Forum Geometricorum
Volume 14 (2014) 203–215.

FORUM GEOM
ISSN 1534-1178

On Some Triads of Homothetic Triangles

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Abstract. To a given reference triangle Δ and three directions $\alpha,\ \beta,\ \gamma$ we construct four triads of homothetic triangles and investigate relations between their homothetic centers, centroids, midway triangles, medial triangles and areas.

1. Two triads of homothetic triangles

Given an arbitrary triangle $\Delta = ABC$ with sidelines $a,\ b,\ c$ and an ordered set $\{\alpha,\ \beta,\ \gamma\}$ of three directions in the plane of Δ. Let

$$p_1 = \alpha\beta\gamma, \quad p_2 = \gamma\alpha\beta, \quad p_3 = \beta\gamma\alpha, \quad \text{and}$$
$$p^1 = \gamma\beta\alpha, \quad p^2 = \beta\alpha\gamma, \quad p^3 = \alpha\gamma\beta$$

be the even and odd permutations of these directions. Each such permutation $p = \pi_1\pi_2\pi_3$ defines three lines with directions π_1 at A, π_2 at B and π_3 at C, which are the sidelines of two triads T_Δ and T^Δ of homothetic triangles $\Delta_i = U_iV_iW_i$ and $\Delta^i = U^iV^iW^i$ (i = 1, 2, 3) with angles U, V, W. The assignment of the indexed symbols U, V, W to the vertices of these triangles is chosen so that homologous vertices have the same symbol (see Figure 1).

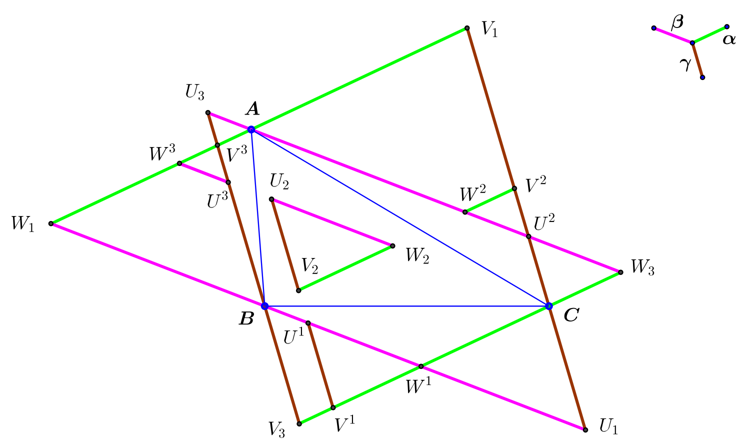

Figure 1

We shall consider some properties of these triads and relationships with other triads of triangles.

Publication Date: September 2, 2014. Communicating Editor: Paul Yiu.

2. Coordinate representations of geometric objects

Geometric objects are described in this paper by homogeneous barycentric coordinates with reference to the triangle Δ. $P = (u : v : w)$ is a point, $l = [p : q : r]$ a line. For a triangle given by its vertices we use the matrix representation (round brackets) with vertice coordinates in the rows. The same triangle can be represented in another matrix form (square brackets), where the rows mean the coordinates of the lines.

We shall regard each direction as a point on the line at infinity with the same name. Then the ordered direction triple $(\alpha,\ \beta, \gamma)$ has the matrix form

$$D = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

with vanishing row sums. By suitable factors in each row it is possible, that not only the row sums of D vanish, but also the column sums, and that all cofactors are equal of unity. In order to verificate analytically the propositions in the sections below, we shall use some other properties of such matrices (see [2]):

$$\beta_1 - \gamma_2 = \gamma_3 - \alpha_1 = \alpha_2 - \beta_3 = \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 =: \lambda_1, \tag{1}$$
$$\gamma_2 - \alpha_3 = \alpha_1 - \beta_2 = \beta_3 - \gamma_1 = \alpha_3\beta_2\gamma_1 - \alpha_1\beta_3\gamma_2 =: \lambda_2, \tag{2}$$
$$\alpha_3 - \beta_1 = \beta_2 - \gamma_3 = \gamma_1 - \alpha_2 = \alpha_2\beta_1\gamma_3 - \alpha_3\beta_2\gamma_1 =: \lambda_3; \tag{3}$$
$$\beta_3 - \gamma_2 = \gamma_1 - \alpha_3 = \alpha_2 - \beta_1 = \alpha_2\beta_3\gamma_1 - \alpha_3\beta_1\gamma_2 =: \mu_1, \tag{4}$$
$$\gamma_2 - \alpha_1 = \alpha_3 - \beta_2 = \beta_1 - \gamma_3 = \alpha_3\beta_1\gamma_2 - \alpha_1\beta_2\gamma_3 =: \mu_2, \tag{5}$$
$$\alpha_1 - \beta_3 = \beta_2 - \gamma_1 = \gamma_3 - \alpha_2 = \alpha_1\beta_2\gamma_3 - \alpha_2\beta_3\gamma_1 =: \mu_3 \tag{6}$$

with

$$\sum_{i=1}^{3} \lambda_i = \sum_{i=1}^{3} \mu_i = 0 \qquad \text{and} \qquad \sum_{k=1}^{3} \mu_i^2 = 6 + \sum_{k=1}^{3} \lambda_i^2\ ; \tag{7}$$

$$\alpha_2\alpha_3 - \beta_3\gamma_2 = \beta_2\beta_3 - \gamma_3\alpha_2 = \gamma_2\gamma_3 - \alpha_3\beta_2 =: \xi_1, \tag{8}$$
$$\alpha_3\alpha_1 - \beta_1\gamma_3 = \beta_3\beta_1 - \gamma_1\alpha_3 = \gamma_3\gamma_1 - \alpha_1\beta_3 =: \xi_2, \tag{9}$$
$$\alpha_1\alpha_2 - \beta_2\gamma_1 = \beta_1\beta_2 - \gamma_2\alpha_1 = \gamma_1\gamma_2 - \alpha_2\beta_1 =: \xi_3; \tag{10}$$
$$\alpha_2\alpha_3 - \beta_2\gamma_3 = \beta_2\beta_3 - \gamma_2\alpha_3 = \gamma_2\gamma_3 - \alpha_2\beta_3 =: \eta_1, \tag{11}$$
$$\alpha_3\alpha_1 - \beta_3\gamma_1 = \beta_3\beta_1 - \gamma_3\alpha_1 = \gamma_3\gamma_1 - \alpha_3\beta_1 =: \eta_2, \tag{12}$$
$$\alpha_1\alpha_2 - \beta_1\gamma_2 = \beta_1\beta_2 - \gamma_1\alpha_2 = \gamma_1\gamma_2 - \alpha_1\beta_2 =: \eta_3. \tag{13}$$

