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A Note on Haga's Theorems in Paper Folding

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Abstract. Haga's three theorems in the mathematics of square paper folding are unified in a simple way.

1. Introduction

Haga's famous theorems in the mathematics of square paper folding consists of three main parts [1, 2, 3]. Let us assume that $ABCD$ is a piece of square paper with a point E on the side AD. We fold the paper so that the corner C coincides with E and the side BC is carried into $B'E$, which intersects the side AB at a point F (see Figure 1). We call this Haga's fold of the first kind. Haga discovered if E is the midpoint of AD, then F divides AB in the ratio $2:1$ internally (first theorem). Also if F is the midpoint of AB, then E divides AD in the ratio $2:1$ internally (third theorem; see Figure 2).

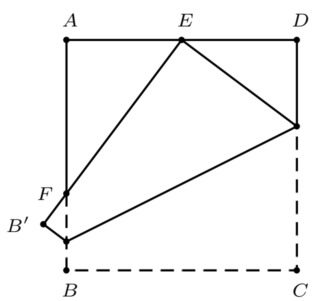

Figure 1

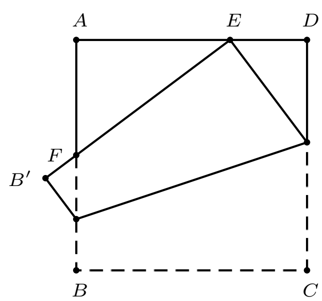

Figure 2

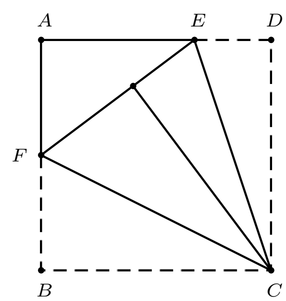

Figure 3

Let F be a point on the side AB such that the reflection of B in the line CF coincides with the reflection of D in the line CE (see Figure 3). This is called Haga's fold of the second kind, with the crease lines CE and CF. He discovered if F is the midpoint of AB, then E divides AD in the ratio $2:1$ internally (second theorem). In this note, we show that these three facts are unified in a simple way.

2. Main theorem

It is easy to show AF as a function of DE. Indeed, the following fact is given in [1]: If $AB = 1$, then $AF = \frac{2DE}{1+DE}$ holds for the fold of the first kind. Also he pointed out that his fold of the first kind derived from the fold of the second kind, and vice versa. In fact for the fold of the first kind, the reflection of B in the line CF coincides with the reflection of D in the line CE (see Figure 4). Haga's results are unified as follows.

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Theorem. *The relation* $\frac{AF}{FB} = 2 \cdot \frac{DE}{EA}$ *holds for Haga's folds of the first and second kinds*.

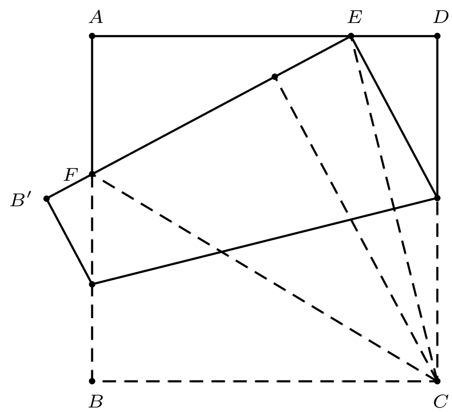

Figure 4

Proof. Let $AB = 1$. The theorem can be proved using the relation $AF = \frac{2DE}{1+DE}$. We give a proof using trigonometry. By the above remark, it is sufficient to prove for the fold of the second kind. Let $\theta = \angle DCE$ and $t = \tan\theta$. Then we get $DE = t$ and $EA = 1 - t$ (see Figure 4). This implies $\frac{DE}{EA} = \frac{t}{1-t}$. While $\angle BCF = \frac{\pi}{4} - \theta$ leads to $FB = \tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-t}{1+t}$ and $AF = 1 - FB = \frac{2t}{1+t}$. Hence we get $\frac{AF}{FB} = \frac{2t}{1-t}$. The theorem is now proved. □

By the theorem $AF : FB = k : 1$ is equivalent to $DE : EA = k : 2$ for a positive real number k. Haga's results are obtained when $k = 1$ and $k = 2$.

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