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Symbolic Substitution Has a Geometric Meaning

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Abstract. By comparing two different metrics in the affine plane, it is shown that symbolic substitution, introduced by Clark Kimberling, has a clear geometric meaning.

1. Introduction

Consider in the real plane $\mathbb{R}^2$ a triangle ABC with vertices

$$A = \left(\frac{a^2 - b^2 + c^2}{2a}, \frac{\sqrt{\sigma}}{2a}\right),\ B = (0, 0),\ C = (a, 0),$$

where a, b and c are positive real numbers with

$$\sigma := \sigma(a, b, c) = (a + b + c)(-a + b + c)(a - b + c)(a + b - c) > 0.$$

If in $\mathbb{R}^2$ the Euclidean distance d of two points $U = (X, Y)$ and $U' = (X', Y')$ is defined by

$$d(U, U') = \sqrt{(X - X')^2 + (Y - Y')^2},$$

then the reference triangle ABC has side lengths a, b, c. In the following we use Conway's triangle notation:

$$S = \frac{1}{2}\sqrt{\sigma}, \quad S_A = \frac{1}{2}(b^2 + c^2 - a^2), \text{etc.}$$

Thus, we can write $A = \left(\frac{S_B}{a}, \frac{S}{a}\right)$.

Besides Cartesian coordinates we also use barycentric coordinates with respect to the reference triangle. The notation $U = (u, v, w)_{ABC}$ is used if (u, v, w) are the absolute coordinates of U with respect to ABC. If $(u : v : w)$ are homogeneous coordinates of U, then we write $U = (u : v : w)_{ABC}$. The conversion from Cartesian coordinates (X, Y) of a point U to its barycentric coordinates (u, v, w) can be achieved by taking

$$u = \frac{aY}{S}, \quad v = \frac{-((X - a)S + Y S_C)}{aS}, \quad w = \frac{XS - Y S_B}{aS} \qquad (*)$$

Given two points U and V with absolute barycentric coordinates (u, v, w) and (u', v', w'), the square of the distance between these two points can be calculated

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by the formula

$$d^2(U,U') = S_A(u-u')^2 + S_B(v-v')^2 + S_C(w-w')^2.$$

In addition to the canonical Euclidean metric, we introduce a second metric d_g, which we call *generalized metric* or *g-metric*, for short. This second metric is obtained by giving new lengths to the sides of the reference triangle:

$$d_g(A,B) = c_g, \quad d_g(B,C) = a_g, \quad d_g(C,A) = b_g.$$

We still demand that these lengths a_g, b_g, c_g are positive real numbers [1] but do not claim that

$$\sigma_g := (a_g + b_g + c_g)(-a_g + b_g + c_g)(a_g - b_g + c_g)(a_g + b_g - c_g)$$

is a positive real number.

For different signs of σ_g we get different geometries: $\sigma_g > 0$ delivers an affine version of the Euclidean metric, for $\sigma_g = 0$ the metric is Galilean, and for $\sigma_g < 0$ the metric is a Lorentz-Minkowski metric. With respect to this generalized metric, the square of the distance between two points U and U' with absolute barycentric coordinates (u,v,w) and (u',v',w') is given by

$$d_g^2(U,U') = S_{A,g}(u-u')^2 + S_{B,g}(v-v')^2 + S_{C,g}(w-w')^2 \qquad (**)$$

with $S_{A,g} = \frac{-a_g^2+b_g^2+c_g^2}{2}$ etc. The point with coordinates $(a_g : b_g : c_g)$ will become the generalized incenter I_g. The new circumcenter O_g has coordinates $(a_g^2(-a_g^2 + b_g^2 + c_g^2) : \cdots : \cdots)$. More generally, if $x(a,b,c)$ is a barycentric center function of a triangle center X, then $X_g = (x(a_g,b_g,c_g) : \cdots : \cdots)_{ABC}$ is the corresponding g-center. The point G_g still agrees with $G = (1:1:1)_{ABC}$. The square of the new triangle area is $\frac{1}{16}\sigma_g$.

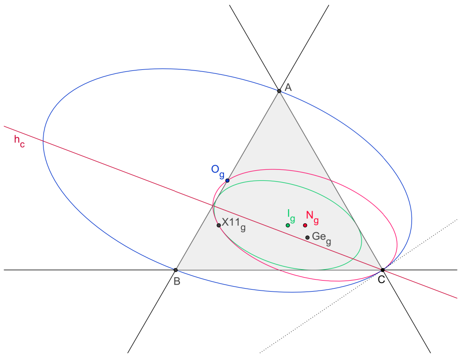

Figure 1. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (3 : 4 : 5)$. The picture shows the circumcircle (blue), the incircle (green) and the nine-point circle (red).

Figures 1 and 2 show the situation for an equilateral triangle ($a = b = c$) on which was imposed a new metric $(a_g, b_g, c_g) = (3,4,5)$ and $(a_g, b_g, c_g) =$

[1] Later on, we dismiss this condition, see 2.4 .

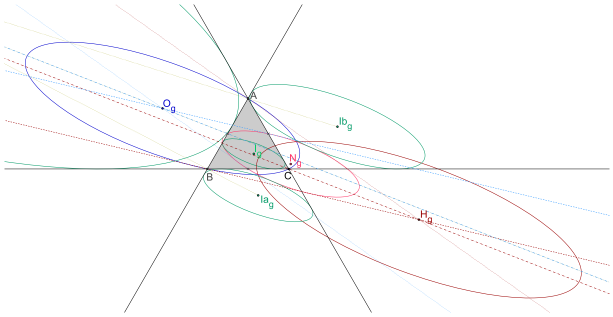

Figure 2. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (6 : 9 : 13)$. Besides the circumcircle, the incircle and the nine-point circle, the picture shows the excircles (green) and the polarcircle (brown).