Furthermore, for each row i of D,

$$\sum_{k=1}^{3} \left(\prod_{j \neq i} d_{jk} \prod_{j \neq k} d_{ij} \right) = 1, \tag{14}$$

and for each column k,

$$\sum_{i=1}^{3}\left(\prod_{j\neq i} d_{jk}\prod_{j\neq k} d_{ij}\right)=1. \tag{15}$$

Here is an example:

$$D=\begin{pmatrix}1 & 2 & -3\\ 1 & 3 & -4\\ -2 & -5 & 7\end{pmatrix}.$$

The 9 lines at the points $A,\ B,\ C$ with the directions $\alpha,\ \beta,\ \gamma$ have following coordinate representations:

	A	B	C
α	$\alpha_A=[0:-\alpha_3:\alpha_2]$	$\alpha_B=[\alpha_3:0:-\alpha_1]$	$\alpha_C=[-\alpha_2:\alpha_1:0]$
β	$\beta_A=[0:-\beta_3:\beta_2]$	$\beta_B=[\beta_3:0:-\beta_1]$	$\beta_C=[-\beta_2:\beta_1:0]$
γ	$\gamma_A=[0:-\gamma_3:\gamma_2]$	$\gamma_B=[\gamma_3:0:-\gamma_1]$	$\gamma_C=[-\gamma_2:\gamma_1:0]$

From this it is easy to determine the matrix forms of the triangles Δ_i and Δ^i in normalized barycentric coordinates (row sums are equal of unity):

$$\Delta_1=\begin{bmatrix}\alpha_A\\ \beta_B\\ \gamma_C\end{bmatrix}\cong\begin{pmatrix}\beta_B\cap\gamma_C\\ \gamma_C\cap\alpha_A\\ \alpha_A\cap\beta_B\end{pmatrix}=\begin{pmatrix}U_1\\ V_1\\ W_1\end{pmatrix}=\begin{pmatrix}\beta_1\gamma_1 & \beta_1\gamma_2 & \beta_3\gamma_1\\ \gamma_1\alpha_2 & \gamma_2\alpha_2 & \gamma_2\alpha_3\\ \alpha_3\beta_1 & \alpha_2\beta_3 & \alpha_3\beta_3\end{pmatrix},$$

$$\Delta_2=\begin{bmatrix}\gamma_A\\ \alpha_B\\ \beta_C\end{bmatrix}\cong\begin{pmatrix}\beta_C\cap\gamma_A\\ \gamma_A\cap\alpha_B\\ \alpha_B\cap\beta_C\end{pmatrix}=\begin{pmatrix}U_2\\ V_2\\ W_2\end{pmatrix}=\begin{pmatrix}\beta_1\gamma_2 & \beta_2\gamma_2 & \beta_2\gamma_3\\ \gamma_3\alpha_1 & \gamma_2\alpha_3 & \gamma_3\alpha_3\\ \alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_3\beta_1\end{pmatrix},$$

$$\Delta_3=\begin{bmatrix}\beta_A\\ \gamma_B\\ \alpha_C\end{bmatrix}\cong\begin{pmatrix}\beta_A\cap\gamma_B\\ \gamma_B\cap\alpha_C\\ \alpha_C\cap\beta_A\end{pmatrix}=\begin{pmatrix}U_3\\ V_3\\ W_3\end{pmatrix}=\begin{pmatrix}\beta_3\gamma_1 & \beta_2\gamma_3 & \beta_3\gamma_3\\ \gamma_1\alpha_1 & \gamma_1\alpha_2 & \gamma_3\alpha_1\\ \alpha_1\beta_2 & \alpha_2\beta_2 & \alpha_2\beta_3\end{pmatrix};$$

$$\Delta^1=\begin{bmatrix}\gamma_A\\ \beta_B\\ \alpha_C\end{bmatrix}\cong\begin{pmatrix}\beta_B\cap\gamma_A\\ \gamma_A\cap\alpha_C\\ \alpha_C\cap\beta_B\end{pmatrix}=\begin{pmatrix}U^1\\ V^1\\ W^1\end{pmatrix}=-\begin{pmatrix}\beta_1\gamma_3 & \beta_3\gamma_2 & \beta_3\gamma_3\\ \gamma_2\alpha_1 & \gamma_2\alpha_2 & \gamma_3\alpha_2\\ \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_1\beta_3\end{pmatrix},$$

$$\Delta^2=\begin{bmatrix}\beta_A\\ \alpha_B\\ \gamma_C\end{bmatrix}\cong\begin{pmatrix}\beta_A\cap\gamma_C\\ \gamma_C\cap\alpha_B\\ \alpha_B\cap\beta_A\end{pmatrix}=\begin{pmatrix}U^2\\ V^2\\ W^2\end{pmatrix}=-\begin{pmatrix}\beta_2\gamma_1 & \beta_2\gamma_2 & \beta_3\gamma_2\\ \gamma_1\alpha_1 & \gamma_2\alpha_1 & \gamma_1\alpha_3\\ \alpha_1\beta_3 & \alpha_3\beta_2 & \alpha_3\beta_3\end{pmatrix},$$

$$\Delta^3=\begin{bmatrix}\alpha_A\\ \gamma_B\\ \beta_C\end{bmatrix}\cong\begin{pmatrix}\beta_C\cap\gamma_B\\ \gamma_B\cap\alpha_A\\ \alpha_A\cap\beta_C\end{pmatrix}=\begin{pmatrix}U^3\\ V^3\\ W^3\end{pmatrix}=-\begin{pmatrix}\beta_1\gamma_1 & \beta_2\gamma_1 & \beta_1\gamma_3\\ \gamma_1\alpha_3 & \gamma_3\alpha_2 & \gamma_3\alpha_3\\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_2\end{pmatrix}.$$

