$

Clearly there are analogous hyperbolas $\mathscr{F}_b$ and $\mathscr{F}_c$ which are isogonal conjugates of the lines OI_b and OI_c. These hyperbolas have centers

$$\begin{aligned}
F_b =&\ ((b + c)^2(a + b - c) :\ -(c - a)^2(a + b + c) :\ (a + b)^2(b + c - a)),\\
F_c =&\ ((b + c)^2(c + a - b) :\ (c + a)^2(b + c - a) :\ -(a - b)^2(a + b + c)).
\end{aligned}$$

Remark. The centers of the triad of ex-Feuerbach hyperbolas, being the points of tangency of the nine-point circle with the excircles, are perspective with ABC at the outer Feuerbach point

$$X(12) = \left(\frac{(b+c)^2}{b+c-a} :\ \frac{(c+a)^2}{c+a-b} :\ \frac{(a+b)^2}{a+b-c}\right).$$

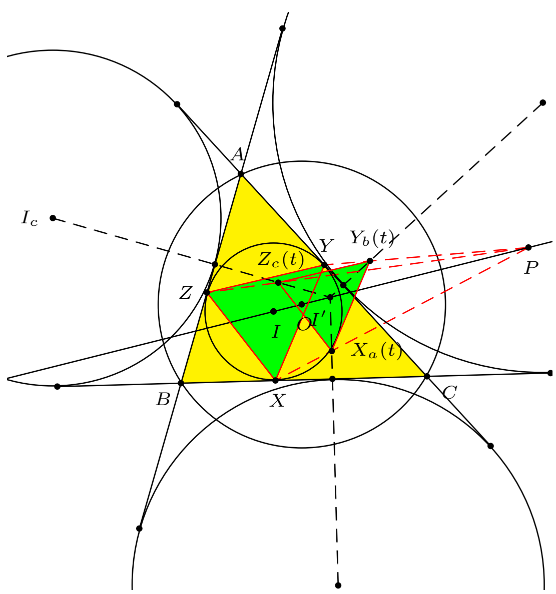

Figure 6.

Proposition 11. *The triangle $X_a(t)Y_b(t)Z_c(t)$ is homothetic to the intouch triangle at the point $(a(r_a - t) : b(r_b - t) : c(r_c - t)$, which divides OI in the ratio* $2R + r - t : -2r$ (see Figure 6).

7. Some special cases

7.1. $t = R$. By Corollary 5(b), the point $P_a(R)$ is the infinite point on the line OI_a. It follows that $Q_a(R)$ is on the circumcircle. It is the (fourth) intersection of the hyperbola $\mathscr{F}_a$ with the circumcircle. These points are the reflections of H in F_a, F_b, F_c respectively (see Figure 7).

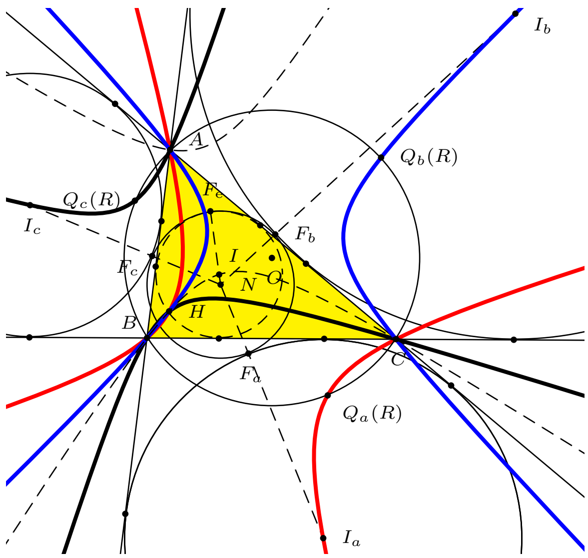

Figure 7.

7.2. $t = 2R$. It is well known that the circumcenter of the excentral triangle is the reflection I' of I in O, and is the point of concurrency of the perpendiculars from the excenters to the respective sidelines of triangle ABC (see Figure 6), and the circumradius is $2R$. It follows that the points $X_a(2R)$, $Y_b(2R)$, $Z_c(2R)$ all coincide with this circumcenter. It follows that the lines $AQ_a(2R)$, $BQ_b(2R)$, $CQ_c(2R)$ are concurrent at this point. $t = 2R$ is the only nonzero value of t for which the triangle $Q_a(t)Q_b(t)Q_c(t)$ is perspective with ABC.