$(6, 9, 13)$, respectively. Considering these two new metrics, the first triangle is right angled, the second obtuse.

A generalization of the incenter, the circumcenter and the orthocenter of a triangle is given by I. Minevich and P. Morton [12, Section 4]. They introduce the name *generalized* triangle centers. We adopt this terminology.

1.1. *Embedding the affine in a projective plane.* The affine plane can be embedded in a projective plane $\mathcal{P}$ by adding the line at infinity $\mathcal{L}_\infty$. This line consists of all points $(u, v, w)_{ABC}$ with $u + v + w = 0$. Points on $\mathcal{L}_\infty$ are called *infinite* or *improper* while those in the affine plane are called *finite* or *proper*. A line is called *proper* if it differs from $\mathcal{L}_\infty$.

1.2. *g-orthogonality of lines.* If two proper lines are g-orthogonal, we shall say that their infinite points form a pair of g-orthopoints. Two infinite points $U = (u : v : w)_{ABC}$ and $U' = (u' : v' : w')_{ABC}$ are g-orthopoints if and only if $S_{A,g}uu' + S_{B,g}vv' + S_{C,g}ww' = 0$. See [15, p. 55] or [2] for the Euclidean case. The following version of Thales' theorem is valid in all metric affine geometries: Given two lines $\mathcal{L}$ and $\mathcal{L}'$ which meet in a finite point P, these lines are orthogonal precisely when there exist two (not necessarily finite) points Q and R, one on $\mathcal{L}$ and the other on $\mathcal{L}'$ and both different from P, so that the center of the (generalized) circle PQR is a point on the line QR.

In the following, we analyze the three cases $\sigma_g > 0$, $\sigma_g = 0$ and $\sigma_g < 0$.

2. The affine Euclidean case

If $\sigma_g > 0$, then the g-incenter I_g lies inside the medial triangle and the g-symmedian K_g lies inside the Steiner inellipse. The metric given by formula $(**)$ is affine Euclidean. *Proof*: By an affine transformation τ with fixed point I_g, the inconic with center I_g can be mapped onto a circle. Let A', B', C' be the image points of A, B, C under τ. This transformation maps a point $U = (u : v : w)_{ABC}$ to the point $U' = (u : v : w)_{A'B'C'}$. I_g is the incenter of $A'B'C'$; therefore the ratio of the sidelengths of this triangle is $a_g : b_g : c_g$. If x is a barycentric center function, then τ maps the g-center $X_g = (x(a_g, b_g, c_g) : x(b_g, c_g, a_g) : x(c_g, a_g, b_g))_{ABC}$ of

ABC to the triangle center $X = (x(a_g, b_g, c_g) : x(b_g, c_g, a_g) : x(c_g, a_g, b_g))_{A'B'C'}$ of $A'B'C'$. $\square$

2.1. *Description of the generalized circumcircle.* In case of $\sigma_g > 0$, the generalized circles are ellipses. We assume that I_g and I are different points, so that these generalized circles are ellipses with two foci. The focal axes of these ellipses have all the same infinite point. We will determine this point. Furthermore, for the g-circumcircle we calculate the lengths of the two principal axes and the distance between the two foci.

The equation of a g-circle with center $M = (m_a, m_b, m_c)_{ABC}$ and radius ρ is

$$\sum_{\text{cyclic}} S_{A,g}((m_b + m_c)x - m_a(y + z))^2 = \rho^2(m_a + m_b + m_c)^2(x + y + z)^2.$$

We concentrate on the g-circumcircle, which is given by the equation

$$a_g^2 yz + b_g^2 zx + c_g^2 xy = 0.$$

Using $(*)$, we can derive an equation in Cartesian coordinates,

$$\alpha(X - X_0)^2 + 2\beta(X - X_0) + \gamma(Y - Y_0)^2 = \left(\frac{\alpha a_g b_g c_g S}{S_g}\right)^2,$$

with $\alpha = (2a_g S)^2$, $\beta = 2S(a_g^2(b^2 - c^2) - a^2(b_g^2 - c_g^2))$,
$\gamma = a_g^2(b^2 - c^2)^2 - 2a^2(b^2 - c^2)(b_g^2 - c_g^2) - a^4(a_g^2 - 2b_g^2 - 2c_g^2)$, and
(X_0, Y_0) the Cartesian coordinates of the g-circumcenter

$$O_g = (a_g^2(-a_g^2 + b_g^2 + c_g^2) : \cdots : \cdots)_{ABC}.$$

The direction vectors of the two principal axes of the circumellipse are the eigenvectors of the matrix $M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$. The two eigenvalues are

$$\lambda_\pm = \frac{1}{2}\left(\alpha + \gamma \pm \sqrt{(\alpha - \gamma)^2 + 4\beta^2}\right) = 2a^2(\theta \pm \phi^2)$$

with

$$\begin{aligned} \theta = {} & \frac{\alpha + \gamma}{4a^2} = a^2 S_{A,g} + b^2 S_{B,g} + c^2 S_{C,g} = a_g^2 S_A + b_g^2 S_B + c_g^2 S_C, \\ \phi = {} & \sqrt[4]{(\theta + 2SS_g)(\theta - 2SS_g)} \quad \text{and } S_g = \frac{1}{2}\sqrt{\sigma_g}. \end{aligned}$$