In §5 we shall use the concept of the *medial map* m of a point $P=(u:v:w)$, defined by $mP=(v+w:w+u:u+v)$. The *medial image* mP lies on the line PG. The centroid G of Δ divides the segment between P and mP in the ratio 2 : 1. We call a triangle PQR the *medial image* of a triangle UVW, if $P=mU,\ Q=mV$ and $R=mW$.

The isotomic conjugate $P^\bullet$ of a point P has the coordinates $(1/u:\ 1/v:\ 1/w)$.

3. Centroids and homothetic centers of the triangles Δ_i and Δ^i

The centroid of Δ is $G = (1 : 1 : 1)$, G_i are the centroids of the triangles Δ_i. Their j-th coordinates are the sums of the j-th column of the matrix Δ_i. The triangle $\Delta_G = (G_{ij})$ is formed by the centroids G_i. Similarly define G^i and $\Delta^G = (G^{ij})$.

Proposition 1. *The triangles Δ_G and Δ^G have the same centroid G as Δ.*

Proof. The sums of the i-th column sums of $\Delta_1,\ \Delta_2$ and Δ_3 resp. $\Delta^1,\ \Delta^2$ and Δ^3 have the same value 3. □

The centers of homothety P_{ij} of the triangle pairs $(\Delta_i,\ \Delta_j)$ are in detail

$$P_{12} = (\beta_1(\gamma_1\alpha_2 - \gamma_3\alpha_3) : \gamma_2(\alpha_3\beta_1 - \alpha_2\beta_2) : \alpha_3(\beta_2\gamma_3 - \beta_1\gamma_1)),$$
$$P_{23} = (\alpha_1(\beta_3\gamma_1 - \beta_2\gamma_2) : \beta_2(\gamma_2\alpha_3 - \gamma_1\alpha_1) : \gamma_3(\alpha_1\beta_2 - \alpha_3\beta_3)),$$
$$P_{31} = (\gamma_1(\alpha_2\beta_3 - \alpha_1\beta_1) : \alpha_2(\beta_1\gamma_2 - \beta_3\gamma_3) : \beta_3(\gamma_3\alpha_1 - \gamma_2\alpha_2)).$$

These three points are collinear because each triad of homothetic triangles has collinear homothetic centers. The line g containing them can be written as

$$g = [\delta_2 - \delta_3 : \delta_3 - \delta_1 : \delta_1 - \delta_2], \tag{16}$$

with abbreviations $\delta_i := \alpha_i\beta_i\gamma_i$. Similarly, the homothetic centers P^{ij} of the pairs $(\Delta^i,\ \Delta^j)$ lie on the same line g. □

Proposition 2. *The line g contains the centroid G of Δ.*

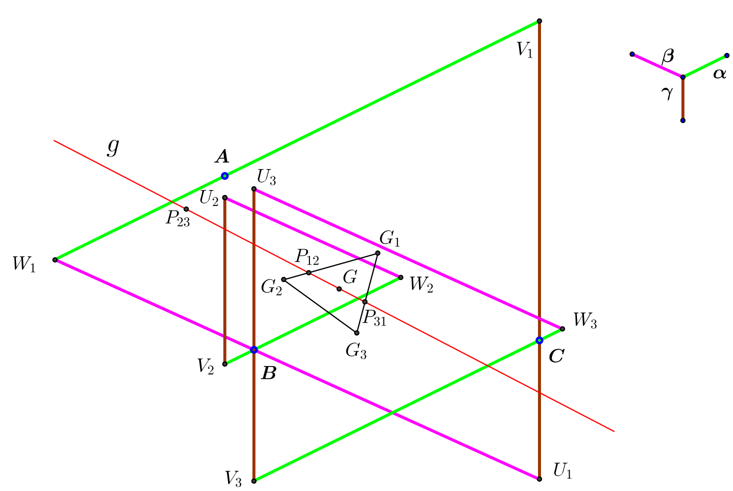

Figure 2

Proof. In accordance with (16) g has a vanishing sum of coordinates. □

Proposition 3. *The homothetic center* P_{ij} *of the pair* $(\Delta_i,\ \Delta_j)$ *lies on the side* G_iG_j *of* Δ_G *and the homothetic center* P^{ij} *on the side* G^iG^j *of* Δ^G.

Proof. Verification. □

4. Areas

Let $|UVW|$ be the area of a triangle UVW and σ the area of the reference triangle Δ. If the vertices of the triangles are given by their normalized barycentric coordinates, then UVW has the area $\sigma \cdot |\det(U,V,W)|$. We shall see below that there is a simple connection between the areas of Δ, Δ_i, Δ^i, Δ_G and Δ^G, independent of $\alpha,\ \beta$ and γ.

