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Properties of Equidiagonal Quadrilaterals

Martin Josefsson

Abstract. We prove eight necessary and sufficient conditions for a convex quadrilateral to have congruent diagonals, and one dual connection between equidiagonal and orthodiagonal quadrilaterals. Quadrilaterals with both congruent and perpendicular diagonals are also discussed, including a proposal for what they may be called and how to calculate their area in several ways. Finally we derive a cubic equation for calculating the lengths of the congruent diagonals.

1. Introduction

One class of quadrilaterals that have received little interest in the geometrical literature are the *equidiagonal quadrilaterals*. They are defined to be quadrilaterals with congruent diagonals. Three well known special cases of them are the isosceles trapezoid, the rectangle and the square, but there are other as well. Furthermore, there exists many equidiagonal quadrilaterals that besides congruent diagonals have no special properties. Take any convex quadrilateral $ABCD$ and move the vertex D along the line BD into a position D' such that $AC = BD'$. Then $ABCD'$ is an equidiagonal quadrilateral (see Figure 1).

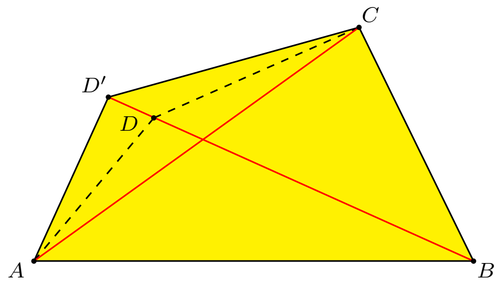

Figure 1. An equidiagonal quadrilateral $ABCD'$

Before we begin to study equidiagonal quadrilaterals, let us define our notations. In a convex quadrilateral $ABCD$, the sides are labeled $a = AB$, $b = BC$, $c = CD$ and $d = DA$, and the diagonals are $p = AC$ and $q = BD$. We use θ for the angle between the diagonals. The line segments connecting the midpoints of opposite sides of a quadrilateral are called the bimedians and are denoted m and n, where m connects the midpoints of the sides a and c.

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2. Characterizations of equidiagonal quadrilaterals

Of the seven characterizations for equidiagonal quadrilaterals that we will prove in this section, three have already appeared in our previous papers [11] and [12]. We include them here anyway for the sake of completeness. One of them is proved in a new way.

It is well known that the midpoints of the sides in any quadrilateral are the vertices of a parallelogram, called Varignon's parallelogram. The diagonals in this parallelogram are the bimedians of the original quadrilateral and the sides in the Varignon parallelogram are half as long as the diagonal in the original quadrilateral that they are parallel to. When studying equidiagonal quadrilaterals, properties of the Varignon parallelogram proves to be useful.

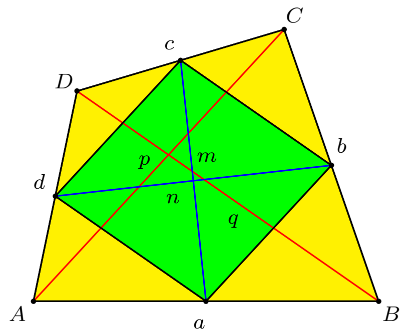

Figure 2. The Varignon parallelogram

Using the parallelogram law in the Varignon parallelogram yields (see Figure 2)

$$m^2+n^2=2\left(\left(\frac{p}{2}\right)^2+\left(\frac{q}{2}\right)^2\right)$$

which is equivalent to

$$p^2+q^2=2(m^2+n^2). \tag{1}$$

This equality is valid in all convex quadrilaterals.

For the product of the diagonals we have a necessary and sufficient condition of equidiagonal quadrilaterals in terms of the bimedians.

Proposition 1. *The product of the diagonals p and q in a convex quadrilateral with bimedians m and n satisfies*

$$pq \le m^2+n^2$$

where equality holds if and only if it is an equidiagonal quadrilateral.

Proof. By adding and subtracting $2pq$ to the left hand side of (1), we get

$$2pq \le (p-q)^2+2pq = 2(m^2+n^2).$$

The inequality follows, with equality if and only if $p=q$. □

The first part in the following theorem was proved by us as Theorem 7 (ii) in [11], but we repeat the short argument here.

Theorem 2. *A convex quadrilateral is equidiagonal if and only if*
(i) *the bimedians are perpendicular, or*
(ii) *the midpoints of its sides are the vertices of a rhombus.*

Proof. (i) It is well known that a quadrilateral has perpendicular diagonals if and only if the sum of the squares of two opposite sides is equal to the sum of the squares of the other two sides (see Theorem 1 in [11]). Hence we get

$$p = q \quad \Leftrightarrow \quad \left(\frac{p}{2}\right)^2 + \left(\frac{p}{2}\right)^2 = \left(\frac{q}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \quad \Leftrightarrow \quad m \perp n$$

since opposite sides in a parallelogram are congruent.

(ii) A parallelogram is a rhombus if and only if its diagonals are perpendicular. Since the diagonals in the Varignon parallelogram are the bimedians of the original quadrilateral (see Figure 2), (ii) is equivalent to (i). □

The next characterization is about the area of the quadrilateral. To prove it in a new way compared to what we did in [12, p.19], we need the following area formula for convex quadrilaterals. We cannot find a reference for this formula, but it is similar to one we derived in [10].

