Forum Geometricorum
Volume 14 (2014) 163–171.

FORUM GEOM
ISSN 1534-1178

Antirhombi

Nikolaos Dergiades

Abstract. First we give the definition and some properties of the non-rhombus quadrilateral that we call antirhombus [1], that is circumscribed around a circle with center the centroid of the quadrilateral. Then we try to cut a triangle ABC with a line to form an antirhombus, we prove that there are three such lines forming with the sides of ABC an hexagon circumscribed around the incircle of ABC and then investigate their interesting configuration.

1. Circumscribed quadrilaterals with the same incenter and centroid

Let $ABCD$ be a quadrilateral circumscribed around a circle with center O and radius r, and x, y, z, w be the distances of the vertices A, B, C, D from the points of tangency T_a, T_b, T_c, T_d (Figure 1).

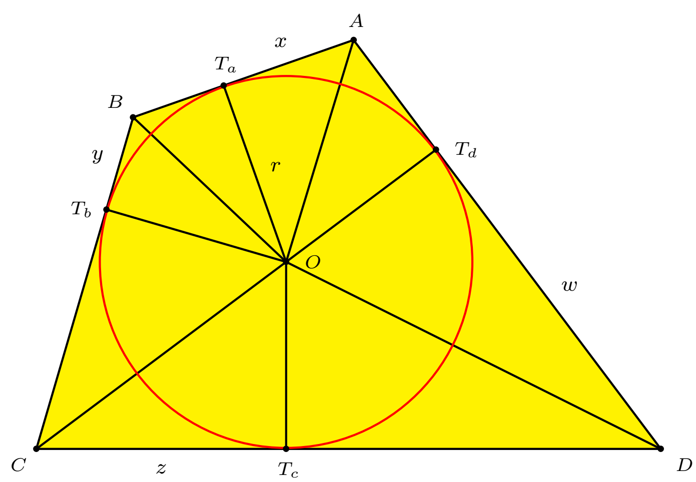

Figure 1

We have

$$x = r\cot\frac{A}{2}, \quad z = r\cot\frac{C}{2}, \qquad OA = \frac{r}{\sin\frac{A}{2}}, \quad OC = \frac{r}{\sin\frac{C}{2}}.$$

Hence,

$$x + z = r\left(\cot\frac{A}{2} + \cot\frac{C}{2}\right) = r\cdot\frac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}},$$

or

$$\frac{OA\cdot OC}{x+z} = \frac{r}{\sin\frac{A+C}{2}}.$$

Publication Date: August 26, 2014. Communicating Editor: Paul Yiu.

Similarly, we have

$$\frac{OB \cdot OD}{y+w} = \frac{r}{\sin \frac{B+D}{2}}.$$

Since $\sin \frac{A+C}{2} = \sin \frac{B+D}{2}$, we have

$$\frac{OA \cdot OC}{x+z} = \frac{OB \cdot OD}{y+w}. \tag{1}$$

It is obvious that every rhombus is a circumscribed parallelogram and its center is both the incenter and centroid. If the centroid of a circumscribed parallelogram is also its incenter, then this parallelogram is a rhombus. We will investigate this double property for other quadrilaterals.

(1) It is easy to see that if the incenter of a circumscribed trapezium (Figure 2) is also the centroid, i.e., the midpoint of its median , then from the right angled triangles OAB and ODC we get $AB = 2MO = 2ON = CD$. Hence the trapezium is isosceles with $AB = CD = MN$.

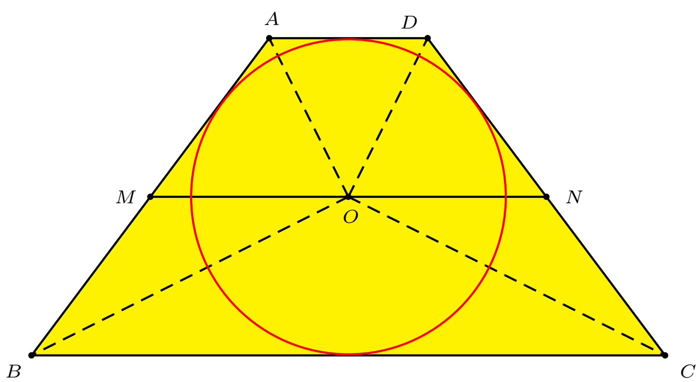

Figure 2

(2) If in a circumscribed trapezium $ABCD$ (with bases BC and AD) we have $OA \cdot OC = OB \cdot OD$, then