Making use of (1) to (6) we find

$$\det(\Delta_i) = \mu_i^2, \qquad \det(\Delta^i) = \lambda_i^2.$$

Let d_{ij} and d^{ij} be the column sums of the matrices Δ_i and Δ^i, respectively, then we have the (normalized) matrices $(G_{ij}) = \frac{1}{3}(d_{ij})$ and $(G^{ij}) = \frac{1}{3}(d^{ij})$, and it is valid

$$\det(\Delta_G) = \frac{1}{27}\det(d_{ij}) = \frac{1}{27}\begin{vmatrix} d_{11} & d_{12} & 3 \\ d_{21} & d_{22} & 3 \\ d_{31} & d_{32} & 3 \end{vmatrix} = \frac{1}{27}\begin{vmatrix} d_{11} & d_{12} & 3 \\ d_{21} & d_{22} & 3 \\ 3 & 3 & 9 \end{vmatrix} = \frac{1}{3}\begin{vmatrix} d_{11}-1 & d_{12}-1 \\ d_{21}-1 & d_{22}-1 \end{vmatrix}.$$

In accordance with

$$\begin{array}{ll} d_{11}-1 = \gamma_1\lambda_1 - \beta_1\lambda_2 & \qquad d_{12}-1 = \alpha_2\lambda_2 - \gamma_2\lambda_3 \\ d_{21}-1 = \beta_1\lambda_2 - \alpha_1\lambda_3 & \qquad d_{22}-1 = \gamma_2\lambda_3 - \beta_2\lambda_1 \end{array}$$

it follows that

$$\det(\Delta_G) = \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) = \frac{1}{6}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

Similarly,

$$\det(\Delta^G) = \frac{1}{3}(\mu_1^2 + \mu_2^2 + \mu_1\mu_2) = \frac{1}{6}(\mu_1^2 + \mu_2^2 + \mu_3^2).$$

From this we obtain

Proposition 4.

$$\sum |\Delta_i| = 6\cdot|\Delta^G| = 6\cdot(|\Delta_G|+|\Delta|), \qquad \sum |\Delta^i| = 6\cdot|\Delta_G| = 6\cdot(|\Delta^G|-|\Delta|),$$

$$|\Delta^G| - |\Delta_G| = |\Delta|, \qquad \sum|\Delta_i| - \sum|\Delta^i| = 6\cdot|\Delta|.$$

5. The triads T_Δ, T^Δ and their midway triangles

Given two labeled triangles $T_1 = X_1Y_1Z_1$ and $T_2 = X_2Y_2Z_2$. Let X_{12} be the midpoint of the line segment X_1X_2, Y_{12} and Z_{12} the midpoints of the segments Y_1Y_2 and Z_1Z_2, respectively. Then the triangle $T_{12} = X_{12}Y_{12}Z_{12}$ is called the *midway triangle of the pair* (T_1, T_2).

The midway triangles of the pairs $(\Delta_i,\ \Delta_j)$ have obvious the normalized representations:

$$\Delta_{12} = \begin{pmatrix} U_{12} \\ V_{12} \\ W_{12} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\beta_1\gamma_3 & -\beta_3\gamma_2 & \beta_3\gamma_1+\beta_2\gamma_3 \\ \gamma_1\alpha_2+\gamma_3\alpha_1 & -\gamma_2\alpha_1 & -\gamma_1\alpha_3 \\ -\alpha_2\beta_1 & \alpha_2\beta_3+\alpha_1\beta_2 & -\alpha_3\beta_2 \end{pmatrix},$$

$$\Delta_{23} = \begin{pmatrix} U_{23} \\ V_{23} \\ W_{23} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \beta_1\gamma_2+\beta_3\gamma_1 & -\beta_2\gamma_1 & -\beta_1\gamma_3 \\ -\gamma_2\alpha_1 & \gamma_2\alpha_3+\gamma_1\alpha_2 & -\gamma_3\alpha_2 \\ -\alpha_1\beta_3 & -\alpha_3\beta_2 & \alpha_3\beta_1+\alpha_2\beta_3 \end{pmatrix},$$

$$\Delta_{31} = \begin{pmatrix} U_{31} \\ V_{31} \\ W_{31} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\beta_2\gamma_1 & \beta_2\gamma_3+\beta_1\gamma_2 & -\beta_3\gamma_2 \\ -\gamma_1\alpha_3 & -\gamma_3\alpha_2 & \gamma_3\alpha_1+\gamma_2\alpha_3 \\ \alpha_1\beta_2+\alpha_3\beta_1 & -\alpha_2\beta_1 & -\alpha_1\beta_3 \end{pmatrix},$$

and we find by calculation

$$|\det(\Delta_{12})| = \frac{1}{4}\mu_3^2, \qquad |\det(\Delta_{23})| = \frac{1}{4}\mu_1^2, \qquad |\det(\Delta_{31})| = \frac{1}{4}\mu_2^2.$$

Proposition 5. *For $i \neq j, k \neq i, j$,*
(a) *the midway triangle Δ_{ij} is the medial image of Δ_k and from this congruent with the medial triangle of Δ_k,*
(b) *G is the homothetic center of the pair $(\Delta_k,\ \Delta_{ij})$.*

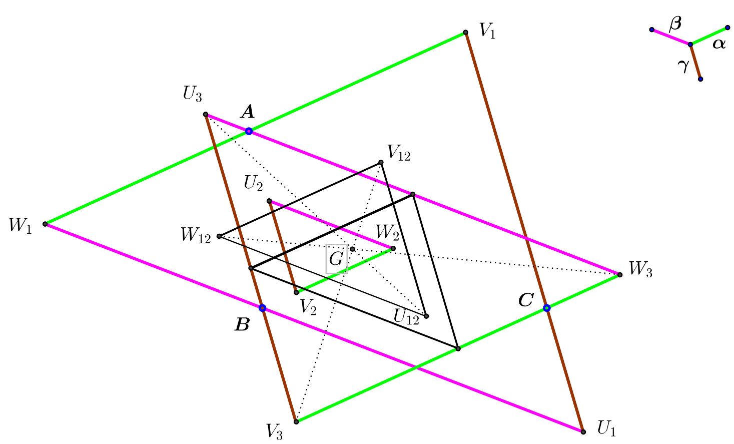

Figure 3

Proof. Verification. □

For the triangles of the triad T^Δ the above proposition is true by analogy.