Theorem 3. *A convex quadrilateral with diagonals p, q and bimedians m, n has the area*

$$K = \sqrt{m^2n^2 - \left(\frac{p^2 - q^2}{4}\right)^2}.$$

Proof. Rewriting (1), we have in all convex quadrilaterals

$$(m^2 - n^2)^2 + 4m^2n^2 = \left(\frac{p^2 + q^2}{2}\right)^2. \tag{2}$$

Theorem 7 in [10] states that a convex quadrilateral has the area

$$K = \tfrac{1}{2}\sqrt{p^2q^2 - (m^2 - n^2)^2}.$$

Inserting (2) yields for the area

$$4K^2 = \frac{4p^2q^2}{4} + 4m^2n^2 - \left(\frac{p^2 + q^2}{2}\right)^2 = 4m^2n^2 - \left(\frac{p^2 - q^2}{2}\right)^2$$

and the formula follows. □

Corollary 4. *The area of a convex quadrilateral is equal to the product of the bimedians if and only if it is an equidiagonal quadrilateral.*

Proof. In Theorem 3, we have that $p = q$ if and only if $K = mn$. □

A direct consequence is another area formula, that also appeared in [12, p.19].

Corollary 5. *A convex quadrilateral with consecutive sides a, b, c, d is equidiagonal if and only if it has the area*

$$K = \tfrac{1}{4}\sqrt{(2(a^2 + c^2) - 4v^2)(2(b^2 + d^2) - 4v^2)}$$

where v is the distance between the midpoints of the diagonals.

Proof. The length of the bimedians in a convex quadrilateral are

$$m = \tfrac{1}{2}\sqrt{2(b^2+d^2)-4v^2} \qquad \text{and} \qquad n = \tfrac{1}{2}\sqrt{2(a^2+c^2)-4v^2} \tag{3}$$

according to [10, p.162]. Using these expressions in Corollary 4 directly yields this formula. □

The next characterization is perhaps not so elegant in itself, but it will be used to derive a more symmetric one later on.

Proposition 6. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d is equidiagonal if and only if*

$$ab\cos B + cd\cos D = ad\cos A + bc\cos C.$$

Proof. The quadrilateral is equidiagonal if and only if $2p^2 = 2q^2$, which, according to the law of cosines, is equivalent to (see Figure 2)

$$a^2+b^2-2ab\cos B+c^2+d^2-2cd\cos D = a^2+d^2-2ad\cos A+b^2+c^2-2bc\cos C.$$

Eliminating common terms and factors on both sides, this is equivalent to the equation in the proposition. □

This lemma, which can be thought of as a law of sines for quadrilaterals and is very similar to the previous proposition, will be used in the next proof.

Lemma 7. *In a convex quadrilateral $ABCD$ with consecutive sides a, b, c, d,*

$$ab\sin B + cd\sin D = ad\sin A + bc\sin C.$$

Proof. By dividing the quadrilateral into two triangles using a diagonal, which can be done in two different ways, we have for its area that (see Figure 2)

$$K = \tfrac{1}{2}ab\sin B + \tfrac{1}{2}cd\sin D = \tfrac{1}{2}ad\sin A + \tfrac{1}{2}bc\sin C.$$

The equation in the lemma follows at once by doubling both sides of the second equality. □

Now we come to our main characterization of equidiagonal quadrilaterals.

Theorem 8. *A convex quadrilateral $ABCD$ with consecutive sides a, b, c, d is equidiagonal if and only if*

$$(a^2-c^2)(b^2-d^2) = 2abcd\big(\cos(A-C) - \cos(B-D)\big).$$

Proof. Squaring both sides of the equation in Lemma 7 yields

$$\begin{aligned} &a^2b^2\sin^2 B + c^2d^2\sin^2 D + 2abcd\sin B\sin D \\ &= a^2d^2\sin^2 A + b^2c^2\sin^2 C + 2abcd\sin A\sin C \end{aligned} \tag{4}$$

which is true in all convex quadrilaterals. Squaring the equation in Proposition 6, we have that a convex quadrilateral is equidiagonal if and only if

$$\begin{aligned} &a^2b^2\cos^2 B + c^2d^2\cos^2 D + 2abcd\cos B\cos D \\ &= a^2d^2\cos^2 A + b^2c^2\cos^2 C + 2abcd\cos A\cos C. \end{aligned} \tag{5}$$

By adding equations (4) and (5) and applying the identity $\sin^2\phi+\cos^2\phi=1$ four times, we get the following equality that is equivalent to the one in Proposition 6 (due to the property that $x=y$ if and only if $x+z=y+z$ for any z)

$$\begin{aligned}&a^2b^2+c^2d^2+2abcd(\sin B\sin D+\cos B\cos D)\\ &=a^2d^2+b^2c^2+2abcd(\sin A\sin C+\cos A\cos C).\end{aligned}$$

Using the subtraction formula for cosine, this is equivalent to

$$a^2b^2-a^2d^2-b^2c^2+c^2d^2=2abcd\cos{(A-C)}-2abcd\cos{(B-D)}$$

which is factored into the equation in the theorem. □

Corollary 9. *Two opposite sides of an equidiagonal quadrilateral are congruent if and only if it is an isosceles trapezoid.*