In the affine Euclidean case we have $\lambda_+ > \lambda_- > 0$. A calculation shows that the lengths of the major and minor half axes are

$$r_{max} = \frac{a a_g b_g c_g S}{S_g \sqrt{\lambda_-}} \quad \text{and} \quad r_{min} = \frac{a a_g b_g c_g S}{S_g \sqrt{\lambda_+}}.$$

The distance f of the foci from the center O_g is therefore

$$f = \sqrt{r_{max}^2 - r_{min}^2} = \frac{a_g b_g c_g \phi}{\sigma_g}.$$

The eigenspaces of λ_+ and λ_- are $\mathbb{R}(\lambda_\pm - \gamma, \beta)$, so that the focal line is $(X_0, Y_0) + \mathbb{R}(\lambda_- - \gamma, \beta)$. The infinite point on this line is

$$(a^2\beta : -S(\lambda_- - \gamma) - S_C\beta : S(\lambda_- - \gamma) - S_B\beta)_{ABC}.$$

The Cartesian coordinates of the two foci $F_\pm$ are $(X_0 \pm fv_1, Y_0 \pm fv_2)$, (v_1, v_2) being a unit eigenvector of λ_-. Subsequently, the barycentric coordinates of the two foci can be calculated:

$$F_\pm = (p \pm p', q \pm q', r \pm r')_{ABC},$$

where (p, q, r) are the absolute barycentric coordinates of O_g and

$$p' = \frac{afv_2}{S}, \quad q' = \frac{f(-v_1S - v_2S_C}{aS}, \quad r' = \frac{f(v_1S - v_2S_B)}{aS}.$$

The foci of a circumconic and an inconic can be obtained by a compass-ruler-construction, see Gibert [8]. Dergiades [3] calculated the lengths of the axes of an inellipse with given center and also described a compass-ruler-construction of the foci.

2.2. *Criterion for the g-orthogonality of lines / Theorem of Brianchon and Poncelet.* Let two different lines meet the line at infinity in the points U, V. These lines are g-orthogonal if and only if there exists a hyperbola through the points A, B, C, U, V and the g-orthocenter $H_g = \left(\frac{1}{-a_g^2+b_g^2+c_g^2} : \cdots : \cdots\right)$.

2.3. *The g-angle between two lines.* We want to determine the g-measure $\phi_g \in \left[0, \frac{\pi}{2}\right]$ of the angle $\angle(\mathcal{L}, \mathcal{L}')$ between the proper lines $\mathcal{L} : lx + my + nz = 0$ and $\mathcal{L}' : l'x + m'y + n'z = 0$. This measure is fixed as soon as we know the value of $\sin\phi_g$ or $\cos\phi_g$. To get these two values, let

$$k := l(m' - n') + m(n' - l') + n(l' - m'),$$

and calculate the g-area $|PQR|_g$ of a triangle PQR, where

$$P = \left(\frac{mn' - m'n}{k}, \frac{l'n - ln'}{k}, \frac{lm' - l'm}{k}\right)_{ABC}$$

is the intersection of $\mathcal{L}$ and $\mathcal{L}'$,

$$Q = \left(\frac{mn' - m'n}{k} + m - n, \cdots, \cdots\right)_{ABC}$$

is a second point on $\mathcal{L}$, and

$$R = \left(\frac{mn' - m'n}{k} + m' - n', \cdots, \cdots\right)_{ABC}$$

is a second point on $\mathcal{L}'$. For this,

$$|PQR|_g = |k| \cdot |ABC|_g = \frac{1}{4}|k|\sqrt{\sigma_g} = \frac{1}{2}|k|S_g.$$

Thus, we get

$$\begin{aligned}
\sin\phi_g &= \left|\frac{2k\cdot|ABC|_g}{d_g(P,Q)d_g(P,R)}\right|,\\
\cos\phi_g &= \left|\frac{S_{A,g}(m-n)(m'-n')+S_{B,g}(n-l)(n'-l')+S_{C,g}(l-m)(l'-m')}{d_g(P,Q)d_g(P,R)}\right|,\\
d_g^2(P,Q) &= S_{A,g}(m-n)^2+S_{B,g}(n-l)^2+S_{B,g}(l-m)^2,\\
d_g^2(P,R) &= S_{A,g}(m'-n')^2+S_{B,g}(n'-l')^2+S_{B,g}(l'-m')^2.
\end{aligned}$$

We interpret $\vec{l}=\begin{pmatrix} m-n\\ n-l\\ l-m\end{pmatrix}$ and $\vec{l'}=\begin{pmatrix} m'-n'\\ n'-l'\\ l'-m'\end{pmatrix}$ as direction vectors of $\mathcal{L}$ and $\mathcal{L}'$ respectively, and the expression

$$S_{A,g}(m-n)(m'-n')+S_{B,g}(n-l)(n'-l')+S_{C,g}(l-m)(l'-m')$$

as a dot product $\vec{l}\cdot\vec{l'} =< \vec{l}|\vec{l'} >_g$ of these two vectors, so that we can write

$$\cos(\phi_g)=\frac{|\vec{l}\cdot\vec{l'}|}{\sqrt{\vec{l}\cdot\vec{l}}\sqrt{\vec{l'}\cdot\vec{l'}}}.$$

2.4. *Negative sidelengths a_g, b_g, c_g.* There is no reason to forbid negative values for a_g, b_g, c_g. For example, if we choose $a_g=-a$, $b_g=b$ and $c_g=c$ or $a_g=a$, $b_g=-b$ and $c_g=-c$, we just swap the incenter I with the excenter I_a and the excenter I_b with the excenter I_c. If a weak center, consisting of the main point and its three mates, is not considered as a quadruple but as a set of four points, changing signs leaves this weak center invariant. Strong centers keep their positions, anyway. This situation does not change if we dismiss the condition $\sigma_g>0$.