In this case, both $Y_c(2R)$ and $Z_b(2R)$ are the reflection of I' in the line I_bI_c. We call this X'. The line AX' is the reflection of AI' in I_bI_c. Since I_bI_c is the external bisector of angle A of triangle ABC, AX' and AI' are isogonal lines with respect to this angle. Likewise, we have $Y' = Z_a(2R) = X_c(2R)$ with BY', BI' isogonal with respect to B, and and $Z' = Y_a(2R) = X_b(2R)$ with CZ', CI' isogonal with respect to C. It follows that AX', BY', CZ' are concurrent at a point P, which is the isogonal conjugate of I', and lies on the Feuerbach hyperbola $\mathscr{F}$ (see Figure 8).

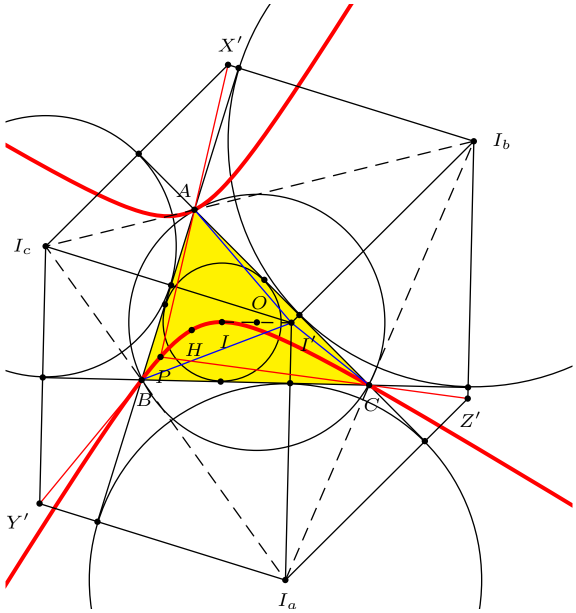

Figure 8.

8. Second tangents from O to the ex-Feuerbach hyperbolas

The hyperbolas $\mathscr{F}_a$, $\mathscr{F}_b$, $\mathscr{F}_c$ also appear in [1], where they are called the ex-central Feuerbach hyperbolas. Neuberg [5] also mentioned these hyperbolas. The A-ex-Feuerbach hyperbola $\mathscr{F}_a$, being the isogonal conjugate of the line OI_a, is tangent to the line at I_a (see Figure 9). If the second tangent from O to $\mathscr{F}_a$ touches it at T_a, then the line I_aT_a is the polar of O with respect to the hyperbola $\mathscr{F}_a$. This is the line

$$\begin{aligned} &bc(b-c)(a+b+c)(a(b^2+c^2-a^2)+(b+c)(c+a-b)(a+b-c))x \\ &+ca(c+a)(a+b-c)(c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b))y \\ &-ab(a+b)(c+a-b)(b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a))z \\ &=0. \end{aligned}$$

Apart from the excenter I_a, this line intersects the hyperbola $\mathscr{F}_a$ again at

$$\begin{aligned} T_a = \Bigg(&\frac{a}{a(b^2+c^2-a^2)+(b+c)(c+a-b)(a+b-c)} \\ &: \frac{b}{c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b)} \\ &: \frac{c}{b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a)} \Bigg). \end{aligned}$$

Similarly, the second tangents from O to $\mathscr{F}_b$ and $\mathscr{F}_c$ (apart from OI_b and OI_c) touch these hyperbolas at

$$T_b = \left(\frac{a}{c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b)}\right.$$
$$: \frac{b}{b(c^2+a^2-b^2)+(c+a)(a+b-c)(b+c-a)}$$
$$\left. : \frac{c}{a(b^2+c^2-a^2)-(b+c)(c+a-b)(a+b-c)}\right),$$

and

$$T_c = \left(\frac{a}{b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a)}\right.$$
$$: \frac{b}{a(b^2+c^2-a^2)-(b+c)(c+a-b)(a+b-c)}$$
$$\left. : \frac{c}{c(a^2+b^2-c^2)+(a+b)(b+c-a)(c+a-b)}\right),$$

These three points of tangency form a triangle perspective with ABC at

$$T = (a(a(b^2+c^2-a^2)-(b+c)(c+a-b)(a+b-c))$$
$$: b(b(c^2+a^2-b^2)-(c+a)(a+b-c)(b+c-a))$$
$$: c(c(a^2+b^2-c^2)-(a+b)(b+c-a)(c+a-b))).$$

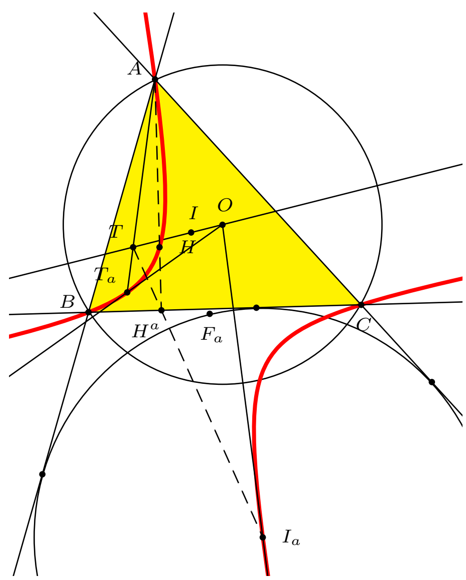

Figure 9.