Proof. Applying the trigonometric formula $\cos\phi-\cos\psi=-2\sin\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2}$ and the sum of angles in a quadrilateral, we have that the equation in Theorem 8 is equivalent to

$$(a+c)(a-c)(b+d)(b-d)=-4abcd\sin{(A+B)}\sin{(A+D)}.$$

Hence $a=c$ or $b=d$ is equivalent to $A+B=\pi$ or $A+D=\pi$, which are well known characterizations of a trapezoid (see [13, p.24]). □

3. A new duality regarding congruent and perpendicular diagonals

Theorem 7 in [11] can be reformulated to say that *a convex quadrilateral is equidiagonal if and only if its Varignon parallelogram is orthodiagonal, and the quadrilateral is orthodiagonal if and only if its Varignon parallelogram is equidiagonal.* Thus it gives a sort of dual connection between a quadrilateral and its Varignon parallelogram. Here we shall prove another duality between a quadrilateral and one quadrilateral associated with it. First let us remind the reader that if squares are erected outwards on the sides of a quadrilateral, then their centers are the vertices of a quadrilateral that is both equidiagonal and orthodiagonal.[1] This result is called van Aubel's theorem. It can be proved using elementary triangle geometry (see the animated proof at [7]) or basic properties of complex numbers as in [2, pp.62–64].

What happens if we exchange the squares for equilateral triangles? Problem 5 on the shortlist for the International Mathematical Olympiad in 1992 asked for a proof that the two line segments connecting opposite centroids of those triangles are perpendicular if the quadrilateral has congruent diagonals [6, p.269]. That problem covered only a quarter of the following theorem, since the converse statement as well as a dual one and its converse are also true. Essentially the same proof of part (i) was given at [15]. We have found no reference to neither the proof nor the statement of part (ii).

[1] An orthodiagonal quadrilateral is a quadrilateral with perpendicular diagonals.

Theorem 10. *Suppose equilateral triangles are erected outwards on the sides of a convex quadrilateral $ABCD$. Then the following characterizations hold:*
(i) *$ABCD$ is an equidiagonal quadrilateral if and only if the triangle centroids are the vertices of an orthodiagonal quadrilateral.*
(ii) *$ABCD$ is an orthodiagonal quadrilateral if and only if the triangle centroids are the vertices of an equidiagonal quadrilateral.*

Proof. (i) Let the triangle centroids be G_1, G_2, G_3 and G_4. In an equilateral triangle with side x, the distance from the centroid to a vertex (equal to the circumradius R) is $R = \frac{x}{\sqrt{3}}$. Applying the law of cosines in triangle G_1AG_4 yields (see Figure 3)

$$\begin{aligned}(G_1G_4)^2 &= \left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{d}{\sqrt{3}}\right)^2 - \frac{a}{\sqrt{3}} \cdot \frac{d}{\sqrt{3}} \cos\left(A + \frac{\pi}{3}\right) \\ &= \frac{a^2}{3} + \frac{d^2}{3} - \frac{ad}{3}\left(\frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A\right).\end{aligned}$$

In the same way we have

$$\begin{aligned}(G_2G_3)^2 &= \frac{b^2}{3} + \frac{c^2}{3} - \frac{bc}{3}\left(\frac{1}{2}\cos C - \frac{\sqrt{3}}{2}\sin C\right), \\ (G_1G_2)^2 &= \frac{a^2}{3} + \frac{b^2}{3} - \frac{ab}{3}\left(\frac{1}{2}\cos B - \frac{\sqrt{3}}{2}\sin B\right), \\ (G_3G_4)^2 &= \frac{c^2}{3} + \frac{d^2}{3} - \frac{cd}{3}\left(\frac{1}{2}\cos D - \frac{\sqrt{3}}{2}\sin D\right).\end{aligned}$$

Thus, simplifying and collecting similar terms yields that

$$\begin{aligned}&(G_1G_2)^2 + (G_3G_4)^2 - (G_2G_3)^2 - (G_1G_4)^2 \\ = \ &\tfrac{1}{6}(ad\cos A + bc\cos C - ab\cos B - cd\cos D) \\ + \ &\tfrac{\sqrt{3}}{6}(ab\sin B + cd\sin D - ad\sin A - bc\sin C).\end{aligned}$$

The last parenthesis is equal to zero in all convex quadrilaterals (Lemma 7). Hence we have

$$\begin{aligned}&(G_1G_2)^2 + (G_3G_4)^2 = (G_2G_3)^2 + (G_1G_4)^2 \\ \Leftrightarrow \quad &ab\cos B + cd\cos D = ad\cos A + bc\cos C,\end{aligned}$$

where the first equality is a well known characterization for $G_1G_3 \perp G_2G_4$ (see Theorem 1 in [11]) and the second equality is true if and only if $ABCD$ is equidiagonal according to Proposition 6.