3. The Galilean case

If $\sigma_g=0$, the point H_g is an infinite point and agrees with O_g and N_g. H_g is the absolute pole of the Galilean plane: two finite points lying on the same line through H_g have Galilean distance 0. A line through the absolute pole is therefore called a null line. In particular, the three generalized triangle altitudes, considered as lines, are null lines. This is in accordance with the area of the triangle: $\frac{\sqrt{\sigma_g}}{4}=0$. The g-symmedian point K_g is a point on the Steiner inellipse. The incenter and the excenters are the barycentric square roots of K_g. One of these roots agrees with H_g, so it is an infinite point. The other three roots lie on the sidelines of the medial triangle of ABC.

3.1. *g-circles in Galilean geometry.* Generalized circles in the Galilean plane are parabolas. Some of them may degenerate. For example, the g-incircle and the g-excircles are double lines (lines with multiplicity 2), and if the incenter is a vertex of the medial triangle, then the g-circumcircle and the g-nine-point circle each consist of two parallel lines. It should be noted that a degenerate g-circle in Galilean geometry has more than one center. All these centers lie on a "centerline" of this g-circle. Michel Bataille [2] calculated the barycentric coordinates of the focus F of

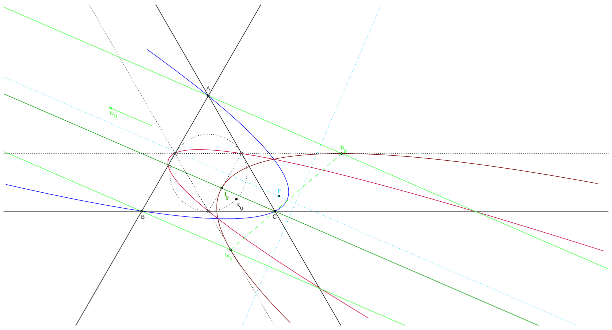

Figure 3. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (2 : 3 : 5)$. The picture shows the g-circumcircle (blue) with focus F, the g-nine-point circle (red) and the g-polar circle (brown). The g-polar circle runs through the g-incenter and the g-excenters. The green lines are the g-altitudes. As double lines they are the g-incircle and two g-excircles.

a circumparabola $\mathcal{C}$ of the triangle ABC: if the infinite point of $\mathcal{C}$ has barycentric coordinates $(u : v : w)$, then the focus is

$$F = \left(\frac{u^2}{vw} + \frac{a^2vw}{a^2vw + b^2wu + c^2uv} : \cdots : \cdots \right)_{ABC}.$$

Figure 3 gives an illustration of a triangle in the Galilean plane.

3.2. *g-Orthogonality of lines in a Galilean plane.* Two lines which meet in a finite point are orthogonal if and only if one of these is a null line.

3.3. *The g-angle between two lines.* It makes sense to define the g-measure of the angle between two non-null-lines $\mathcal{L}$ and $\mathcal{L}'$ to be the g-distance of the poles of these lines with respect to the g-circumcircle of the triangle ABC. *Proof*: We may assume that the finite part of the g-circumcircle of the triangle ABC in $\mathbb{R}^2$ is given by the equation $y = kx^2$, $k \neq 0$, and that $y = mx + n$ and $y = m'x + n'$ are the equations of the lines $\mathcal{L}$ and $\mathcal{L}'$. The poles of these two lines with respect to the g-circumcircle are $P = (\frac{m}{2k}, -n)$ resp. $P' = (\frac{m'}{2k}, -n')$. There exists a real number $k' \neq 0$ so that the Galilean distance between any two points (x, y) and (x', y') is $k'|x - x'|$. Therefore, the distance between P and P' is $\frac{k'}{2|k|}|m - m'|$. This is consistent with the usual definition of the measure of the angle between the two lines $\mathcal{L}$ and $\mathcal{L}'$ in Galilean geometry, see for example [7, Chap 4] or [13, Chap 23].$\square$

Remarks. (1) Using barycentric coordinates, the pole of the line $\mathcal{L} : lx+my+nz = 0$ with respect to the circumcircle $\mathcal{C} : a_g^2yz + b_g^2zx + c_g^2xy = 0$ is

$$P = \left(a_g^2(-a_g^2l + b_g^2m + c_g^2n) : \cdots : \cdots\right)_{ABC}.$$

(2) For the reference triangle ABC, the ratio of the measure of an angle and the length of its opposite side is $\frac{1}{2}$. (The measure α of the angle between AB and AC is $\frac{1}{2}a$ *etc*).

(3) For the medial triangle of ABC, the measure of an angle equals the measure of its opposite side.

