

## Three Constructions of Archimedean Circles in an Arbelos

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**Abstract.** We give ruler and compass constructions of three Archimedean circles in an arbelos, each with the endpoints of a diameter on the smaller semicircles. In the first case, the diameter contains the intersection of the defining smaller semicircles of the arbelos. In the second case, these endpoints are the intersections of the smaller semicircles with the lines joining the endpoints of the base of the arbelos to a fixed point on the dividing perpendicular line. In the third case, the diameter containing these endpoints is parallel to the base line of the arbelos.

### 1. Introduction

We consider three constructions of Archimedean circles in an arbelos. Given a segment  $AB$  with an interior point  $C$ , the semicircles  $(O)$ ,  $(O_1)$ ,  $(O_2)$  with diameters  $AB$ ,  $AC$ ,  $CB$  on the same side of  $AB$  bound the arbelos, with dividing line  $CD$  perpendicular to  $AB$ . Let  $a$  and  $b$  be the radii of the semicircles  $(O_1)$  and  $(O_2)$ . Circles with radius  $t := \frac{ab}{a+b}$  are called Archimedean. They are congruent to the Archimedean twin circles [1, 2, 3]. We shall make use of the Archimedean circles with centers  $O_1$  and  $O_2$  respectively.

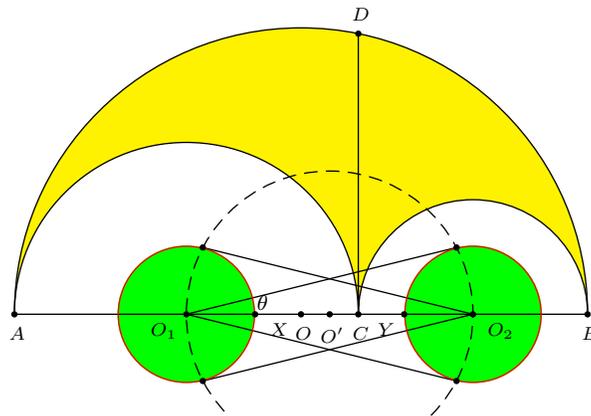


Figure 1.

In particular, we shall encounter below lines making an angle  $\theta$  with  $AB$  defined by

$$\sin \theta = \frac{t}{a+b}. \quad (1)$$

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Publication Date: September 25, 2014. Communicating Editor: Floor van Lamoen.

The author thanks Floor van Lamoen for his excellent suggestion of searching for the Archimedean circle in the last section of this paper.

Such lines are parallel to the tangents from  $O_1$  to the Archimedean circle with center  $O_2$  or vice versa. The points of tangency are the intersections with the circle with diameter  $O_1O_2$  and center  $O'$  (see Figure 1). We adopt a Cartesian coordinate system with origin at  $C$ , so that the points  $A$  and  $B$  have coordinates  $(-2a, 0)$  and  $(2b, 0)$  respectively. The equations of the circles  $(O_1)$  and  $(O_2)$  are

$$\begin{aligned} (x + a)^2 + y^2 &= a^2, \\ (x - b)^2 + y^2 &= b^2. \end{aligned}$$

**2. Archimedean circles with diameter through  $C$  and endpoints on  $(O_1)$  and  $(O_2)$**

Consider the construction of a line  $\mathcal{L}$  through  $C$  intersecting the circles  $(O_1)$  and  $(O_2)$  at  $A'$  and  $B'$  respectively so that the segment  $A'B'$  has length  $2t$ , and the circle with diameter  $A'B'$  is Archimedean. If  $\mathcal{L}$  has slope  $m$ , then these intersections are  $A' = \left(\frac{-2a}{1+m^2}, \frac{-2am}{1+m^2}\right)$  and  $B' = \left(\frac{2b}{1+m^2}, \frac{2bm}{1+m^2}\right)$ . Since the difference between the  $x$ -coordinates is  $\frac{2(a+b)}{1+m^2}$ ,  $A'B'^2 = \frac{4(a+b)^2}{1+m^2}$ . This is equal to  $(2t)^2$  if and only if

$$1 + m^2 = \frac{(a + b)^4}{a^2b^2} = \frac{(a + b)^2}{t^2} = \csc^2 \theta$$

for the angle  $\theta$  defined by (1). It follows that the slope  $m = \pm \cot \theta$ , and the line  $\mathcal{L}$  is perpendicular to a tangent from  $O_1$  to the Archimedean circle at  $O_2$ .

**Theorem 1.** *A line through  $C$  intersecting  $(O_1)$  and  $(O_2)$  at the endpoints of a diameter of an Archimedean circle if and only if it is perpendicular to a tangent from  $O_1$  to the Archimedean circle with center  $O_2$ .*

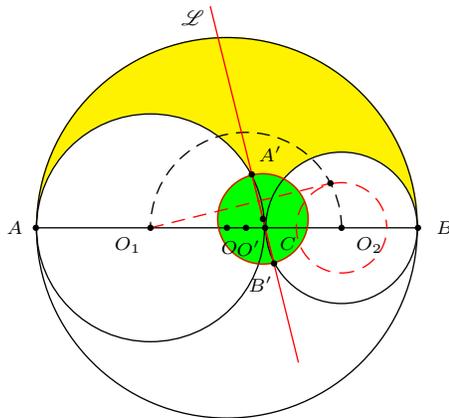


Figure 2(a)