$$\begin{aligned} & \frac{r}{\sin \frac{A}{2}} \cdot \frac{r}{\sin \frac{C}{2}} = \frac{r}{\sin \frac{B}{2}} \cdot \frac{r}{\sin \frac{D}{2}} \\ \Longrightarrow \quad & \sin \frac{A}{2} \cos \frac{D}{2} = \cos \frac{A}{2} \sin \frac{D}{2} \\ \Longrightarrow \quad & \tan \frac{A}{2} = \tan \frac{D}{2} \\ \Longrightarrow \quad & A = D. \end{aligned}$$

Hence the trapezium is isosceles and the incenter O is also its centroid.

We will generalize this property by adopting the following definition formulated by George Baloglou [1].

Definition. An antirhombus is a circumscribed quadrilateral $ABCD$ with incenter O that satisfies the condition

$$OA \cdot OC = OB \cdot OD.$$

Theorem 1. *A circumscribed quadrilateral ABCD with incenter O is an antirhombus if and only if* $x + z = y + w$.

Proof. It is obvious from equality (1). □

Theorem 2 ([2, Theorem 13])**.** *A circumscribed quadrilateral with no parallel sides has the same incenter and centroid if and only if it is an antirhombus.*

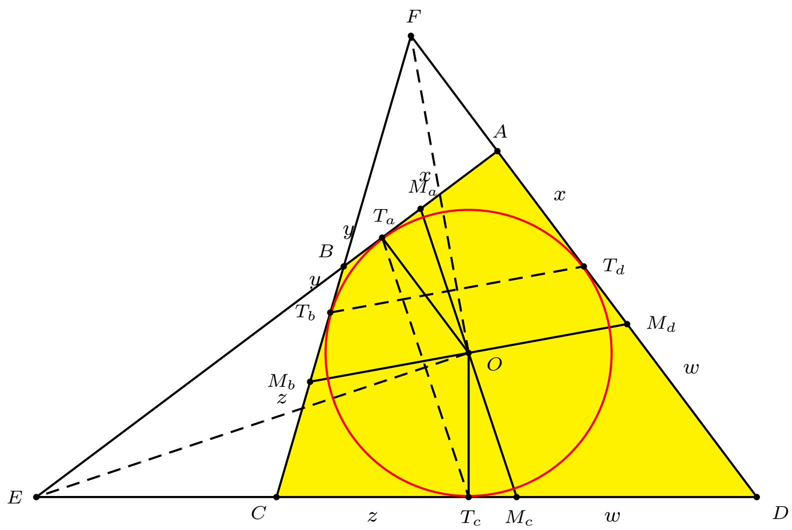

Figure 3

Proof. (1) Let $ABCD$ be a circumscribed quadrilateral with centroid and incenter the point O, and E, F be the intersection points of the opposite sides AB, CD and BC, AD (Figure 3). This point O must be the midpoint of the bimedians M_aM_c, M_bM_d. Let T_a, T_b, T_c, T_d be the tangency points with the incircle. The triangle EM_aM_c is isosceles because EO is a bisector and a median. The triangle ET_aT_c is also isosceles. Therefore, M_aM_c and T_aT_c are parallel, and are both perpendicular to EO. If $x > y$, then $w > z$, and we have $\frac{x-y}{2} = T_aM_a = T_cM_c = \frac{w-z}{2}$. It follows that $x + z = y + w$, and $ABCD$ is an antirhombus. Similarly, M_bM_d and T_bT_d are parallel, and are both perpendicular to FO.