This is the triangle center $X(46)$ in [4]. It has a number of interesting properties. It divides OI externally in the ratio $R+r : -2r$, and can be constructed as the cevian quotient H/I. In other words, it is the perspector of the orthic triangle and the excentral triangle. Therefore, the point T_a, and similarly T_b and T_c, can be easily constructed as follows.
(1) Join I_a and H_a to intersect the line OI at T.
(2) Join A and T to intersect the hypebola $\mathscr{F}_a$ at T_a (see Figure 9).

9. A correspondence between the Euler line and the Feuerbach hyperbola

Let $P = (u : v : w)$ be an aribitrary point. The lines PI_a, PI_b, PI_c intersect the respective ex-Feuerbach hyperbolas at

$$W_a = \left(\frac{(b-c)(a+b+c)}{cv-bw} : \frac{c+a)(a+b-c)}{aw+cu} : \frac{(a+b)(c+a-b)}{bu+av}\right),$$
$$W_b = \left(\frac{(b+c)(a+b-c)}{cv+bw} : \frac{c-a)(a+b+c)}{aw-cu} : \frac{(a+b)(b+c-a)}{bu+av}\right),$$
$$W_c = \left(\frac{(b+c)(c+a-b)}{cv+bw} : \frac{c+a)(b+c-a)}{aw+cu} : \frac{(a-b)(a+b-c)}{bu-av}\right).$$

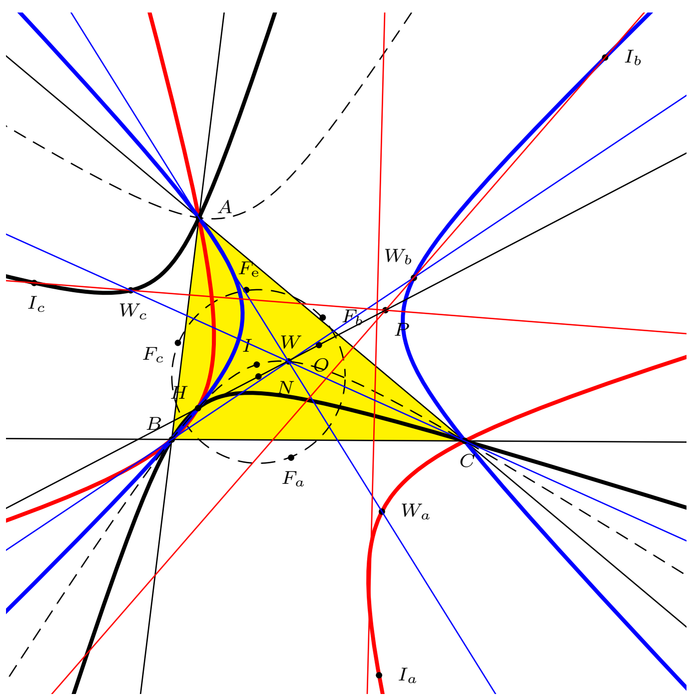

Figure 10.

These form a triangle perspective with ABC. The perspector is the point

$$W = \left(\frac{b+c}{(b+c-a)(cv+bw)} : \frac{c+a}{(c+a-b)(aw+cu)} : \frac{a+b}{(a+b-c)(bu+av)}\right).$$

Proposition 12. *The perspector W is on the Feuerbach hyperbola if and only if P lies on the Euler line.*

Proof. The perspector W is on the Feuerbach hyperbola if and only if its isogonal conjugate

$$W^* = \left(\frac{a^2(b+c-a)(cv+bw)}{b+c} : \frac{b^2(c+a-b)(aw+cu)}{c+a} : \frac{c^2(a+b-c)(bu+av)}{a+b}\right)$$

lies on the line OI with equation

$$\sum_{\text{cyclic}} bc(b-c)(b+c-a)x = 0.$$

By substitution, we have

$$\begin{aligned}
0 = & \sum_{\text{cyclic}} bc(b-c)(b+c-a) \cdot \frac{a^2(b+c-a)(cv+bw)}{b+c} \\
= & \; abc \sum_{\text{cyclic}} \frac{a(b-c)(b+c-a)^2(cv+bw)}{b+c} \\
= & \; \frac{abc}{(b+c)(c+a)(a+b)} \sum_{\text{cyclic}} a(b-c)(c+a)(a+b)(b+c-a)^2(cv+bw).
\end{aligned}$$