6. Two triads of homothetic inscribed triangles

The above described construction of two triads of homothetic in Δ *circumscribed* triangles Δ_i and Δ^i raises the question whether or not exist triads of homothetic in Δ *inscribed* triangles whose sides have the given directions $\alpha,\ \beta,\ \gamma$. For this purpose we form groups of vertices of Δ^i and Δ_i to new triangles Φ^i and Φ_i, respectively:

$$\Phi^1=\begin{pmatrix}U^3\\V^1\\W^2\end{pmatrix},\ \Phi^2=\begin{pmatrix}U^2\\V^3\\W^1\end{pmatrix},\ \Phi^3=\begin{pmatrix}U^1\\V^2\\W^3\end{pmatrix};$$
$$\Phi_1=\begin{pmatrix}U_3\\V_1\\W_2\end{pmatrix},\ \Phi_2=\begin{pmatrix}U_2\\V_3\\W_1\end{pmatrix},\ \Phi_3=\begin{pmatrix}U_1\\V_2\\W_3\end{pmatrix}.$$

Then we intersect the sidelines of these triangles with certain sidelines of the reference triangle (see Figure 4). The points of intersection A_i, B_i, C_i and A^i, B^i, C^i form triangles Ω_i and Ω^i, respectively, with following normalized barycentric coordinates:

$$\Omega_1=\begin{pmatrix}A_1\\B_1\\C_1\end{pmatrix}:=\begin{pmatrix}V^1W^2\cap a\\W^2U^3\cap b\\U^3V^1\cap c\end{pmatrix}=-\frac{1}{\mu_1}\begin{pmatrix}0&\gamma_2&-\beta_3\\-\gamma_1&0&\alpha_3\\\beta_1&-\alpha_2&0\end{pmatrix},$$
$$\Omega_2=\begin{pmatrix}A_2\\B_2\\C_2\end{pmatrix}:=\begin{pmatrix}U^2V^3\cap a\\V^3W^1\cap b\\W^1U^2\cap c\end{pmatrix}=-\frac{1}{\mu_2}\begin{pmatrix}0&\beta_2&-\alpha_3\\-\beta_1&0&\gamma_3\\\alpha_1&-\gamma_2&0\end{pmatrix},$$
$$\Omega_3=\begin{pmatrix}A_3\\B_3\\C_3\end{pmatrix}:=\begin{pmatrix}W^3U^1\cap a\\U^1V^2\cap b\\V^2W^3\cap c\end{pmatrix}=-\frac{1}{\mu_3}\begin{pmatrix}0&\alpha_2&-\gamma_3\\-\alpha_1&0&\beta_3\\\gamma_1&-\beta_2&0\end{pmatrix};$$
$$\Omega^1=\begin{pmatrix}A^1\\B^1\\C^1\end{pmatrix}:=\begin{pmatrix}V_1U_3\cap a\\U_3W_2\cap b\\W_2V_1\cap c\end{pmatrix}=\frac{1}{\lambda_1}\begin{pmatrix}0&\alpha_2&-\beta_3\\-\alpha_1&0&\gamma_3\\\beta_1&-\gamma_2&0\end{pmatrix},$$
$$\Omega^2=\begin{pmatrix}A^2\\B^2\\C^2\end{pmatrix}:=\begin{pmatrix}U_2W_1\cap a\\W_1V_3\cap b\\V_3U_2\cap c\end{pmatrix}=\frac{1}{\lambda_2}\begin{pmatrix}0&\gamma_2&-\alpha_3\\-\gamma_1&0&\beta_3\\\alpha_1&-\beta_2&0\end{pmatrix},$$
$$\Omega^3=\begin{pmatrix}A^3\\B^3\\C^3\end{pmatrix}:=\begin{pmatrix}W_3V_2\cap a\\V_2U_1\cap b\\U_1W_3\cap c\end{pmatrix}=\frac{1}{\lambda_3}\begin{pmatrix}0&\beta_2&-\gamma_3\\-\beta_1&0&\alpha_3\\\gamma_1&-\alpha_2&0\end{pmatrix}.$$

Then we have (see Figure 5)

Proposition 6. (a) *The sidelines of the inscribed triangles* Ω_i *and* Ω^i *have the directions* $\alpha,\ \beta$ *and* γ.
(b) *The sides of* Δ_i *are parallels of the sides* a_i, b_i, c_i *of* Ω_i *at* A, B, C *(in this order). An analogous statement is true for* Δ^i.

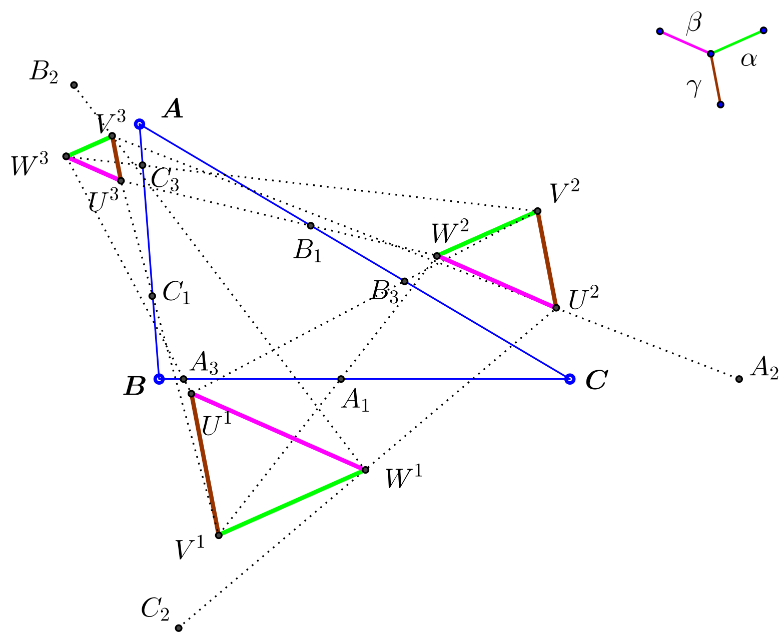

Figure 4

(c) *The points in each of the triples* $\{A_1, B_3, C_2\}$, $\{A_2, B_1, C_3\}$, $\{A_3, B_2, C_1\}$, $\{A^1, B^2, C^3\}$, $\{A^2, B^3, C^1\}$ *and* $\{A^3, B^1, C^2\}$ *are collinear.*
(d) *The product of the areas of* Δ_i *and* Ω_i *and of* Δ^i *and* Ω^i *is independent of* $\alpha,\ \beta,\ \gamma$*:*

$$|\Delta_i| \cdot |\Omega_i| = |\Delta^i| \cdot |\Omega^i| = |\Delta|^2.$$

Proof. Verification.
(a) For instance the line A_1B_1 intersects the infinite line in accordance with (4) at $-\mu_1(\gamma_1 : \gamma_2 : \gamma_3)$, therefor it has the direction γ.