4. The Lorentz-Minkowski case

If $\sigma_g < 0$ then the metric d_g is a Lorentz-Minkowski metric or an LM metric, for short. In this case, generalized circles are hyperbolas which might eventually degenerate. Given a g-metric, all the g-circles run through the same two points on the line at infinity $\mathcal{L}_\infty$. These two points P_1 and P_2 are the absolute poles of the LM plane. The distance between two different finite points U and V is 0 if and only if (either) P_1 or P_2 is a point on the line UV; in this case UV is called a null line. In LM geometry, not only the distance between two points can be negative but also the square of the distance. Following A. Einstein [6], we will define: Two points U and V are in a *spacelike* position if $d_g^2(U,V) > 0$, and they are in a *timelike* position if $d_g^2(U,V) < 0$. If two points U, V are in a spacelike (respectively timelike) position, then all finite points P, Q on the line UV with $P \neq Q$, are in a spacelike (respectively timelike) position. Therefore, we can call the line UV *spacelike* (respectively *timelike*). Obviously, in LM geometry triangles can have real as well as imaginary sidelengths. Triangles having both, a real and an imaginary sidelength $\neq 0$, do not have weak triangle centers. All strong centers still exist.

4.1. *Description of the generalized circles.* Generalized circles in the LM plane are hyperbolas. The focal axes of two g-circles do not have to be parallel but may be orthogonal, as can be seen in Figure 7. The principal axes and the foci can be calculated the same way as in §2.1 for the Euclidean case, except for the distance between the foci and the center O_g. This is now

$$f = \sqrt{r_{max}^2 + r_{min}^2} = -\frac{a_g b_g c_g \sqrt{|\theta|}}{\sigma_g}.$$

The two absolute poles are

$$P_1 = \ (-a^2(\omega+\beta) : (a^2+S_B)(\omega+\beta)+S\gamma : -S_B(\omega+\beta)-S\gamma)_{ABC},$$
$$P_2 = \ (-a^2(\omega-\beta) : (a^2-S_B)(\omega-\beta)+S\gamma : S_B(\omega-\beta)-S\gamma)_{ABC}.$$

Here we use the notations of §2.1 and $\omega := \sqrt{\beta^2-\alpha\gamma} = a^2\sqrt{-\sigma\sigma_g}$.

4.2. *g-Orthogonality of lines.* A pair (U,V) of points on the line at infinity is a pair of g-orthopoints if and only if the quadruple (U,V,P_1,P_2) is harmonic. It can be easily checked that if a line is spacelike then all its g-orthogonal lines are timelike. Particularly, if a sideline of a triangle is spacelike then the corresponding altitude, considered as a line, is timelike and vice versa. Figure 5 shows a triangle with $d_g(B,C) = 0$, and Figure 6 illustrates the situation for a triangle with $\mathrm{d}_g(A,B) = \mathrm{d}_g(A,C) = 0$. In both cases, the generalized circumcircle as well as the generalized nine point circle degenerate to a pair of lines. In Figure 7 the generalized symmedian K_g is outside the triangle, therefore weak generalized centers do not exist in the real plane.

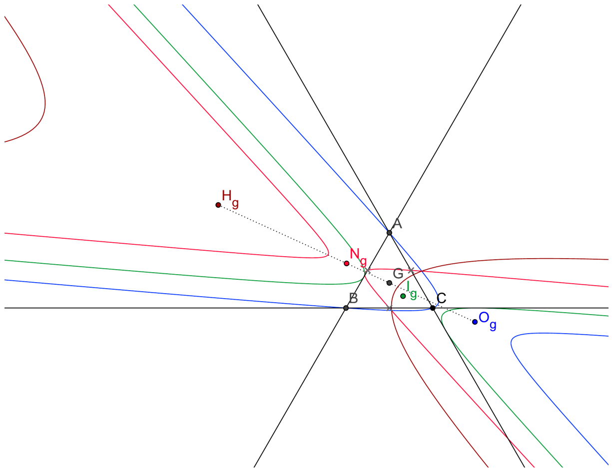

Figure 4. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (3 : 5 : 11)$. The picture shows the g-circumcircle (blue), the g-incircle (green), the g-nine-point circle (red) and the g-polar circle (brown). All g-circles go through the same infinite points.

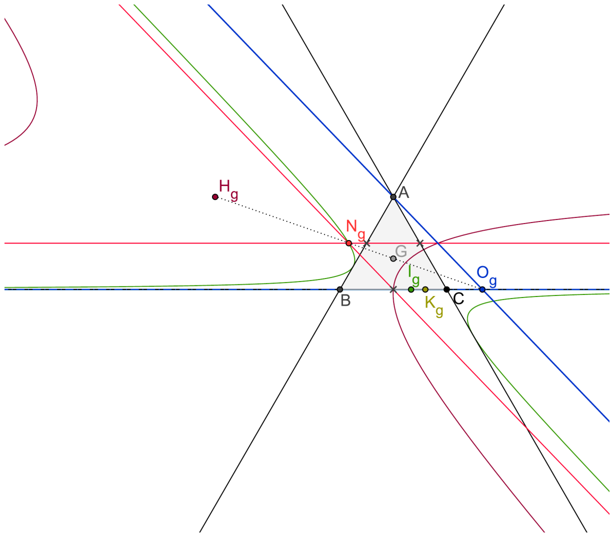

Figure 5. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (0 : 1 : 2)$. The picture shows the following generalized circles: the circumcircle (it consists of two blue lines intersecting in O_g), the incircle (green), the nine-point circle (it consists of two red lines intersecting in N_g) and the polarcircle (brown). I_g, O_g and K_g are points on the sideline BC, N_g is a point on the g-incircle.