Ignoring the nonzero factor, we have

$$\begin{aligned}
0 = & \sum_{\text{cyclic}} a(b-c)(c+a)(a+b)(b+c-a)^2(cv+bw) \\
= & \sum_{\text{cyclic}} (b(c-a)(a+b)(b+c)(c+a-b)^2 \cdot cu \\
& \qquad + c(a-b)(b+c)(c+a)(a+b-c)^2 \cdot bu) \\
= & \sum_{\text{cyclic}} bc(b+c)((c-a)(a+b)(c+a-b)^2 + (a-b)(c+a)(a+b-c)^2)u \\
= & \sum_{\text{cyclic}} bc(b+c) \cdot (-2a(b-c)(b^2+c^2-a^2))u \\
= & \; -2abc \sum_{\text{cyclic}} (b+c)(b-c)(b^2+c^2-a^2)u.
\end{aligned}$$

This means that $P = (u : v : w)$ lies on the Euler line (with equation given in (6)). □

If P divides OH in the ratio $OP : PH = t : 1-t$, then the isogonal conjugate of W is the point dividing OI in the ratio $OW^* : W^*I = \frac{R}{2}(1-t) : rt$. A simple application of Menelaus' theorem yields the following construction of W^*. Let P' be the inferior of P. Then the line F_eP' intersects OI at W^* (see Figure 11).

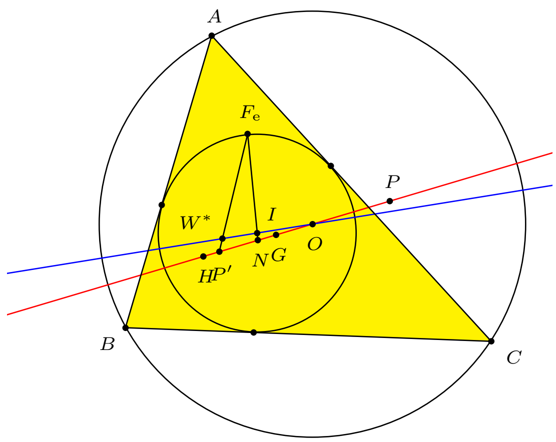

Figure 11.

If we put $W = (x : y : z)$, then P is the point with coordinates

$$\begin{aligned}(u : v : w) = &\left(a\left(-\frac{a(b+c)}{(b+c-a)x} + \frac{b(c+a)}{(c+a-b)y} + \frac{c(a+b)}{(a+b-c)z}\right)\right.\\ &: \ b\left(\frac{a(b+c)}{(b+c-a)x} - \frac{b(c+a)}{(c+a-b)y} + \frac{c(a+b)}{(a+b-c)z}\right)\\ &: \ \left.c\left(\frac{a(b+c)}{(b+c-a)x} + \frac{b(c+a)}{(c+a-b)y} - \frac{c(a+b)}{(a+b-c)z}\right)\right).\end{aligned}$$

10. The asymptotes of the Feuerbach hyperbolas

As is well known, the asymptotes of a rectangular circum-hyperbola which is the isogonal conjugate of a line through O are the Simson lines of the intersections of the line with the circumcircle. For the Feuerbach hyperbola and the ex-Feuerbach hyperbolas, we give an easier construction based on the fact that the lines joining the circumcenter to the incenter and the excenters are tangent to the respective Feuerbach hyperbolas.

Lemma 13. *Let P be a point on a rectangular hyperbola with center O. The tangent to the hyperbola at P intersects the asymptotes at two points on the circle with center P, passing through O.*

Proof. Set up a Cartesian coordinate system with the asymptotes as axes. The equation of the rectangular hyperbola is $xy = c^2$ for some c. If $P\left(ct, \frac{c}{t}\right)$ is a point on the hyperbola, the tangent at P is the line $\frac{1}{2}\left(\frac{c}{t}x + cty\right) = c^2$, or $\frac{x}{t} + yt = 2c$. It intersects the asymptotes (axes) at $X(2ct, 0)$ and $Y\left(0, \frac{2c}{t}\right)$. Since P is the midpoint of XY, $PO = PX = PY$. □

Proposition 14. (a) *The lines joining the Feuerbach point F_e to the intersections of the incircle with the line OI are the asymptotes of the Feuerbach hyperbola $\mathscr{F}$.*

(b) *The lines joining the point F_a to the intersections of the A-excircle with the line OI_a are the asymptotes of the A-ex-Feuerbach hyperbola $\mathscr{F}_a$; similarly for the hyperbolas $\mathscr{F}_b$ and $\mathscr{F}_c$* (see Figure 12).