(2) If $ABCD$ is an antirhombus with incenter O, then we have $x+z = y+w$. If $x > y$, then $w > z$ and $T_aM_a = \frac{x-y}{2} = \frac{w-z}{2} = T_cM_c$. The right angled triangles OT_aM_a and OT_cM_c are congruent. Hence, $OM_a = OM_c$. This means that the incenter O lies on the perpendicular bisector $\mathcal{L}_1$ of the bimedian M_aM_c. Similarly O also lies on the perpendicular bisector $\mathcal{L}_2$ of the bimedian M_bM_d. But the only common point of $\mathcal{L}_1$ and $\mathcal{L}_2$ is the centroid of $ABCD$, the common midpoint of the bimedians. Hence the antirhomus has the same incenter and centroid. Again we have $M_aM_c \| T_aT_c$, $M_aM_c \perp EO$ and the same for the other bimedian M_bM_d. □

Corollary 3. *A circumscribed quadrilateral is an antirhombus if and only if a bimedian connecting two opposite sides is perpendicular to the bisector of the angle of these sides and hence parallel to the chord of the corresponding contact points.*

2. Antirhombi from a triangle

Let ABC be a triangle with intouch triangle $A_1B_1C_1$. We begin with the construction of the unique line $\mathscr{L}_a$ which cuts the sidelines of ABC at the points A_a, A_b, A_c and tangent to the incircle of ABC at a point A_2 such that the quadrilateral BCA_bA_c is an antirhombus.

Construction 4. *Let the perpendicular to AI at I intersect AB at J and AC at K. For the quadrilateral BCA_bA_c to be an antirhombus it is sufficient from Theorem 3 that the line JK* (Figure 4) *be a bimedian of the antirhombus. Hence the points A_c and A_b are the symmetrics of B, C in J, K respectively. The line A_bA_c is the line $\mathscr{L}_a$.*

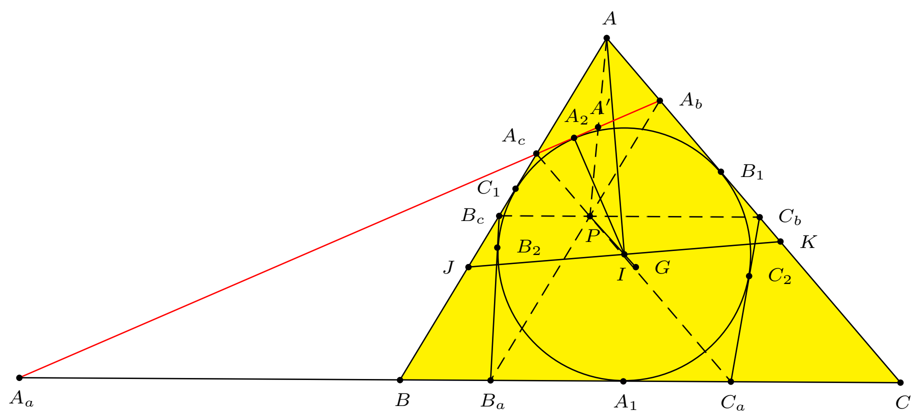

Figure 4

Similarly, we construct the lines $\mathscr{L}_b$ and $\mathscr{L}_c$ intersecting the sidelines at B_a, B_b, B_c, and C_a, C_b, C_c respectively such that CAB_cB_a and ABC_aC_b are antirhomobi.

The following property gives easily the barycentric coordinates of the above points.

Proposition 5. *The lines B_cC_b, C_aA_c, A_bB_a are parallel to the sides of ABC and are concurrent at the image of the centroid G under the homothety* $\mathsf{h}(I, -3)$.

Proof. The parallel from A_b to AC meets the parallel from A_c to AB at a point P. Let A' be the midpoint of A_bA_c (Figure 4). G is the centroid of ABC so we have

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \overrightarrow{0}.$$

Since I is the centroid of the quadrilateral BCA_bA_c,

$$\begin{aligned}
&\overrightarrow{IA_b} + \overrightarrow{IA_c} + \overrightarrow{IB} + \overrightarrow{IC} = \overrightarrow{0},\\
&2\overrightarrow{IA'} + \overrightarrow{IB} + \overrightarrow{IC} = \overrightarrow{0},\\
&\overrightarrow{IP} + \overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC} = \overrightarrow{0},\\
&\overrightarrow{IP} + (\overrightarrow{IG} + \overrightarrow{GA}) + (\overrightarrow{IG} + \overrightarrow{GB}) + (\overrightarrow{IG} + \overrightarrow{GC}) = \overrightarrow{0}.
\end{aligned}$$