4.3. *The g-angle between two lines.* We want to calculate the g-measure ϕ_g of the angle $\angle_g(\mathcal{L}, \mathcal{L}')$ between the proper lines $\mathcal{L} : lx + my + nz = 0$ and $\mathcal{L}' : l'x+m'y+n'z = 0$. To get a real value for ϕ_g, these lines have to be both spacelike or both timelike. The value of $\phi_g \geq 0$ is determined by the value of $\cosh(\phi_g)$. A calculation similar to the one in the Euclidean case gives the following result (we

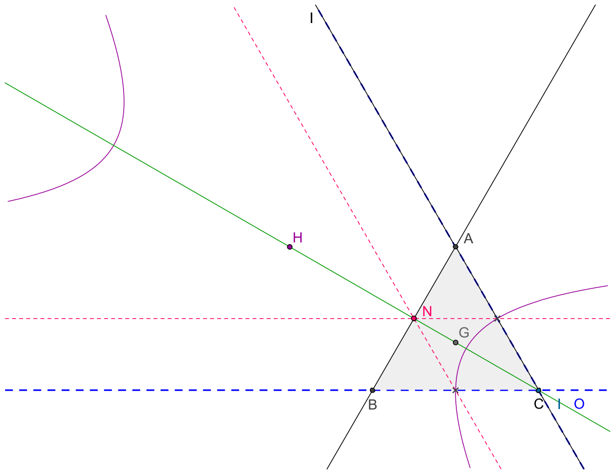

Figure 6. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (0 : 0 : 1)$. The points I_g, O_g and K_g agree with the vertex C.

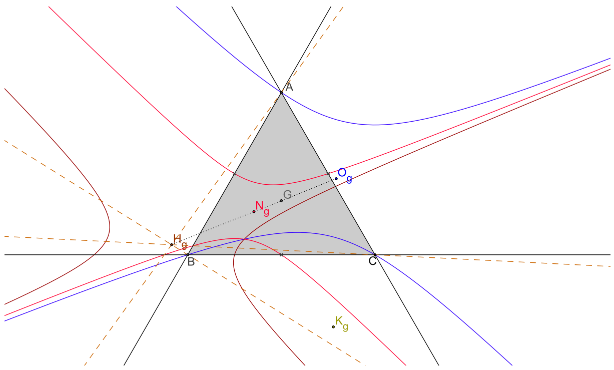

Figure 7. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (2i : 2 : 3)$.

use notations defined in §2.3):

$$\cosh(\phi_g) = \frac{|\vec{l} \cdot \vec{l'}|}{\sqrt{|\vec{l} \cdot \vec{l}|}\sqrt{|\vec{l'} \cdot \vec{l'}|}}.$$

5. Symbolic substitutions

In [10], C. Kimberling gives the following definition of the transfigured plane of a triangle:

> Suppose a, b, c are variables (or indeterminates) over the field of complex numbers and that x, y, z are homogeneous algebraic functions of $(a, b, c) : x = x(a, b, c)$, $y = y(a, b, c)$, $z = z(a, b, c)$, all of the same degree of homogeneity and not all identically zero. Triples (x, y, z) and (x', y', z') are equivalent if $xy' = yx'$ and

> $yz' = zy'$. The equivalence class containing any particular (x, y, z) is denoted by $x : y : z$ and is a 'point'. Let $A = 1 : 0 : 0$, $B = 0 : 1 : 0$, $C = 0 : 0 : 1$. These three points define the reference triangle ABC. The set of all points is the transfigured plane.

We denote this transfigured plane by $\mathcal{P}$. A model of $\mathcal{P}$ can be obtained by taking a, b, c as the side lengths of the triangle ABC and $(x(a, b, c) : y(a, b, c) : z(a, b, c))$ as barycentric coordinates of a point. A second model is constructed by taking $(x : y : z)$ as trilinear coordinates of a point. If x', y', z' are functions of a, b, c, all of the same degree of homogeneity and not all identically zero, then the substitution $(a, b, c) \mapsto (x'(a, b, c), y'(a, b, c), z'(a, b, c))$ induces a transformation on $\mathcal{P}$. While in the first model, in addition to the vertices A, B, C of the reference triangle, its centroid is a fixed point of the transformations, in the second the vertices and the incenter stay fixed. The advantage of the first model, as compared to the second, is that the transformations leave the line at infinity invariant. Kimberling [9] (see also [4, §2.3]) names these substitutions and the hereby induced transformations *symbolic* because, as he says, they suffer from a geometric meaning:

> The adjective *symbolic* is applied to substitutions $(\alpha, \beta, \gamma) \to (\alpha', \beta', \gamma')$ in order to distinguish between these and geometric transformations. Consider the substitution $(a, b, c) \to (bc, ca, ab)$; if a, b, c are the sidelengths $2, 4, 5$ of a triangle then bc, ca, ab are not sidelengths of a triangle. Moreover, for general a, b, c, a geometric construction of a point $(x(a, b, c) : y(a, b, c) : z(a, b, c))$ offers no clues for constructing the point $x(bc, ca, ab) : y(bc, ca, ab) : z(bc, ca, ab)$.

The first of these two arguments can be dismissed, as in the former sections was shown that every triangle can be given any side lengths a_g, b_g, c_g except for $a_g = b_g = c_g = 0$ by taking a suitable generalized metric d_g. Furthermore, this new metric can be used for the construction of new triangle centers. We give two examples:

(1) The substitution

$$\begin{aligned}&(x\ , y\ , z\ ) = (-a, b, c) \mapsto (x', y', z')\\ =&(a^2(b^2+c^2) - (b^2-c^2)^2, b^2(a^2+c^2) - (c^2-a^2)^2, c^2(a^2+b^2) - (a^2-b^2)^2)\end{aligned}$$

maps the excenter $I_a = (-a, b, c)_{ABC}$ to the nine-point center N, the incenter I to the anticevian point ${}_aN$ of N, and the symmedian point K to N^2 (the barycentric square of N). If we take ${}_aN$ as the generalized incenter I_g, then N is going to be the generalized excenter $I_{a,g}$ and N^2 the perspector of the g-circumcircle.

(2) Construction of the contact triangle of the MacBeath inconic, starting from its center N. The substitution

$$\begin{aligned}&(x\ , y\ , z\ ) = (a, b, c) \mapsto (x', y', z')\\ =&(a^2(b^2+c^2) - (b^2-c^2)^2, b^2(a^2+c^2) - (c^2-a^2)^2, c^2(a^2+b^2) - (a^2-b^2)^2)\end{aligned}$$

maps I to N and the Gergonne point G_{e} to G/O, the isotomic conjugate of the circumcenter [2]. I is the center and G_{e} is the perspector of the incircle. Therefore, G_{e} is the isotomic conjugate of the anticomplement of I. Thus, we can construct G/O as isotomic conjugate of the anticomplement of N. The cevian triangle of G/O is the contact triangle of the MacBeath inconic.

5.1. *The geometric meaning of a symbolic substitution.* Let $x(a,b,c)$ be a barycentric center function. If in the affine plane the square of the distance between two points $U=(u,v,w)_{ABC}$ and $U'=(u',v',w')_{ABC}$ is given by

$$d^2(U,U') = \sum_{\text{cyclic}} \frac{1}{2}(-a^2+b^2+c^2)(u-u')^2,$$

the point X that corresponds to this center function has barycentric coordinates $(x(a,b,c) : x(b,c,a) : x(c,a,b))$. The same point is the incenter of the reference triangle when using a distance function d_x with

$$d_x^2(U,U') = \sum_{\text{cyclic}} \frac{1}{2}(-(x(a,b,c))^2+(x(b,c,a))^2+(x(c,a,b))^2)(u-u')^2.$$

Let x', y', z' be functions of a,b,c, all of the same degree of homogeneity and not all identically zero. The point

$$X' = (x(x'(a,b,c), y'(a,b,c), z'(a,b,c)) : \cdots : \cdots)_{ABC}$$

can be interpreted as a point with center function x, when the square of the distance d' of two points $U=(u,v,w)$ and $U'=(u',v',w')$ is given by

$$d'^2(U,U') = \sum_{\text{cyclic}} \frac{1}{2}(-(x'(a,b,c))^2+(y'(a,b,c))^2+(z'(a,b,c))^2)(u-u')^2.$$

6. Generalized versions of Feuerbach's conic theorem

Feuerbach's conic theorem establishes a connection between two different metrics of the affine plane.

6.1. *Feuerbach's conic theorem* (see [5, 1]). Each circumconic of a triangle running through the orthocenter H has its center on the nine point circle.

If we interpret this circumconic as a g-circumcircle of the reference triangle with center O_g, then the g-orthocenter H_g is a point on the Euclidean circumcircle. Thus, we can formulate the following generalized versions (6.1.1) and (6.1.2) of Feuerbach's conic theorem:

6.1.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be two circumconics of triangle ABC. We interpret $\mathcal{C}$ and $\mathcal{C}'$ as generalized circumcircles with center O_g and $O_{g'}$. Then H_g is a point on $\mathcal{C}'$ if and only if H'_g is a point on $\mathcal{C}$.

[2] G/O is X_{284} in [9].

6.1.2. A conic runs through all four points of a g-orthocentric system if and only if its center lies on the g-orthic circle, which is the g-circumcircle of the g-orthic triangle.

6.2. *Corollaries.*

6.2.1. Each conic through the vertices of a quadrangle has its center on the nine-point conic of this (complete) quadrangle, see for example [4, Lemma 19.1.2], [12] and [15].

6.2.2. Together with the incenter I, the excenters I_a, I_b, I_c of the triangle ABC form an orthocentric system, the reference triangle ABC being the orthic triangle. Each conic running through these four points has its center on the circumcircle of the triangle ABC. Taking one of these conics and interpreting its center as a g-orthocenter, this conic is the g-polar circle of ABC. The triangle ABC is self polar with respect to it.

6.2.3. We shall call an LM metric *canonical* if H_g is a point on the (common) circumcircle of ABC. In this case the g-circumconic is a rectangular circumhyperbola of ABC, thus running through H. The center O_g of this conic is the complement of H_g and lies on the nine-point circle. The g-nine-point circle is a rectangular circumhyperbola through the vertices of the medial triangle and its orthocenter (= circumcenter of ABC). The perspector of the g-circumconic, the g-symmedian point K_g, is a point on the tripolar line of H. If H is inside the triangle, K_g has to lie outside, and weak g-centers do not exist in the real plane. We now assume that K_g lies inside the triangle ABC. In this case ABC is obtuse, the g-incenter and the g-excenters exist, and these four points lie on the polar circle of ABC.

6.2.4. If $P = (p : q : r)$ and $P' = (p' : q' : r')$ are the centers of two different circumconics $\mathcal{C}$ and $\mathcal{C}'$ of ABC, then the perspectors of these conics have coordinates $(u : v : w) = (p(-p+q+r) : \cdots : \cdots)$ respectively $(u' : v' : w') = (p'(-p'+q'+r') : \cdots : \cdots)$. The fourth (nontrivial) intersection point of $\mathcal{C}$ and $\mathcal{C}'$ has coordinates

$$\left(\frac{1}{vw'-v'w} : \frac{1}{wu'-w'u} : \frac{1}{uv'-u'v}\right).$$

In the case of $P' = O$, the fourth intersection point is called the Collings transform of P, see [9] for a definition and a bibliography.

6.2.5. Let $ABCD$ be a not degenerate quadrangle, the point D having barycentric coordinates $(d : e : f)$ with respect to ABC. Then the centers of circumconics of $ABCD$ lie on the nine-point conic of the complete quadrangle $ABCD$. This conic is also the bicevian conic of D and G with respect to the triangle ABC. The barycentric equation of this conic is

$$efx(-x+y+z) + fdy(x-y+z) + dez(x+y-z) = 0. \qquad (***)$$

Its center is the midpoint of the Varignon parallelogram of the quadrangle $ABCD$ and has coordinates $(2d+e+f : \cdots : \cdots)$. We will denote this midpoint by M_D. If by a homothety χ with center G and scale factor $\frac{1}{4}$, the quadrangle $ABCD$ is mapped onto a quadrangle $A'B'C'D'$, and D' agrees with M_D, as can be proved by a simple calculation. If we now assume that D is a point on a circumconic $\mathcal{C}$ of ABC, the point $D' = M_D$ lies on the image $\mathcal{C}'$ of $\mathcal{C}$ under χ. Suppose that the center of $\mathcal{C}$ has coordinates $(k : l : m)$, then the center of $\mathcal{C}'$ has coordinates $(2k + l + m : \cdots : \cdots)$. *Proof* of $(***)$: Taking $(d : e : f)_{ABC}$ as the g-orthocenter H_g of ABC, the symmedian point K_g has coordinates $(a_g^2 : b_g^2 : c_g^2) = (d(e+f) : e(f+d) : f(d+e))$. We can now use the equation of the g-nine-point circle $\sum_{\text{cyclic}}(S_{A,g}x^2 - a_g^2 yz = 0)$ (see [16] for the ordinary version) to get the equation $(***)$. $\square$

7. Additional change of the centroid

In the plane $\mathcal{P}$ of the reference triangle ABC we choose a point $P = pA + qB + rC$ with $p + q + r = 1$ and $pqr \neq 0$. The mapping, which assigns each point Q the barycentric product $P \cdot Q$, is a bijective projective transformation of the plane. It maps the quadrangle $ABCG$ (G being the centroid) to the quadrangle $ABCP$ and the line at infinity to the tripolar line $\mathcal{T}_P$ of P. Making P the *new* centroid G_n of triangle ABC and $\mathcal{T}_P$ the *new* line at infinity, we can define a *new* metric on $\mathcal{P} - \mathcal{T}_P$: The triangle ABC is given side lengths a_g, b_g, c_g, where a_g, b_g, c_g may take any real or purely imaginary values except for $a_g = b_g = c_g = 0$. The square of the *new* distance between two points $U = (u : v : w)_{ABC}$ and $V = (u' : v' : w')_{ABC}$ is now defined by

$$d_{\text{new}}^2(U, U') = \frac{\sum_{\text{cyclic}} S_{A,g}((uv' - u'v)r + (uw' - u'w)q)^2}{(uqr + vrp + wpq)^2(u'qr + v'rp + w'pq)^2}.$$

If X_g is a triangle center of ABC with respect to the metric d_g^2 as defined in Section 1, then $X_g \cdot P$ is the corresponding center of triangle ABC with respect to the *new* metric. Figure 8 gives an illustration by showing the most important triangle centers and triangle circles.

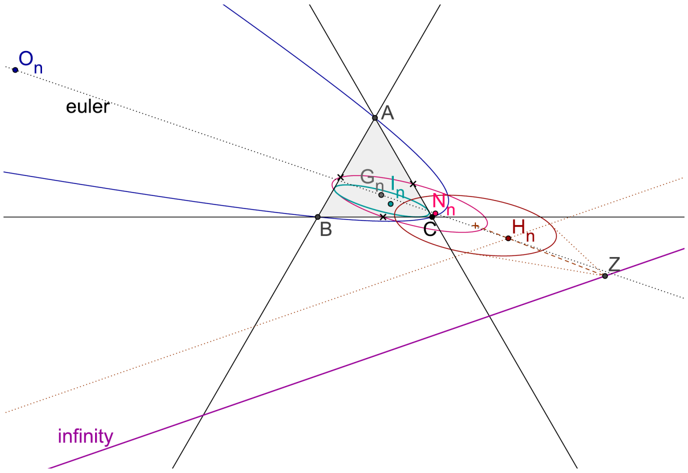

Figure 8. An equilateral triangle is given new side lengths $(a_g : b_g : c_g) = (6 : 9 : 13)$. Here, the *new* line at infinity is the polar line $\mathcal{T}_P$ of the point $P = G_n$ with barycentric coordinates $(2 : 3 : 4)$. There exists a point Z on the *new* line at infinity with the following properties: The mirror image of Z in a *new* circle is its *new* center, and the polar line of Z with respect to this *new* circle is parallel to $\mathcal{T}_P$.

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