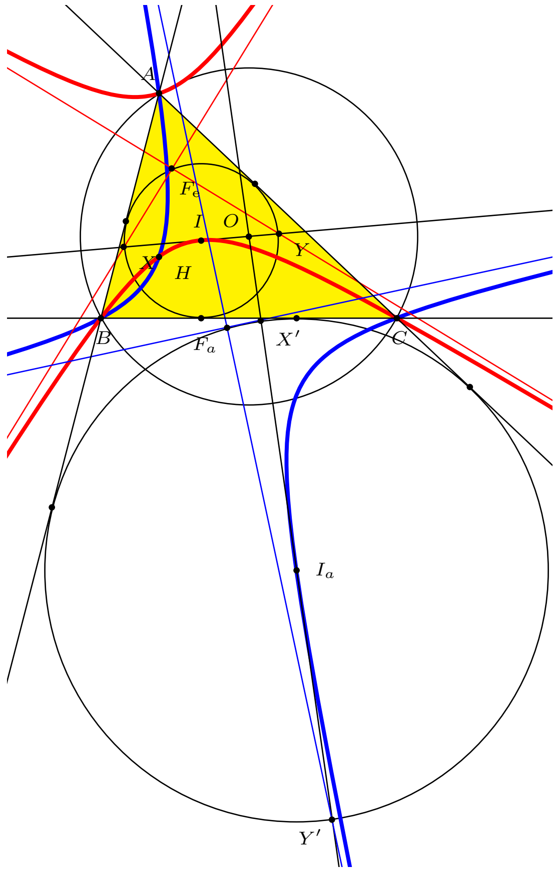

Figure 12.

11. More on the touchpoints triangles

11.1. *The symmedian points of the touchpoints triangles.* Since the A-excircle is the circumcircle of the touchpoints triangle $\mathbf{T}_a$, and the lines BC, CA, AB are the tangents at its vertices, the symmedian point of $\mathbf{T}_a$ is the point of concurrency of BY_a, CZ_a, and AX_a, i.e.,

$$K_a = (-(c+a-b)(a+b-c) : (a+b+c)(c+a-b) : (a+b+c)(a+b-c)).$$

Note that K_a is a point on the A-ex-Feuerbach hyperbola $\mathscr{F}_a$.

The line joining K_a to I_a is the Brocard axis of $\mathbf{T}_a$. It has equation

$$(b-c)(a+b+c)^2x + (c+a)(a+b-c)^2y - (a+b)(c+a-b)^2z = 0.$$

Proposition 15. (a) *The Brocard axes of the touchpoints triangles and the intouch triangle are concurrent at the deLongchamps point L, which is the point on the Euler line of triangle ABC dividing OH in the ratio $-1 : 2$.*

(b) *The van Aubel lines* (joining the orthocenter and the symmedian point) *of the touchpoints triangles and the intouch triangle are concurrent at $H^\bullet$, the isotomic conjugate of the orthocenter of ABC* (see Figure 13).

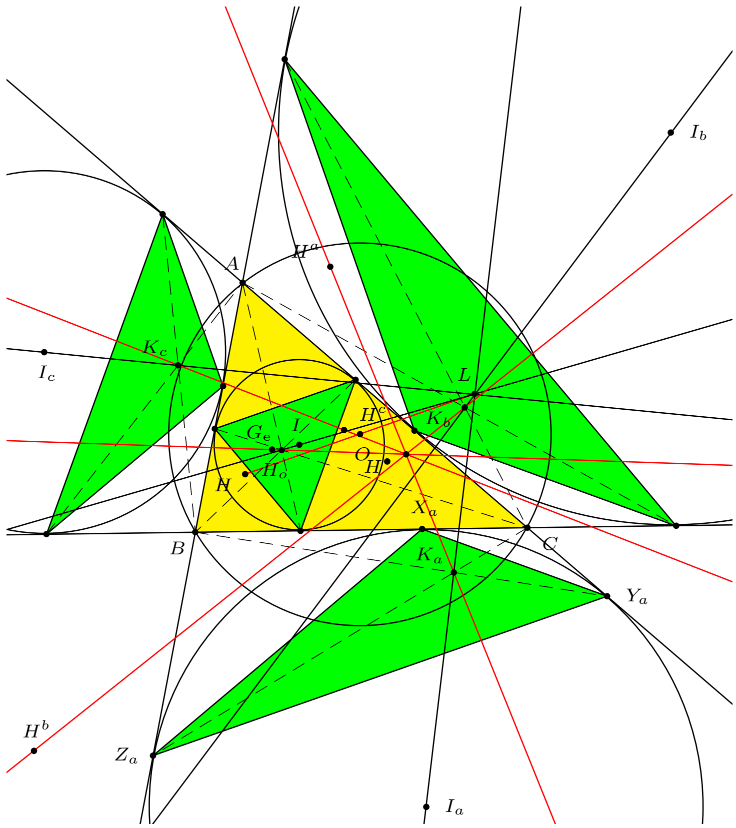

Figure 13.