Hence $\overrightarrow{IP} = -3\overrightarrow{IG}$, which means that P is the image of G under the homothety $\mathsf{h}(I, -3)$. From $P = 4I - 3G$, we obtain

$$P = (3a - b - c :\ 3b - c - a :\ 3c - a - b) \tag{2}$$

in homogeneous barycentric coordinates. Similarly, the lines B_cC_b, C_aA_c, and A_bB_a are parallel to the sides of ABC and concurrent at P. $\square$

So we have an interesting special case mentioned in [5, §12.1.2] and hence in homogeneous barycentric coordinates we have

$$\begin{aligned} C_b &= (u : 0 : v + w), \quad B_c = (u : v + w : 0);\\ A_c &= (w + u : v : 0), \quad C_a = (0 : v : w + u);\\ B_a &= (0 : u + v : w), \quad A_b = (u + v : 0 : w). \end{aligned}$$

The equations of the lines are

$$\begin{aligned} \mathscr{L}_a := A_bA_c : \quad & -x + \frac{w+u}{v}y + \frac{u+v}{w}z = 0,\\ \mathscr{L}_b := B_cB_a : \quad & \frac{v+w}{u}x - y + \frac{u+v}{w}z = 0,\\ \mathscr{L}_c := C_aC_b : \quad & \frac{v+w}{u}x + \frac{w+u}{v}y - z = 0, \end{aligned}$$

Since the lines B_cC_b, C_aA_c, A_bB_a are concurrent at P, from the converse of Brianchon's theorem we conclude that there is a conic $\mathscr{C}_1$ inscribed in the hexagon $A_bA_cB_cB_aC_aC_b$. It is known [5] that the coordinates of the lines BC, CA, AB, $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ correspond to points on the dual conic. Since these points are the vertices of ABC and the points $\left(-1 :\ \frac{w+u}{v} :\ \frac{u+v}{w}\right)$, $\left(\frac{v+w}{u} :\ -1 :\ \frac{u+v}{w}\right)$, $\left(\frac{v+w}{u} :\ \frac{w+u}{v} :\ -1\right)$, that all lie on the circumconic

$$\frac{v+w}{x} + \frac{w+u}{y} + \frac{u+v}{z} = 0$$

the dual conic, which is tangent to the 6 lines is the conic with equation

$$\sum_{\text{cyclic}} (v+w)^2x^2 - 2(w+u)(u+v)yz = 0.$$

This conic has center

$$O_1 = (2u + v + w :\ u + 2v + w :\ u + v + 2w).$$

In our case with $P = X_{145}$ given in (2) above, we have

$$\sum_{\text{cyclic}} (s-a)^2x^2 - 2(s-b)(s-c)yz = 0,$$

which is clearly the incircle with center I. Hence, the incenter I of ABC is also the incenter of the quadrilaterals CAB_cB_a and ABC_aC_b.

These quadrilaterals are convex if and only if for the sides $a \le b \le c$ of triangle ABC, we have $3a - b - c > 0$.

Proposition 6. *The tangency points A_2, B_2, C_2 of the lines $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ are the vertices of a triangle perspective with ABC.*

Proof. The point A_2 is the pole of $\mathscr{L}_a$ with respect to $\mathscr{C}_1$:

$$\begin{aligned} A_2 = & \begin{pmatrix} -vw & w(w+u) & v(u+v) \end{pmatrix} \begin{pmatrix} 0 & u+v & w+u \\ u+v & 0 & v+w \\ w+u & v+w & 0 \end{pmatrix} \\ = & \begin{pmatrix} (v+w)(w+u)(u+v) & v^2(u+v) & w^2(w+u) \end{pmatrix}, \end{aligned}$$

or

$$A_2 = \left(v+w : \frac{v^2}{w+u} : \frac{w^2}{u+v} \right).$$

Similarly, from the coordinates of the points B_2, C_2 we conclude the triangle $A_2B_2C_2$ is perspective with ABC at the point

$$\left(\frac{u^2}{v+w} : \frac{v^2}{w+u} : \frac{w^2}{u+v} \right).$$

□

In our case with $P = X_{145}$, the perspector is

$$Q = \left(\frac{(3a-b-c)^2}{b+c-a} : \frac{(3b-c-a)^2}{c+a-b} : \frac{(3c-a-b)^2}{a+b-c} \right).$$