(b) For instance the translation of the line A_1B_1 at C yields the line $[-\gamma_2 : \gamma_1 : 0] = \gamma_C$.

(c) For instance $\det(A_1, B_3, C_2) = \begin{vmatrix} 0 & \gamma_2 & -\beta_3 \\ -\alpha_1 & 0 & \beta_3 \\ \alpha_1 & -\gamma_2 & 0 \end{vmatrix} = 0.$

(d) For instance according to (4)

$$\det(\Omega_1) \cdot \det(\Delta_1) = -\frac{1}{\mu_1^3}(\alpha_3\beta_1\gamma_2 - \alpha_2\beta_3\gamma_1) \cdot \mu_1^2 = 1. \qquad \square$$

For the comparison of homothetic triangles it is useful to give homologous vertices the same position:

$$\Omega_1 = A_1B_1C_1, \quad \Omega_2 = B_2C_2A_2, \quad \Omega_3 = C_3A_3B_3;$$
$$\Omega^1 = C^1B^1A^1, \quad \Omega^2 = B^2A^2C^2, \quad \Omega^3 = A^3C^3B^3.$$

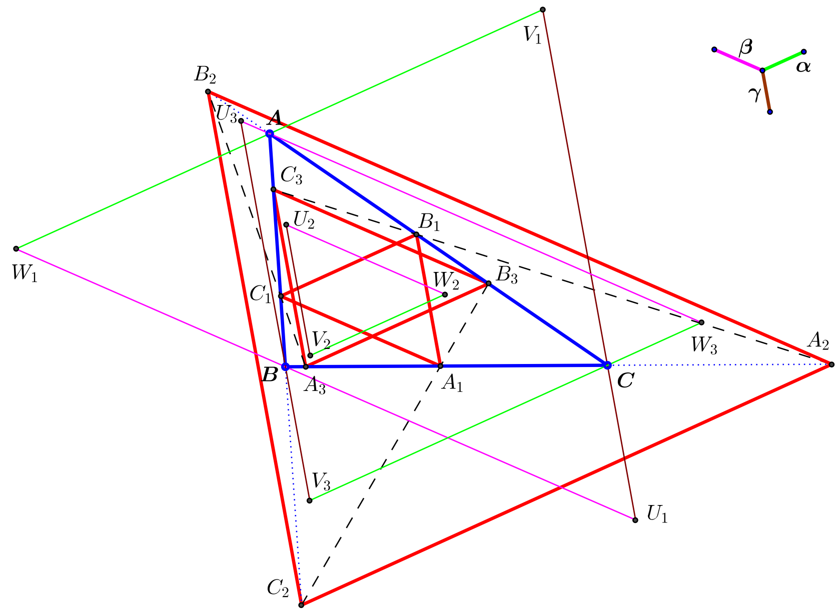

Figure 5

Then for instance the homothetic center Q_{12} of the pair (Ω_1, Ω_2) is the common point of intersection of the lines $A_1B_2,\ B_1C_2$ and C_1A_2. By a simple calculation it follows for the homothetic centers Ω_{ij} and Ω^{ij}

$$Q_{12} = (\beta_1 : \gamma_2 : \alpha_3),\quad Q_{23} = (\alpha_1 : \beta_2 : \gamma_3),\quad Q_{31} = (\gamma_1 : \alpha_2 : \beta_3),$$
$$Q^{12} = (\alpha_1 : \gamma_2 : \beta_3),\quad Q^{23} = (\gamma_1 : \beta_2 : \alpha_3),\quad Q^{31} = (\beta_1 : \alpha_2 : \gamma_3).$$

It is clear, that the homothetic centers $Q_{12},\ Q_{23},\ Q_{31}$ and $Q^{12},\ Q^{23},\ Q^{31}$ are collinear on lines g_Ω and g^Ω, respectively, and these lines have according to (8)-(13) the simple representations

$$g_\Omega = [\xi_1 : \xi_2 : \xi_3],\quad g^\Omega = [\eta_1 : \eta_2 : \eta_3].$$

The centroids S_i of Ω_i and S^i of Ω^i, respectively, are

$$S_1 = (\beta_1 - \gamma_1 : \gamma_2 - \alpha_2 : \alpha_3 - \beta_3),\qquad S^1 = (\beta_1 - \alpha_1 : \alpha_2 - \gamma_2 : \gamma_3 - \beta_3),$$
$$S_2 = (\alpha_1 - \beta_1 : \beta_2 - \gamma_2 : \gamma_3 - \alpha_3),\qquad S^2 = (\alpha_1 - \gamma_1 : \gamma_2 - \beta_2 : \beta_3 - \alpha_3),$$
$$S_3 = (\gamma_1 - \alpha_1 : \alpha_2 - \beta_2 : \beta_3 - \gamma_3),\qquad S^3 = (\gamma_1 - \beta_1 : \beta_2 - \alpha_2 : \alpha_3 - \gamma_3).$$

A simple computation proves the following proposition:

Proposition 7. *The three centroids of the Ω_i and the three homothetic centers Q_{ij} lie on the line g_Ω, the three centroids of the Ω^i and the three homothetic centers Q^{ij} on the line g^Ω* (see Figure 6).