(ii) This statement is trickier to prove with trigonometry, so instead we will use complex numbers. Let the vertices A, B, C and D of a convex quadrilateral be represented by the complex numbers z_1, z_2, z_3 and z_4 respectively. Also, let the centroids G_1, G_2, G_3 and G_4 of the equilateral triangles be represented by the

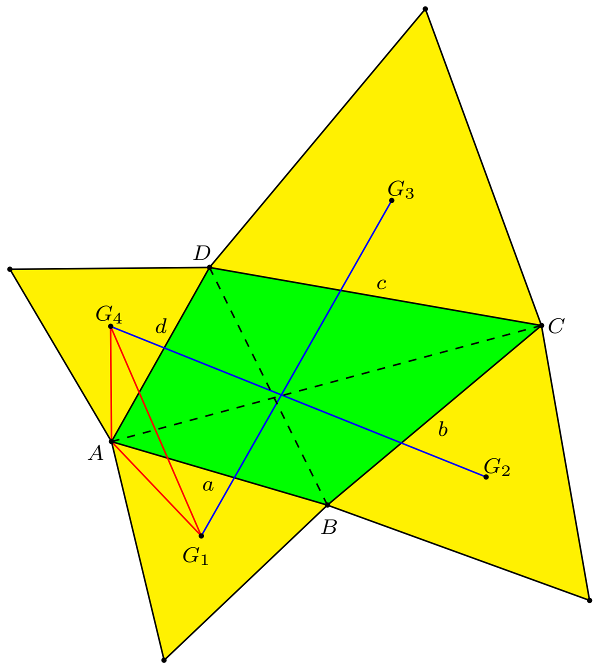

Figure 3. Four equilateral triangles and their centroids

complex numbers g_1, g_2, g_3 and g_4 respectively. The latter are related to the former according to

$$g_1 = \frac{z_1 - z_2}{\sqrt{3}} e^{i\frac{\pi}{6}}, \quad g_2 = \frac{z_2 - z_3}{\sqrt{3}} e^{i\frac{\pi}{6}}, \quad g_3 = \frac{z_3 - z_4}{\sqrt{3}} e^{i\frac{\pi}{6}}, \quad g_4 = \frac{z_4 - z_1}{\sqrt{3}} e^{i\frac{\pi}{6}}.$$

The proof will be in two parts.

($\Rightarrow$) If $ABCD$ is orthodiagonal, then $z_3 - z_1 = i\mathscr{R}(z_4 - z_2)$ for some real number $\mathscr{R} \neq 0$. Using the expressions for the centroids, we get

$$\begin{aligned} \left|\frac{g_3 - g_1}{g_4 - g_2}\right| &= \left|\frac{z_3 - z_4 - (z_1 - z_2)}{z_4 - z_1 - (z_2 - z_3)}\right| = \left|\frac{(z_3 - z_1) - (z_4 - z_2)}{(z_3 - z_1) + (z_4 - z_2)}\right| \\ &= \left|\frac{i\mathscr{R}(z_4 - z_2) - (z_4 - z_2)}{i\mathscr{R}(z_4 - z_2) + (z_4 - z_2)}\right| = \left|\frac{i\mathscr{R} - 1}{i\mathscr{R} + 1}\right| = \frac{\sqrt{1 + \mathscr{R}^2}}{\sqrt{1 + \mathscr{R}^2}} = 1 \end{aligned}$$

where the exponential functions and the $\sqrt{3}$ were canceled out in the first equality. This proves that the line segments connecting opposite centroids are congruent, so $G_1G_2G_3G_4$ is an equidiagonal quadrilateral.

($\Leftarrow$) If $G_1G_2G_3G_4$ is equidiagonal, then according to the rewrite in the first part,

$$|(z_3 - z_1) - (z_4 - z_2)| = |(z_3 - z_1) + (z_4 - z_2)|.$$

We shall prove that this implies that $z_3 - z_1$ and $z_4 - z_2$ are perpendicular. Let us define the two new complex numbers w_1 and w_2 according to $w_1 = z_3 - z_1$ and $w_2 = z_4 - z_2$. Thus we are to prove that if $|w_1 - w_2| = |w_1 + w_2|$, then w_1 and w_2

are perpendicular. This is quite obvious from a geometrical perspective considering the vector nature of complex numbers, but we give an algebraic proof anyway. To this end we use the polar form. Thus we have $w_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $w_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$. We square the two equal absolute values and rewrite $|w_1 - w_2|^2 = |w_1 + w_2|^2$ to get

$$(r_1\cos\varphi_1 - r_2\cos\varphi_2)^2 + (r_1\sin\varphi_1 - r_2\sin\varphi_2)^2$$
$$= (r_1\cos\varphi_1 + r_2\cos\varphi_2)^2 + (r_1\sin\varphi_1 + r_2\sin\varphi_2)^2.$$

Expanding these expressions and canceling equal terms, this is equivalent to

$$4r_1r_2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) = 0 \quad \Leftrightarrow \quad \cos(\varphi_1 - \varphi_2) = 0.$$

The last equation has the valid solutions $\varphi_1 - \varphi_2 = \pm\frac{\pi}{2}$, which proves that the angle between w_1 and w_2 is a right angle. Hence $ABCD$ is orthodiagonal. □

Other generalizations of van Aubel's theorem concerning rectangles, rhombi and parallelograms can be found in [5] and [17].