Remark. The intersection of the Euler line with the line IG_e at the deLongchamps point L is a well known fact. See [6].

Proposition 16. *The triangle $H^aH^bH^c$ is perspective with the cevian triangle of Q if and only if Q lies on the line*

$$\mathscr{L}: \frac{(b+c)(b^2+c^2-a^2)}{b+c-a}x + \frac{(c+a)(c^2+a^2-b^2)}{c+a-b}y + \frac{(a+b)(a^2+b^2-c^2)}{a+b-c}z = 0$$

or the Feuerbach hyperbola $\mathscr{F}$.

(a) *If Q traverses $\mathscr{L}$, the perspector traverses the line* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(b) *If Q is on the Feuerbach hyperbola, the perspector P lies on the Jerabek hyperbola $\mathscr{J}$. The line joining QP passes through the orthocenter H* (see Figure 14).

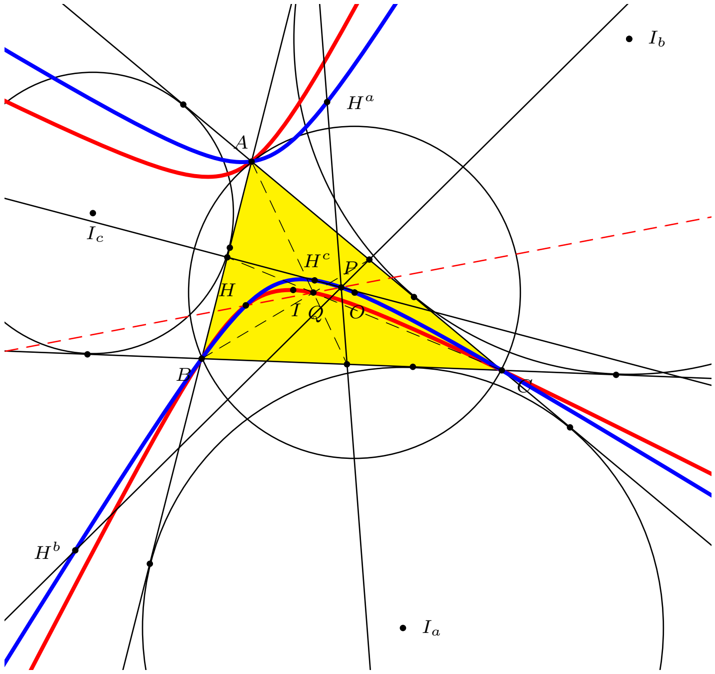

Figure 14.

Proof. Let $Q = (u : v : w)$ with cevian triangle $Q_aQ_bQ_c$ where $Q_a = (0 : v : w)$, $Q_b = (u : 0 : w)$, $Q_c = (u : v : 0)$. The equations of the lines H^aQ_a, H^bQ_b, H^cQ_c are

$$(a+b+c)(c(a-b)(a+b-c)v + b(c-a)(c+a-b)w)x \\ -a(c+a-b)(a+b-c)(b+c)(wy-vz) = 0, \tag{13}$$

$$(a+b+c)(a(b-c)(b+c-a)w + c(a-b)(a+b-c)u)y \\ -b(a+b-c)(b+c-a)(c+a)(uz-wx) = 0, \tag{14}$$

$$(a+b+c)(b(c-a)(c+a-b)u + a(b-c)(b+c-a)v)z \\ -c(b+c-a)(c+a-b)(a+b)(vx-uy) = 0. \tag{15}$$

Eliminating x, y, z from equations (13), (14), (15), we have

$$2abc(a+b+c)\left(\sum_{\text{cyclic}}(b+c)(c+a-b)(a+b-c)(b^2+c^2-a^2)u\right) \\ \left(\sum_{\text{cyclic}} a(b-c)(b+c-a)vw\right) = 0.$$

Therefore the lines H^aQ_a, H^bQ_b, H^cQ_c are concurrent if and only if $Q = (u : v : w)$ lies on the line $\mathscr{L}$ or the Feuerbach hyperbola $\mathscr{F}$.