Marginal node: The triangles Φ_i and Φ^i have else some peculiarities:

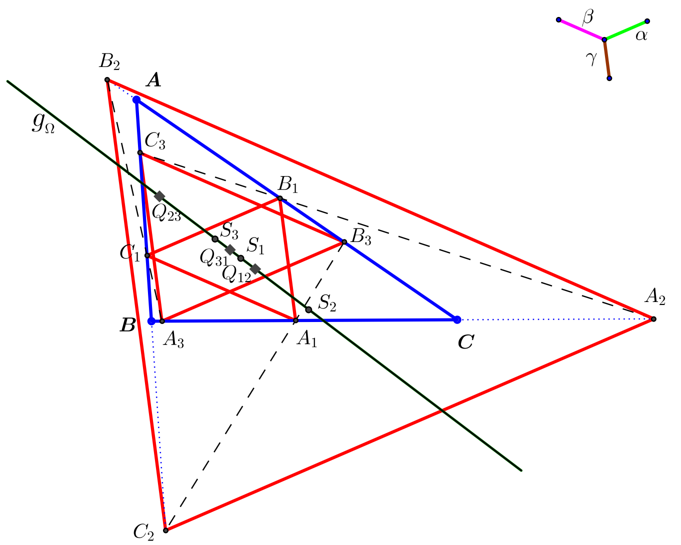

Figure 6

Proposition 8. (a) *The triangles* Φ_i *and* Φ^i *have the same area as the reference triangle* (see Figure 7).

(b) *The vertices of* Φ_i *and* Φ^i *lie on a circumconic* $\mathcal{C}_i$ *and* $\mathcal{C}^i$ *of* Δ*, respectively. The* $\mathcal{C}_i$ *are concurrent at* $\eta^\bullet$*, the* $\mathcal{C}^i$ *at* $\xi^\bullet$ (see Figure 8).

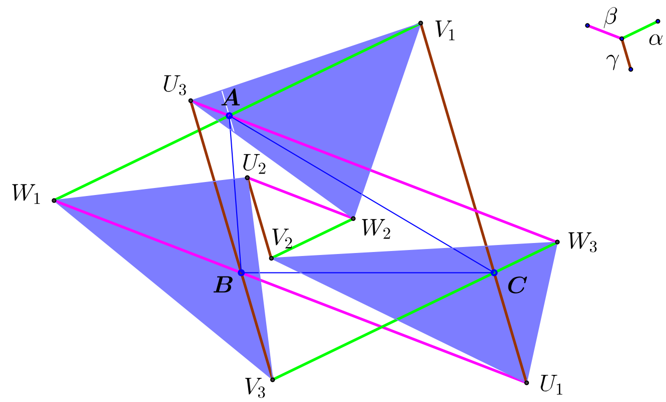

Figure 7

Proof. (a) By use of normalized barycentric coordinates of the vertices of Φ_i and Φ^i it is easy to show, that $|\det(\Phi_i)| = |\det(\Phi^i)| = 1$.

(b) It is simple to verify, that the isotomic conjugates of the vertices of each triangle Φ_i and Φ^i are collinear and that the concerned lines are concurrent at η and ξ, respectively. From this follows the assertion directly. □

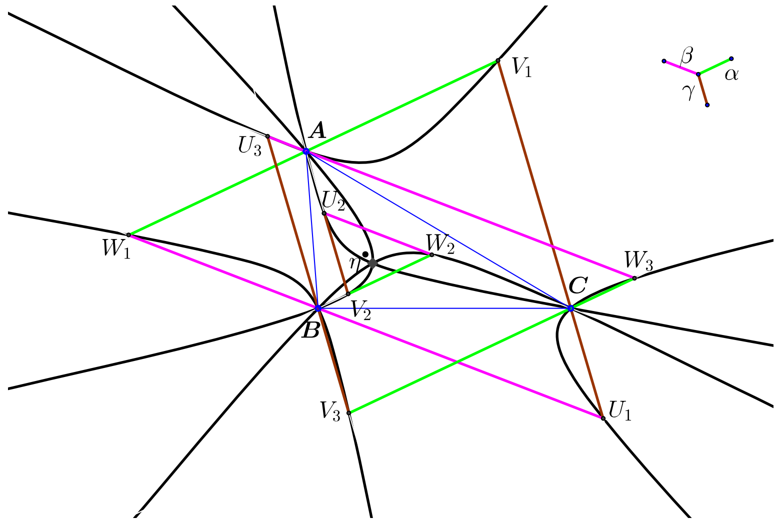

Figure 8

7. A special direction triple ($\alpha,\ \beta,\ \gamma$)

In the end we consider an interesting special case. Given a direction $\alpha = (\alpha_1 : \alpha_2 : \alpha_3)$. We choose β and γ as iterate Brocardians: $\beta = \alpha_{\leftarrow\leftarrow} = (\alpha_3 : \alpha_1 : \alpha_2)$ and $\gamma = \alpha_{\rightarrow\rightarrow} = (\alpha_2 : \alpha_3 : \alpha_1)$ [1]. The isotomic conjugates $\alpha^\bullet, \beta^\bullet, \gamma^\bullet$ are points on the Steiner ellipse, they form a Brocardian triple, that is the line at two of these points is the dual of the third point.

Let $A',\ B',\ C'$ be the reflections of $A,\ B,\ C$ in G and

$$\mathcal{C}_A:\ x^2 - yz = 0, \quad \mathcal{C}_B:\ y^2 - zx = 0, \quad \mathcal{C}_C:\ z^2 - xy = 0$$

three ellipses created by translation of the Steiner ellipse with centers $A',\ B',\ C'$ and passing through the common point G.

In the following we mention without proofs some properties of points and triangles defined above.

(1) The line triples $(\gamma_A,\ \beta_B,\ \alpha_C),\ (\beta_A,\ \alpha_B,\ \gamma_C)$ and $(\alpha_A,\ \gamma_B,\ \beta_C)$ are concurrent at the points

$P_1 = (\alpha_3\alpha_1 : \alpha_2\alpha_3 : \alpha_1\alpha_2),\ P_2 = (\alpha_1\alpha_2 : \alpha_3\alpha_1 : \alpha_2\alpha_3),\ P_3 = (\alpha_2\alpha_3 : \alpha_1\alpha_2 : \alpha_3\alpha_1)$,

respectively, that is each of the triangles Δ^i degenerates into a point on the Steiner ellipse.