4. Quadrilaterals that are both equidiagonal and orthodiagonal

Consider Table 1, where three well known properties of the diagonals in seven of the most basic quadrilaterals are shown. The answer "no" refers to the general case for each quadrilateral. One thing is obvious, there is something missing here. No quadrilateral with just the two properties of perpendicular and congruent diagonals is included. This is because no name seems to have been given to this class of quadrilaterals.[2]

Quadrilateral	Bisecting diagonals	Perpendicular diagonals	Congruent diagonals
Trapezoid	No	No	No
Isosceles trapezoid	No	No	Yes
Kite	No	Yes	No
Parallelogram	Yes	No	No
Rhombus	Yes	Yes	No
Rectangle	Yes	No	Yes
Square	Yes	Yes	Yes

Table 1. Diagonal properties in basic quadrilaterals

Before we proceed, we quote in Table 2 in a somewhat expanded form a theorem we proved in [11, p.19]. The four properties on each line in this table are equivalent. The Varignon parallelogram properties follows directly from the fact

[2] In [14, p.50] Gerry Leversha claims that such a quadrilateral is sometimes called a pseudo-square. We can however not find any other reference for that use of the name (neither on the web nor in any geometry books or papers we know of). Instead a Google search indicates that a pseudo-square is a squares with four cut off vertices.

that a parallelogram is a rhombus if and only if its diagonals are perpendicular, and it is a rectangle if and only if its diagonals are congruent [4, p.53].

Original quadrilateral	Diagonal property	Bimedian property	Varignon parallelogram
Equidiagonal	$p = q$	$m \perp n$	Rhombus
Orthodiagonal	$p \perp q$	$m = n$	Rectangle

Table 2. Special cases of the Varignon parallelogram

The bimedians of a convex quadrilateral are the diagonals of its Varignon parallelogram, so the original quadrilateral has congruent and perpendicular diagonals if and only if the Varignon parallelogram has perpendicular and congruent diagonals (see Table 2). For such quadrilaterals, the Varignon parallelogram is a square, and this is a characterization of those quadrilaterals with congruent and perpendicular diagonals since a parallelogram is a square if and only if it is both a rhombus and a rectangle. Thus we have the following two necessary and sufficient conditions.

Theorem 11. *A convex quadrilateral has congruent and perpendicular diagonals if and only if*
(i) *the bimedians are perpendicular and congruent, or*
(ii) *the midpoints of its sides are the vertices of a square.*

So what shall we call these quadrilaterals? They are both equidiagonal and orthodiagonal, but trying to combine the two words yields no good name. The individual words describe the defining properties of these quadrilaterals. With that and Theorem 11 in mind, we propose that a quadrilateral with congruent and perpendicular diagonals is called a *midsquare quadrilateral* (see Figure 4).

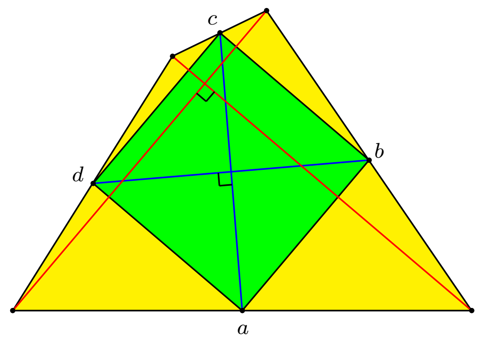

Figure 4. A midsquare quadrilateral and its Varignon square

Three special cases of midsquare quadrilaterals are orthodiagonal isosceles trapezoids, equidiagonal kites and squares.

Proposition 12. *A midsquare quadrilateral is a square if and only if its diagonals bisect each other.*

Proof. If the diagonals of a midsquare quadrilateral bisect each other, it is obvious that it is a square since the diagonals divide it into four congruent right triangles with equal legs.

Conversely it is a well known property that in a square, the diagonals bisect each other. □

After having given a name for this neglected type of quadrilateral, we now consider its area. The first formula in the following proposition has been known at least since 1962 according to [3, p.132].

Proposition 13. *A convex quadrilateral with diagonals p, q and bimedians m, n is a midsquare quadrilateral if and only if its area is given by*

$$K = \tfrac{1}{4}(p^2 + q^2) \qquad \text{or} \qquad K = \tfrac{1}{2}(m^2 + n^2).$$

Proof. Using the identity $(p-q)^2 = p^2+q^2-2pq$, the area of a convex quadrilateral satisfies (see [8])

$$K = \tfrac{1}{2}pq \sin\theta = \tfrac{1}{4}\big(p^2 + q^2 - (p-q)^2\big)\sin\theta \le \tfrac{1}{4}(p^2 + q^2)$$

where equality holds if and only if $p = q$ and $p \perp q$.

The second formula follows at once from the first by using equality (1). □

Since the two diagonals and the two bimedians are individually congruent in a midsquare quadrilateral, its area can be calculated with the four simple formulas

$$K = \tfrac{1}{2}p^2 = \tfrac{1}{2}q^2 = m^2 = n^2. \tag{6}$$

The next proposition gives more area formulas for midsquare quadrilaterals.