This point is not in the current edition of the ENCYCLOPEDIA OF TRIANGLE CENTERS [4]. It has (6-9-13)-search number $0.0267031360104\cdots$. This divides the line IG_e in the ratio

$$IQ : QG_e = 4R + r : -8r.$$

The point of tangency with BC is

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & u+v & w+u \\ u+v & 0 & v+w \\ w+u & v+w & 0 \end{pmatrix} = \begin{pmatrix} 0 & u+v & w+u \end{pmatrix}.$$

Proposition 7. *The hexagon $A_bA_cB_cB_aC_aC_b$ is inscribed in a conic.*

Proof. Since

$$\begin{aligned} & \frac{BB_a}{B_aC} \cdot \frac{BC_a}{C_aC} \cdot \frac{CC_b}{C_bA} \cdot \frac{CA_b}{A_bA} \cdot \frac{AA_c}{A_cB} \cdot \frac{AB_c}{B_cB} \\ = & \frac{w}{u+v} \cdot \frac{w+u}{v} \cdot \frac{u}{v+w} \cdot \frac{u+v}{w} \cdot \frac{v}{w+u} \cdot \frac{v+w}{u} \\ = & 1, \end{aligned}$$

from Carnot's theorem we conclude that the hexagon $A_bA_cB_cB_aC_aC_b$ is inscribed in a conic $\mathscr{C}_2$ that has [5, p.141] equation

$$\sum_{\text{cyclic}} vw(v+w)x^2 - u(vw + (w+u)(u+v))yz = 0.$$

The center of this conic is the point

$$O_2 = (u(2vw+u(v+w-u)) : v(2wu+v(w+u-v)) : w(2uv+w(u+v-w))),$$

which is the midpoint of P and G/P. □

Proposition 8. *The points A_a, B_b, C_c lie on the trilinear polar of X_{5435}.*

Proof. The lines $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ meet the sides BC, CA, AB at the points A_a, B_b, C_c respectively. These are collinear on the Pascal line of the hexagon $A_bA_cB_cB_aC_aC_b$. The line $\mathscr{L}_a$ meets BC at the point $A_a = \left(0 : -\frac{v}{w+u} : \frac{w}{u+v}\right)$, which is the intersection of BC with the trillinear polar of the point $Q = \left(\frac{u}{v+w} : \frac{v}{w+u} : \frac{w}{u+v}\right)$. For $P = X_{145}$, this is the point

$$X_{5435} = \left(\frac{3a-b-c}{b+c-a} : \frac{3b-c-a}{c+a-b} : \frac{3c-a-b}{a+b-c}\right);$$

see [3]. The same holds for B_b and C_c. □

Proposition 9. *The Brianchon points of the quadrilaterals A_bA_cBC, B_cB_aCA, C_aC_bAB are the vertices of a triangle $P_1P_2P_3$ which is perspective with ABC at X_{5435}.*

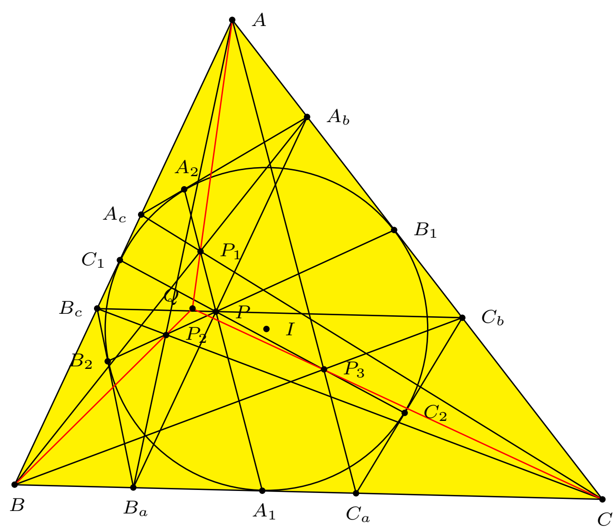

Figure 5

Proof. The Brianchon point P_1 of the circumscribed quadrilateral A_bA_cBC is the common point of the diagonals BA_b, CA_c and the contact chords A_1A_2, B_1C_1 (Figure 5). Hence

$$P_1 = ((w+u)(u+v) : v(u+v) : w(w+u))$$

and the line AP_1 passes through X_{5435}. Similarly, the lines BP_2 and CP_3 also pass through the same point. □

Proposition 10. *The triangle* $P_1P_2P_3$ *is perspective with the contacts triangle* $A_1B_1C_1$ *at* P.