Eliminating u, v, w from equations (13), (14), (15), we have

$$32\Delta^2(bcx + cay + abz)\left(\sum_{\text{cyclic}} a^2(b^2-c^2)(b^2+c^2-a^2)yz\right) = 0.$$

Therefore the locus of the point of concurrency is the union of the line $\mathscr{L}(I)$: $bcx + cay + abz = 0$ (the trilinear polar of the incenter) and the Jerabek hyperbola $\mathscr{J}$.

Now the line $\mathscr{L}$ contains the point

$$Q_0 = \left(\frac{a(b-c)(b+c-a)^2}{b+c} : \frac{b(c-a)(c+a-b)^2}{c+a} : \frac{c(a-b)(a+b-c)^2}{a+b}\right)$$

as is easily verified. Choosing $Q = (u : v : w)$ to be this point, and solving equations (13), (14), (15), we have the perspector

$$P_0 = (a(b^2-c^2) : b(c^2-a^2) : c(a^2-b^2))$$

on the line $\mathscr{L}(I)$. Therefore, by continuity, when Q traverses the line $\mathscr{L}$, P traverses $\mathscr{L}(I)$.

On the other hand, if Q lies on the Feuerbach hyperbola, then P lies on the Jerabek hyperbola. If we take Q to be the point

$$\left(\frac{1}{bc+t(b^2+c^2-a^2)} : \frac{1}{ca+t(c^2+a^2-b^2)} : \frac{1}{ab+t(a^2+b^2-c^2)}\right)$$

on the Feuerbach hyperbola, then P is the point

$$\left(\frac{a(b+c)(b+c-a)}{(b^2+c^2-a^2)(2rbc+t(b^2+c^2-a^2))} : \cdots : \cdots\right)$$

on the Jerabek hyperbola. The line joining Q and P contains the orthocenter H. □

Remarks. (1) The triangle center Q_0 appears in [4] as $X(1021)$.

(2) The triangle center P_0 is the intersection of the lines $bcx + cay + abz = 0$ and $ax + by + cz = 0$. It appears in [4] as $X(661)$.

(3) The line $\mathscr{L}$ can be constructed as the line containing the harmonic conjugates of $I_aH \cap BC$ in BC, $I_bH \cap CA$ in CA, and $I_cH \cap AB$ in AB. It is the trilinear polar of the triangle center $X(29)$.

Proposition 17. *The triangle $H^aH^bH^c$ is perspective with the anticevian triangle of Q if and only if Q lies on the orthic axis*

$$\mathscr{L}(H) : \quad (b^2+c^2-a^2)x + (c^2+a^2-b^2)y + (a^2+b^2-c^2)z = 0$$

or the circumconic

$$\mathscr{C} : \quad a(b^2-c^2)yz + b(c^2-a^2)zx + c(a^2-b^2)xy = 0$$

passing through I and its isotomic conjugate $I^\bullet = (bc : ca : ab)$.

(a) *If Q traverses the orthic axis, the perspector traverses the line* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (again).

(b) *If Q is on the circumconic $\mathscr{C}$, the perspector P lies on the Jerabek hyperbola* (again) *. The line QP passes through $H_{\rm i} = \left(\frac{a(b+c)}{b+c-a} : \cdots : \cdots\right)$, the common point of the two conics* (see Figure 15).

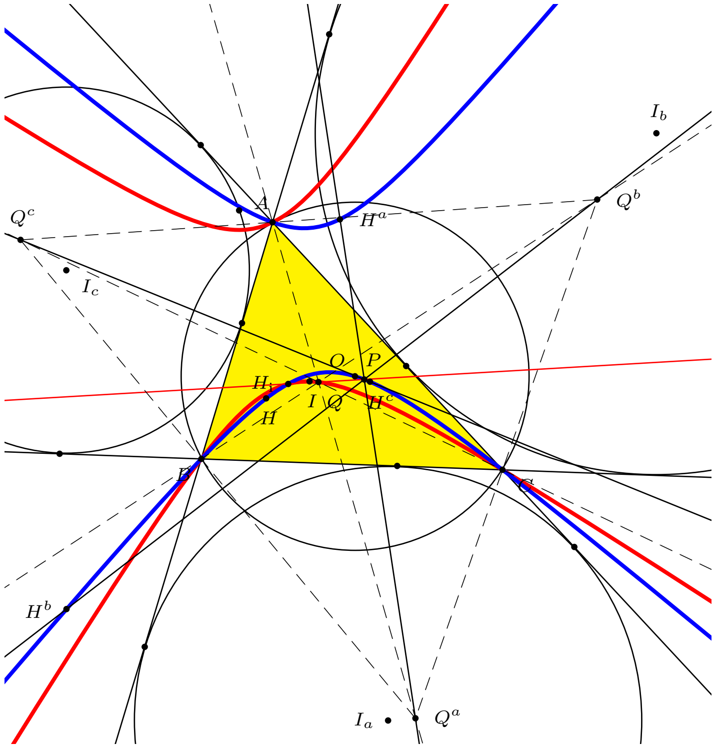

Figure 15.