Proposition 14. *A convex quadrilateral with consecutive sides a, b, c, d is a midsquare quadrilateral if and only if its area is given by*

$$K = \tfrac{1}{4}\big(2(a^2 + c^2) - 4v^2\big) = \tfrac{1}{4}\big(2(b^2 + d^2) - 4v^2\big)$$

where v is the distance between the midpoints of the diagonals.

Proof. A convex quadrilateral has congruent diagonals if and only if its area is the product of the bimedians according to Corollary 4. Since the diagonals are perpendicular if and only if the bimedians are congruent (Table 2), the two area formulas follows at once using (3). □

Corollary 15. *A convex quadrilateral with consecutive sides a, b, c, d is a square if and only if its area is*

$$K = \tfrac{1}{2}(a^2 + c^2) = \tfrac{1}{2}(b^2 + d^2).$$

Proof. These formulas are a direct consequence of the last proposition since a convex quadrilateral is a square if and only if it is a midsquare quadrilateral with bisecting diagonals ($v = 0$) according to Proposition 12. □

Now we come to an interesting question. Can we calculate the area of a midsquare quadrilateral knowing only its four sides? The answer is yes. The origin for the next theorem is a solved problem we found in the pleasant book [9, pp.179–180]. There Heilbron states that this area is given by

$$K = \frac{1}{4}\left(a^2 + c^2 + \sqrt{2(a^2c^2 + b^2d^2)}\right).$$

He starts his derivation thoroughly, but at the end, when he obtains a quadratic equation, he merely claims that solving it will provide the formula he was supposed to derive. When we started to analyze the solutions to this equation in more detail, we began to smell a rat, and eventually realized that Heilbrons formula is in fact incorrect. We will motivate this after our proof of the correct formula.

Theorem 16. *A midsquare quadrilateral with consecutive sides a, b, c, d has the area*

$$K = \frac{1}{4}\left(a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\right).$$

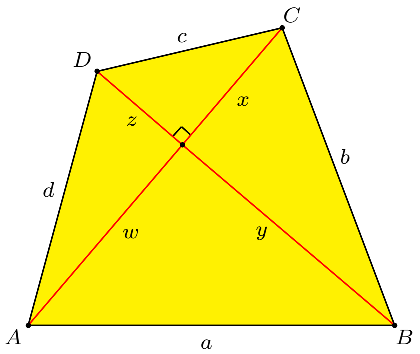

Figure 5. The diagonal parts in a midsquare quadrilateral

Proof. We use notations on the sides and the diagonal parts as in Figure 5, where $w + x = y + z = p$ since the diagonals are congruent. The area is given by $K = \frac{1}{2}p^2$, so we need to express a diagonal p in terms of the sides. Using the Pythagorean theorem, we get

$$a^2 - b^2 = w^2 - x^2 = (w + x)(w - x) = p(2w - p)$$

and similar $b^2 - c^2 = p(2y - p)$. Thus we have

$$a^2 - b^2 + p^2 = 2pw \qquad \text{and} \qquad b^2 - c^2 + p^2 = 2py.$$

Squaring and adding these yields

$$(a^2 - b^2 + p^2)^2 + (b^2 - c^2 + p^2)^2 = 4p^2(w^2 + y^2) = 4p^2a^2$$

where we used the Pythagorean theorem again in the last equality. Expanding and simplifying results in a quadratic equation in p^2:

$$2p^4 - 2(a^2 + c^2)p^2 + (a^2 - b^2)^2 + (b^2 - c^2)^2 = 0.$$

This has the solutions

$$p^2 = \frac{a^2 + c^2 \pm \sqrt{-a^4 - c^4 + 2a^2c^2 - 4b^4 + 4a^2b^2 + 4b^2c^2}}{2}.$$

The radicand can be simplified to

$$-a^4 - c^4 + 2a^2c^2 + 4b^2d^2 = 4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2$$

where we used $a^2 - b^2 + c^2 = d^2$ (see Theorem 1 in [11]). Thus

$$p^2 = \frac{1}{2}\Big(a^2 + c^2 \pm \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\Big). \qquad (7)$$

To decide the correct sign we study the special case when the quadrilateral is a square. Using $a = b = c = d$ in (7) yields

$$p^2 = \tfrac{1}{2}(2a^2 \pm 2a^2)$$

where we see that the solution with the negative sign is obviously false. The area formula now follows when inserting (7) into $K = \frac{1}{2}p^2$. □

Note that it is easy to get formulas for the lengths of the diagonals and the bimedians in a midsquare quadrilateral in terms of the sides. We simply have to combine (6) and Theorem 16.

Remark. Let us comment on the formula suggested by Heilbron. It gives the correct area for a square, so we need to do a more thorough investigation. If his formula were correct, it would mean that $2(a^2c^2 + b^2d^2) = (a^2 + c^2)^2$. But then his formula could be simplified to $K = \frac{1}{2}(a^2 + c^2)$. According to Corollary 15, this is a characterization for a square. Hence his formula must be incorrect, since the quadrilateral has perpendicular and congruent diagonals, but need not to be a square. Another way to dispute it is by considering a right kite with $a = d$ and $c = b$. It has the area $K = ac$, but Heilbrons formula gives $K = \frac{1}{4}(a + c)^2$. Equating these expressions yields $(a-c)^2 = 0$ which again imply the quadrilateral must be a square, which it is not.