Proof. The points P, A_1, A_2 are collinear because

$$\begin{vmatrix} u & v & w \\ 0 & u+v & w+u \\ v+w & \frac{v^2}{w+u} & \frac{w^2}{u+v} \end{vmatrix} = u(w^2-v^2)+(v+w)(v(w+u)-w(u+v)) = 0.$$

Hence the line A_1P_1 passes through P, and the same holds for the lines B_1P_2 and C_1P_3.

Since the line A_1A_2 is the polar of the point A_a and passes through P, we conclude that the line $\mathscr{L}$ of the points A_a, B_b, C_c (which is the tripolar of X_{5435}) is the polar of P relative to the incircle of ABC. □

Proposition 11. *The lines* $\mathscr{L}_a$, $\mathscr{L}_b$, $\mathscr{L}_c$ *bound a triangle perspective with* ABC *at* P.

Proof. It is easy to see that the lines $\mathscr{L}_b$ and $\mathscr{L}_c$ meet at (Figure 6) the point $A_3 = \left(-\frac{u^2}{v+w} : v : w\right)$. Similarly $\mathscr{L}_c$ and $\mathscr{L}_a$ meet at $B_3 = \left(u : -\frac{v^2}{w+u} : w\right)$, and $\mathscr{L}_a$ and $\mathscr{L}_b$ meet at $C_3 = \left(u : v : -\frac{w^2}{u+v}\right)$. From these coordinates it is clear that $A_3B_3C_3$ and ABC are perspective at $P = (u : v : w)$. □

Proposition 12. *The points* A, B, C, A_3, B_3, C_3 *lie on a conic* $\mathscr{C}_3$, *and the centers of the conics* $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ *are collinear.*

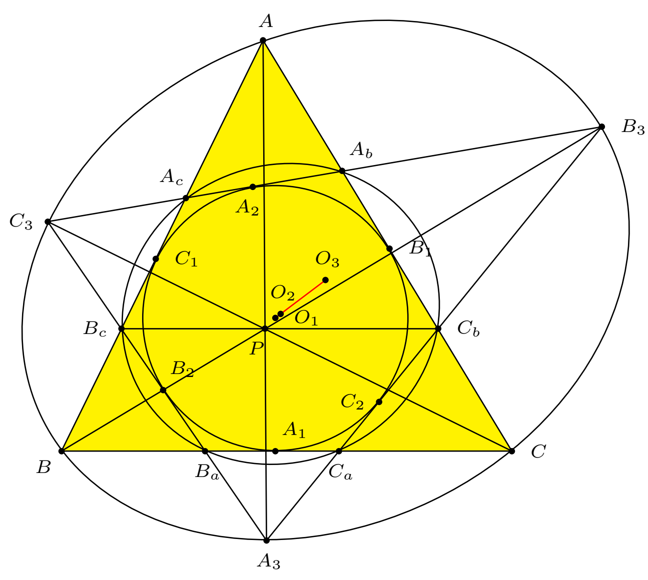

Figure 6

Proof. It is easy to see that the conic $\mathscr{C}_3$ in question is

$$\frac{u^2}{x} + \frac{v^2}{y} + \frac{w^2}{z} = 0$$

with center

$$O_3 = (u^2(v^2 + w^2 - u^2) : \ v^2(w^2 + u^2 - v^2) : \ w^2(u^2 + v^2 - w^2)).$$

The centers O_1, O_2, O_3 are all on the line

$$\sum_{\text{cyclic}} (v - w)(u(v + w) - (v^2 + vw + w^2))x = 0.$$

□

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Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr