Forum Geometricorum
Volume 14 (2014) 243–246.

FORUM GEOM
ISSN 1534-1178

Dao's Theorem on Six Circumcenters associated with a Cyclic Hexagon

Nikolaos Dergiades

Abstract. We reformulate and give an elegant proof of a wonderful theorem of Dao Thanh Oai concerning the centers of the circumcircles of the six triangles each bounded by the lines containing three consecutive sides of the hexagon.

In slightly different notations Dao Thanh Oai [3] has posed the problem of proving the following remarkable theorem.

Theorem (Dao). *Let A_i, $i = 1, 2, \dots, 6$, be six points on a circle. Taking subscripts modulo 6, we denote, for $i = 1, 2, \dots, 6$, the intersection of the lines A_iA_{i+1} and $A_{i+2}A_{i+3}$ by B_{i+3}, and the circumcenter of the triangle $A_iA_{i+1}B_{i+2}$ by C_{i+3}. The lines C_1C_4, C_2C_5, C_3C_6 are concurrent.*

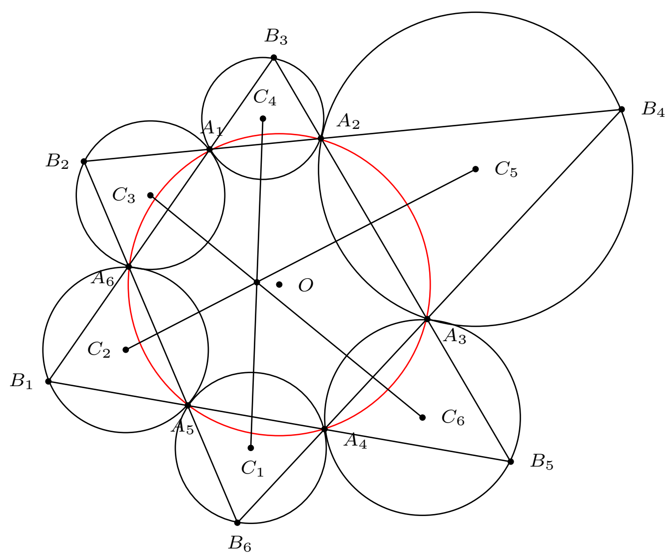

Figure 1

Indeed a proof with tedious computer aided calculations with barycentric coordinates has been given in [4]. In this note we give an elegant proof using complex numbers by considering the given hexagon as inscribed on the unit circle with center 0 in the complex plane.

Publication Date: September 15, 2014. Communicating Editor: Paul Yiu.

Lemma 1. *If A, B, C, D are points on the unit circle with affixes a, b, c, d respectively, and the lines AB, CD intersect at E, then the circumcenter P of triangle ACE has affix $p = \frac{ac(b-d)}{ab-cd}$.*

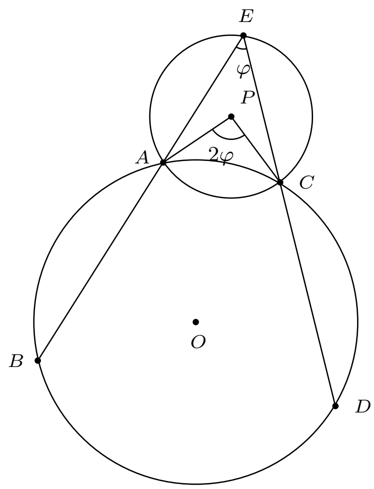

Figure 2

Proof. If the oriented angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is $\varphi = \angle AEC$, then $\angle APC = 2\varphi$. If $z = \cos\varphi + i\sin\varphi$, then since for a point A on the unit circle, the conjugate of its affix is $\overline{a} = \frac{1}{a}$, we conclude that

$$(c-p) = (a-p)z^2. \qquad (*)$$

Now,

$$\frac{d-c}{|d-c|} = \frac{b-a}{|b-a|}z \implies \frac{(d-c)^2}{|d-c|^2} = \frac{(b-a)^2}{|b-a|^2}z^2 \implies \frac{d-c}{\overline{d}-\overline{c}} = \frac{b-a}{\overline{b}-\overline{a}}z^2$$
$$\implies \frac{d-c}{\frac{1}{d}-\frac{1}{c}} = \frac{b-a}{\frac{1}{b}-\frac{1}{a}}z^2 \implies cd = abz^2.$$

This reduces to $cd = abz^2$, and from $(*)$, $(c-p)ab = (a-p)cd$. From this, $p = \frac{ac(b-d)}{ab-cd}$. □

To avoid excessive use of subscripts, we reformulate and prove Dao's Theorem in the following form.

Theorem 2. *Let A, B, C, X, Y, Z be arbitrary points on the unit circle with complex affixes a, b, c, x, y, z respectively. The lines ZB, XC, YA and CY, AZ, BX bound the triangles $A'B'C'$ and $A''B''C''$. If A_1, B_1, C_1, A_2, B_2, C_2 are the circumcenters of the circles $(A'YC)$, $(B'ZA)$, $(C'XB)$, $(A''BZ)$, $(B''CX)$, $(C''AY)$, then the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.*

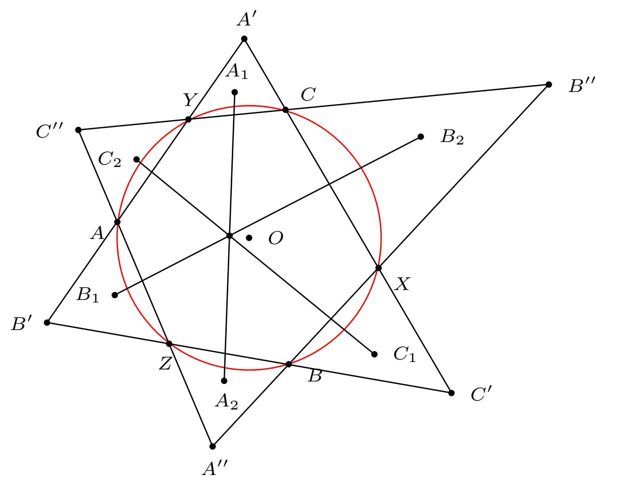

Figure 3

Proof. From Lemma 1, we have the affixes of the circumcenters:

$$a_1 = \frac{cy(x-a)}{cx-ay}, \qquad a_2 = \frac{bz(a-x)}{az-bx},$$
$$b_1 = \frac{az(y-b)}{ay-bz}, \qquad b_2 = \frac{cx(b-y)}{bx-cy},$$
$$c_1 = \frac{bx(z-c)}{bz-cx}, \qquad c_2 = \frac{ay(c-z)}{cy-az}.$$

For every point W on the line A_1A_2, the number $t = \frac{w-a_1}{w-a_2}$ is real. Therefore, $t = \frac{\overline{w}-\overline{a_1}}{\overline{w}-\overline{a_2}}$. This gives the equation of the line A_1A_2 as $\begin{vmatrix} w & \overline{w} & 1 \\ a_1 & \overline{a_1} & 1 \\ a_2 & \overline{a_2} & 1 \end{vmatrix} = 0$ (see [1]). Since a, b, c, x, y, z are unit complex numbers,

$$\overline{a_1} = \frac{\overline{cy}(\overline{x}-\overline{a})}{\overline{cx}-\overline{ay}} = \frac{\frac{1}{c}\cdot\frac{1}{y}\left(\frac{1}{x}-\frac{1}{a}\right)}{\frac{1}{c}\cdot\frac{1}{x}-\frac{1}{a}\cdot\frac{1}{y}} = \frac{x-a}{cx-ay}.$$

Similarly, $\overline{a_2} = \frac{a-x}{az-bx}$. From these we obtain the equation of the line A_1A_2, and likewise those of B_1B_2 and C_1C_2. These are

$$(az+cx-ay-bx)w + (bcxy+abyz-cayz-bczx)\overline{w} + (a-x)(cy-bz) = 0,$$
$$(bx+ay-bz-cy)w + (cayz+bczx-abzx-caxy)\overline{w} + (b-y)(az-cx) = 0,$$
$$(cy+bz-cx-az)w + (abzx+caxy-bcxy-abyz)\overline{w} + (c-z)(bx-ay) = 0.$$

The three lines are concurrent if and only if the determinant

$$\begin{vmatrix} az+cx-ay-bx & bcxy+abyz-cayz-bczx & (a-x)(cy-bz) \\ bx+ay-bz-cy & cayz+bczx-abzx-caxy & (b-y)(az-cx) \\ cy+bz-cx-az & abzx+caxy-bcxy-abyz & (c-z)(bx-ay) \end{vmatrix} = 0.$$

This clearly is true since each column sum is equal to 0. □

From the equations of the lines it is clear that the point of concurrency is O if and only if

$$(a-x)(cy-bz) = (b-y)(az-cx) = (c-z)(bx-ay) = 0.$$

Assume the points A, B, C, X, Y, Z distinct. This condition is satisfied precisely when the unit complex affixes satisfy $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. From this we conclude that XYZ is obtained from ABC by a rotation.

References

[1] T. Andreescu and D. Andrica, *Complex Numbers from A to ... Z*, Birkhäuser, 2014.
[2] A. Bogomolny, `http://www.cut-the knot.org/m/Geometry/AnotherSevenCircles.shtml`
[3] T. O. Dao, Advanced Plane Geometry, message 1531, August 28, 2014.
[4] N. Dergiades, Advanced Plane Geometry, message 1539, August 29, 2014.

Nikolaos Dergiades: I. Zanna 27, Thessaloniki 54643, Greece
E-mail address: ndergiades@yahoo.gr