5. When are certain quadrilaterals equidiagonal?

So far we have several ways of determining when a convex quadrilateral is equidiagonal. An isosceles trapezoid, a rectangle and a square are always equidiagonal, but how can we know when the diagonals are congruent in other basic quadrilaterals, such as a parallelogram or a cyclic quadrilateral?

Theorem 17. *The following characterizations hold:*
(i) *A parallelogram is equidiagonal if and only if it is a rectangle.*
(ii) *A rhombus is equidiagonal if and only if it is a square.*
(iii) *A trapezoid is equidiagonal if and only if it is an isosceles trapezoid.*
(iv) *A cyclic quadrilateral is equidiagonal if and only if it is an isosceles trapezoid.*

Proof. (i) In a parallelogram $ABCD$ with the two different side lengths a and b, the law of cosines yields that

$$p^2 = q^2 \quad \Leftrightarrow \quad a^2 + b^2 - 2ab\cos B = a^2 + b^2 - 2ab\cos A \quad \Leftrightarrow \quad A = B.$$

Two adjacent angles in a parallelogram are equal if and only if it is a rectangle.

(ii) The first part of the proof is the same as in (i) except that $a = b$, which does not effect the outcome. Two adjacent angles in a rhombus are equal if and only if it is a square.

(iii) The lengths of the diagonals in a trapezoid with consecutive sides a, b, c, d are given by (see [13, p.31])

$$p = \sqrt{\frac{ac(a-c) + ad^2 - cb^2}{a-c}} \qquad \text{and} \qquad q = \sqrt{\frac{ac(a-c) + ab^2 - cd^2}{a-c}}$$

where $a \parallel c$ and $a \neq c$. Thus we get

$$p^2 = q^2 \quad \Leftrightarrow \quad ad^2 - cb^2 = ab^2 - cd^2 \quad \Leftrightarrow \quad (a-c)(d^2 - b^2) = 0.$$

Since $a \neq c$, the only valid solution is $b = d$, so we have an isosceles trapezoid.

(iv) In a cyclic quadrilateral we can apply Ptolemy's second theorem, according to which (see [1, p.65])

$$\frac{p}{q} = \frac{ad + bc}{ab + cd}.$$

Hence

$$p = q \quad \Leftrightarrow \quad ab + cd = ad + bc \quad \Leftrightarrow \quad (a-c)(b-d) = 0$$

where the last equality has the two possible solutions $a = c$ and $b = d$. Any cyclic quadrilateral with a pair of opposite congruent sides is an isosceles trapezoid. One way of realizing this is by connecting the vertices to the circumcenter and thus conclude that this cyclic quadrilateral has a line of symmetry (see Figure 6).

Conversely it is well known that an isosceles trapezoid has congruent diagonals. $\square$

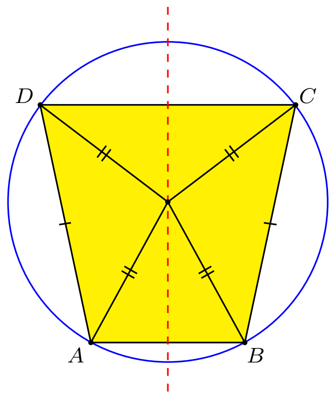

Figure 6. This is an isosceles trapezoid

In the previous section we concluded that an orthodiagonal quadrilateral is also equidiagonal if and only if the midpoints of the sides are the vertices of a square. There don't seem to be any similar easy ways of determining when a kite or a tangential quadrilateral are equidiagonal.

6. The diagonal length in equidiagonal quadrilaterals

We conclude this paper by discussing how the equal length of the diagonals in a general equidiagonal quadrilateral can be calculated given only the four sides, and also how this is related to finding the area of the quadrilateral. Thus this will lead up to a generalization of Theorem 16.

There is a formula relating the four sides and the two diagonals of a convex quadrilateral, sometimes known as Euler's four point relation. It is quite rare to find this relation in geometry books and even rarer to find a proof of it that does not involve determinants, so we start by deriving it here. For this purpose we need the following trigonometric formula.

Lemma 18. *For any two angles α and β we have the identity*

$$\cos^2\alpha + \cos^2\beta + \cos^2(\alpha+\beta) - 2\cos\alpha\cos\beta\cos(\alpha+\beta) = 1.$$

Proof. The addition formula for cosines can be rewritten in the form

$$\cos\alpha\cos\beta - \cos(\alpha+\beta) = \sin\alpha\sin\beta.$$

Squaring both sides, we have

$$(\cos\alpha\cos\beta - \cos(\alpha+\beta))^2 = (1-\cos^2\alpha)(1-\cos^2\beta).$$

Now the identity follows after expansion and simplification. □

The following relation has been derived independently by several mathematicians. It cannot be factored, but there are several ways to collect the terms. The version we present with only four terms is definitely one of the most compact, and except for some basic algebra we only use the law of cosines in the short proof.

Theorem 19 (Euler's four point relation)**.** *In all convex quadrilaterals with consecutive sides a, b, c, d and diagonals p, q, it holds that*

$$\begin{aligned}&p^2q^2(a^2+b^2+c^2+d^2-p^2-q^2) - (a^2-b^2+c^2-d^2)(a^2c^2-b^2d^2)\\ &-p^2(a^2-d^2)(b^2-c^2) + q^2(a^2-b^2)(c^2-d^2) = 0.\end{aligned}$$

Proof. Let $\alpha = \angle BAC$ and $\beta = \angle DAC$ in quadrilateral $ABCD$. The law of cosines applied in triangles BAC, DAC and ABD yields respectively (see Figure 7)

$$\cos\alpha = \frac{a^2+p^2-b^2}{2ap}, \quad \cos\beta = \frac{d^2+p^2-c^2}{2dp}, \quad \cos(\alpha+\beta) = \frac{a^2+d^2-q^2}{2ad}.$$

Inserting these into the identity in Lemma 18 and multiplying both sides of the equation by the least common multiple $4a^2d^2p^2$, we get after simplification

$$\begin{aligned}&d^2(a^2+p^2-b^2)^2 + a^2(d^2+p^2-c^2)^2 + p^2(a^2+d^2-q^2)^2\\ &-(a^2+p^2-b^2)(d^2+p^2-c^2)(a^2+d^2-q^2) = 4a^2d^2p^2.\end{aligned}$$

Now expanding these expressions and collecting similar terms results in Euler's four point relation.[3] □

[3]The history of this six variable polynomial dates back to the 15th century, and it is closely related to the volume of a tetrahedron. The Italian painter Piero della Francesca was also interested in

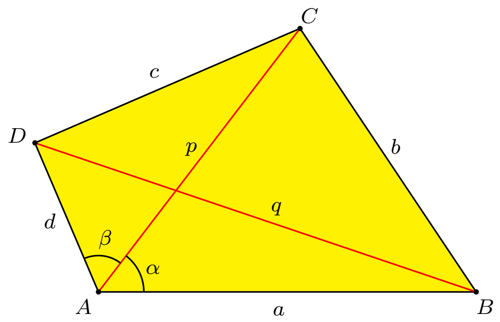

Figure 7. Using the law of cosines in three subtriangles

Returning to the initial goal of calculating the length of the equal diagonals, we set $p = q$ in Theorem 19. This results in the following cubic equation in p^2:

$$2p^6 - (a^2 + b^2 + c^2 + d^2)p^4 + ((a^2 + c^2)(b^2 + d^2) - 2(a^2c^2 + b^2d^2))p^2 + (a^2 - b^2 + c^2 - d^2)(a^2c^2 - b^2d^2) = 0.$$

Cubic equations have been solved for five centuries and there are several different solution methods known. However they all have one thing in common as anyone who has used one of them has noticed: the expressions for the roots they produce are very complicated and in most of the times completely useless. In fact, solving a cubic equation with coefficients like the one above with a computer algebra system can produce several pages of output formulas. Should it be necessary in a practical situation, a numerical solution (on a calculator or computer) is almost always preferable.

After having solved the cubic equation numerically, the area of the equidiagonal quadrilateral (with $p = q$) is given by the formula of Staudt (see [16, p.35])

$$K = \tfrac{1}{4}\sqrt{4p^4 - (a^2 - b^2 + c^2 - d^2)^2}.$$

So it may come as a little disappointment that we did not get a nice formula for the diagonals and the area like in the case when the diagonals are also perpendicular. There are however lots of cubic equations arising when solving problems in the geometry of triangles and quadrilaterals, so this is quite a common occurrence. On

geometry and derived a formula for the volume V of a tetrahedron expressed in terms of its six edges. The formula states that the left hand side of the equation in Theorem 19 is equal to $144V^2$. The formula was rediscovered in the 16th century by the Italian mathematician Niccolò Fontana Tartaglia, who was also involved in the first solution of the cubic equation. In the 18th century the famous Swiss mathematician Leonhard Euler solved the same problem. The invention of determinants made it possible for the 19th century British mathematician Arthur Cayley to express the tetrahedron volume in a very compact form using the so called Cayley-Menger determinant. We do not know which one of these gentlemen was the first to conclude that setting the tetrahedron volume equal to zero would result in an interesting identity for quadrilaterals. A trigonometric derivation that did not involve the tetrahedron has surely been known at least since the 19th century when several mathematicians made thorough trigonometric studies of the geometry of quadrilaterals.

the other hand, we can now get a second derivation of Theorem 16. If the diagonals are both congruent and perpendicular, the constant term of the cubic equation vanishes (since $a^2 + c^2 = b^2 + d^2$), so after simplifying the equation and dividing it by the positive number p^2 we get

$$2p^4 - 2(a^2 + c^2)p^2 + (a^2 + c^2)^2 - 2(a^2c^2 + b^2d^2) = 0.$$

This directly yields the solution

$$p^2 = \frac{1}{2}\left(a^2 + c^2 + \sqrt{4(a^2c^2 + b^2d^2) - (a^2 + c^2)^2}\right)$$

which we recognize from (7).

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Martin Josefsson: Västergatan 25d, 285 37 Markaryd, Sweden
E-mail address: `martin.markaryd@hotmail.com`