(2) The triangles Δ_i have the same centroid G, it is at the same time the homothetic center of each pair $(\Delta_i,\ \Delta_j)$.

(3) The points of each triple $\{U_1, U_2, U_3\}, \{V_1, V_2, V_3\}, \{W_1, W_2, W_3\}$ are collinear. The three lines concurrent in G are the duals of $\alpha,\ \beta,\ \gamma$.

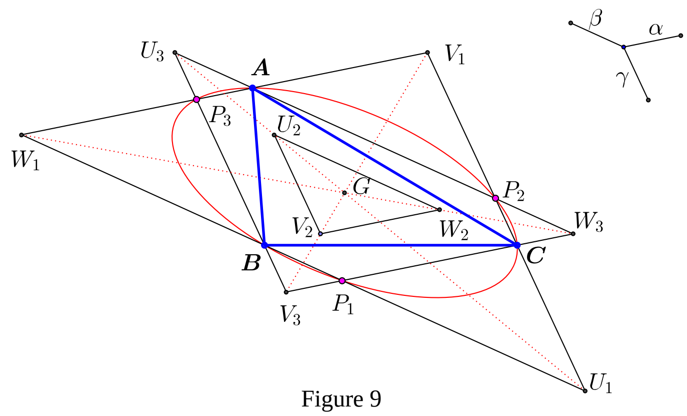

Figure 9

(4) For the vertices of the triangles Δ_i is valid:

$$U_1, V_3, W_2 \in \mathcal{C}_A, \quad U_2, V_1, W_3 \in \mathcal{C}_B, \quad U_3, V_2, W_1 \in \mathcal{C}_C.$$

These triangles are homothetic, congruent and equal in area with Δ.

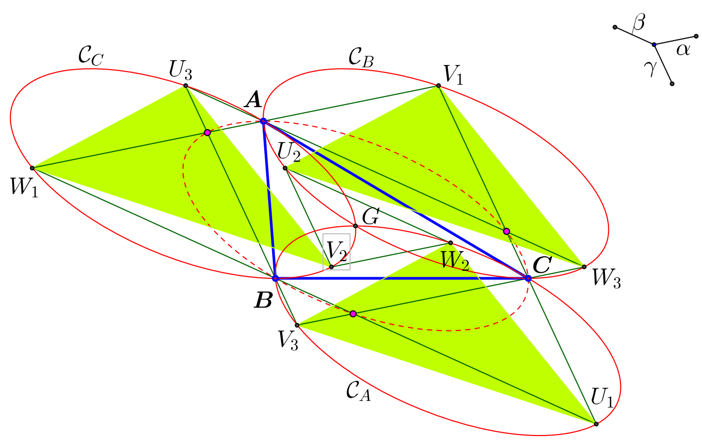

Figure 10

(5) The triangles Φ_k are homothetic and congruent with Δ. The homothetic centers of the pairs $(\Delta,\ \Phi_k)$ are the medial images mP_k of P_k. They are the midpoints of the segments P_iP_j, lie on the Steiner inellipse, have the representations

$$mP_1 = (\alpha_2^2 : \alpha_1^2 : \alpha_3^2),\ mP_2 = (\alpha_3^2 : \alpha_2^2 : \alpha_1^2),\ mP_3 = (\alpha_1^2 : \alpha_3^2 : \alpha_2^2)$$

and form a Brocardian Triple (see Figure 11).

(6) The sum of the areas of Δ_i is sixfold the area of Δ:

$$\sum |\Delta_i| = 6 \cdot |\Delta|.$$

(7) The midway triangle of the pair $(\Delta_i,\ \Delta_j)$ coincides with the medial triangle of Δ_k.

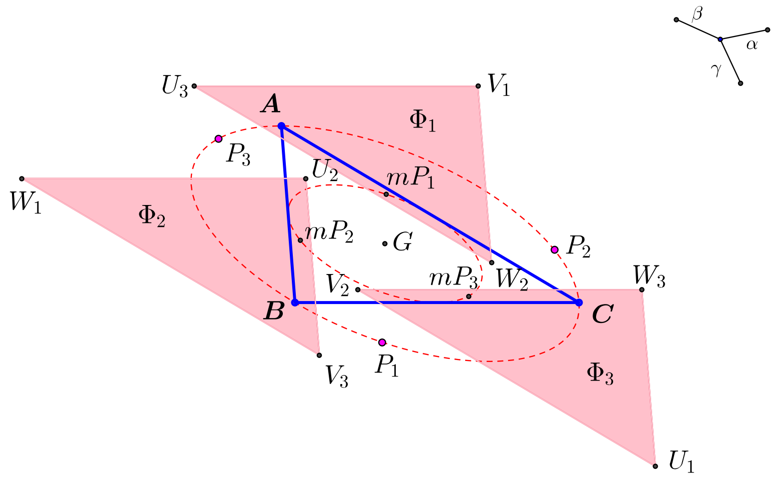

Figure 11

(8) The triangles Ω_i have the representations

$$\Omega_1 = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_2 & 0 & \alpha_3 \\ \alpha_3 & -\alpha_2 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & \alpha_1 & -\alpha_3 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_1 & -\alpha_3 & 0 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\alpha_1 & 0 & \alpha_2 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}$$

and the centroid G. The homothetic centers of the pairs $(\Omega_i,\ \Omega_j)$ coincide with G.

(9) The triangles Ω^i degenerate, their vertices lie on the infinite line.

References

[1] G. Weise, Iterates of Brocardian points and lines, *Forum Geom.*, 10 (2010) 109–118.

[2] G. Weise, Generalization and extension of the Wallace theorem, *Forum Geom.*, 12 (2012) 1–11.

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