Proof. Let $Q = (u : v : w)$ with anticevian triangle $Q^aQ^bQ^c$ where $Q^a = (-u : v : w)$, $Q^b = (u : -v : w)$, $Q^c = (u : v : -w)$. The equations of the lines H^aQ^a, H^bQ^b, H^cQ^c are

$$\begin{aligned}
&(a+b+c)(c(a-b)(a+b-c)v + b(c-a)(c+a-b)w)x \\
&+(a+b-c)(-a(b+c)(c+a-b)w + c(a-b)(a+b+c)u)y \\
&+(c+a-b)(b(c-a)(a+b+c)u + a(b+c)(a+b-c)v)z = 0, \qquad (16)\\
&(a+b-c)(c(a-b)(a+b+c)v + b(c+a)(b+c-a)w)x \\
&+(a+b+c)(a(b-c)(b+c-a)w + c(a-b)(a+b-c)u)y \\
&+(b+c-a)(-b(c+a)(a+b-c)u + a(b-c)(a+b+c)v)z = 0, \qquad (17)\\
&(c+a-b)(-c(a+b)(b+c-a)v + b(c-a)(a+b+c)w)x \\
&+(b+c-a)(a(b-c)(a+b+c)w + c(a+b)(c+a-b)u)y \\
&+(a+b+c)(b(c-a)(c+a-b)u + a(b-c)(b+c-a)v)z = 0. \qquad (18)
\end{aligned}$$

Eliminating x, y, z, we have

$$64abc\Delta^2\left(\sum_{\text{cyclic}}(b^2+c^2-a^2)u\right)\left(\sum_{\text{cyclic}}a(b^2-c^2)vw\right)=0.$$

Therefore the lines H^aQ^a, H^bQ^b, H^cQ^c are concurrent if and only if $Q=(u:v:w)$ lies on the orthic axis $\mathscr{L}(H)$ or the circumconic $\mathscr{C}$.

Eliminating u, v, w from equations (16), (17), (18), we have

$$64\Delta^2(bcx+cay+abz)\left(\sum_{\text{cyclic}}a^2(b^2-c^2)(b^2+c^2-a^2)yz\right)=0.$$

Therefore the locus of the point of concurrency is again the union of the line $\mathscr{L}(I)$: $bcx+cay+abz=0$ (the trilinear polar of the incenter) and the Jerabek hyperbola $\mathscr{J}$.

Now the circumconic $\mathscr{C}$ is the circum-hyperbola which is the isotomic conjugate of the line joining the incenter I to its isotomic conjugate. Its center is the point

$$(a(b-c)^2:b(c-a)^2:c(a-b)^2).$$

If we choose Q to be the point $\left(\frac{1}{at+bc}:\frac{1}{bt+ca}:\frac{1}{ct+ab}\right)$, then the perspector is the point

$$\left(\frac{a(b^2+c^2-a^2)}{at+bc}:\frac{b(c^2+a^2-b^2)}{bt+ca}:\frac{c(a^2+b^2-c^2)}{ct+ab}\right)$$

on the Jerabek hyperbola. In fact, $\mathscr{C}$ intersects $\mathscr{J}$ at $H_{\text{i}}=\left(\frac{a(b+c)}{b+c-a}:\cdots:\cdots\right)$, and the line joining Q and P passes through H_{i}.

$\square$

References

[1] R. C. Alperin, The Poncelet pencil of rectangular hyperbolas, *Forum Geom.*, 10 (2010) 15–20.

[2] F. G.-M., *Exercices de Géométrie*, 6th ed., 1920; Gabay reprint, Paris, 1991.

[3] B. Gibert, *Catalogue of Triangle Cubics*, available at http://bernard.gibert.pagesperso-orange.fr/ctc.html.

[4] C. Kimberling, *Encyclopedia of Triangle Centers*, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[5] J. Neuberg, Sur l'hyperbole de Feuerbach, *Mathesis*, série 2, 3 (1893) 81–89.

[6] A. Vandegehn, Soddy's circles and the de Longchamps point of a triangle, *Amer. Math. Monthly*, 71 (1964) 167–170.

[7] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, 2013, available at http://math.fau.edu/Yiu/Geometry.html

Sándor N. Kiss: "Constantin Brâncuşi" Technology Lyceum, Satu Mare, Romania
E-mail address: d.sandor.kiss@gmail.com

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA
E-mail address: yiu@fau.edu