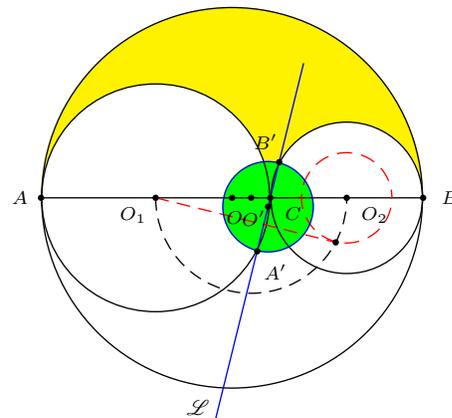


Figure 2(b)

**3. Archimedean circle from intersections of  $QA, QB$  with  $Q$  on  $CD$**

We construct a point  $Q$  on the line  $CD$  such that the intersections of  $AQ$  with  $(O_1)$  and  $BQ$  with  $(O_2)$  are the endpoints of a diameter of an Archimedean circle (see Figure 3). Let  $Q = (0, q)$ . These intersections are  $A'' = \left(\frac{-2aq^2}{q^2+4a^2}, \frac{4a^2q}{q^2+4a^2}\right)$  and  $B'' = \left(\frac{2bq^2}{q^2+4b^2}, \frac{4b^2q}{q^2+4b^2}\right)$ . From these coordinates,

$$A''B''^2 = \frac{4(a+b)^2q^4}{(q^2+4a^2)(q^2+4b^2)}.$$

This is equal to  $(2t)^2$  if and only if  $(a+b)^4q^4 - a^2b^2(q^2+4a^2)(q^2+4b^2) = 0$ . Rewriting this as

$$((a+b)^4 - a^2b^2)q^4 - 4a^2b^2(a^2+b^2)q^2 - 16a^4b^4 = 0, \tag{2}$$

we see that there is a unique positive root.

**Theorem 2.** *There is a unique point  $Q$  on the dividing line  $CD$  such that the intersections of  $QA$  with  $(O_1)$  and  $QB$  with  $(O_2)$  are the endpoints of a diameter of an Archimedean circle.*

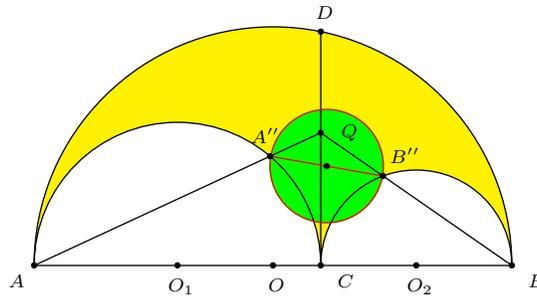


Figure 3.

From (2), we obtain explicitly

$$q^2 = \frac{a^2b^2}{(a+b)^4 - a^2b^2} (2(a^2+b^2) + 2(a+b)\sqrt{(a-b)^2 + 4(a+b)^2}).$$

Now,  $\frac{a^2b^2}{(a+b)^4 - a^2b^2} = \frac{t^2}{(a+b)^2 - t^2} = \tan^2 \theta$  for  $\theta$  defined by (1). It is enough to construct a segment  $CX$  on  $AB$  with

$$CX^2 = 2(a^2+b^2) + 2(a+b)\sqrt{(a-b)^2 + 4(a+b)^2}. \tag{3}$$

Let  $M$  be the “highest” point of  $(O)$ , i.e., the intersection of  $(O)$  with the perpendicular to  $AB$  at  $O$  (see Figure 4). For the construction of  $X$ , we make use of the following.

- (i)  $2(a^2+b^2) = (a+b)^2 + (a-b)^2 = OM^2 + OC^2 = CM^2$ ,
- (ii)  $(a-b)^2 + 4(a+b)^2 = 4(O'O^2 + OM^2) = 4 \cdot O'M^2$ .

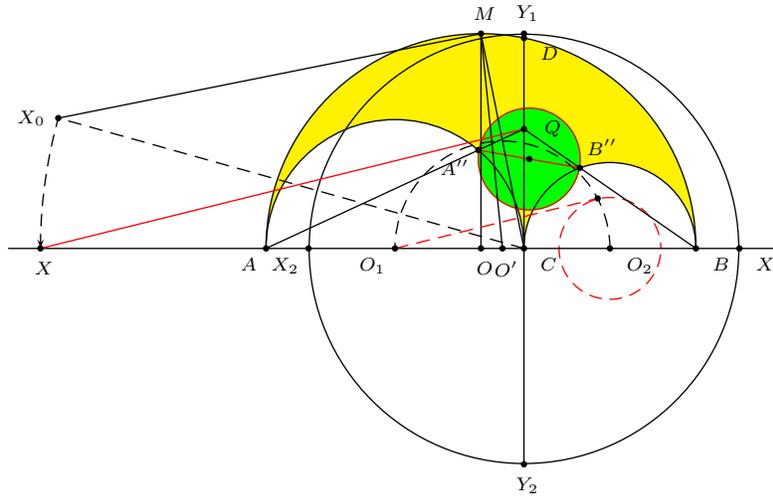


Figure 4.

**Construction.** (1) On different sides of  $AB$  on the perpendicular at  $C$ , choose points  $Y_1$  and  $Y_2$  such that  $CY_1 = OM$  and  $CY_2 = O'M$ . Construct the circle with diameter  $Y_1Y_2$ , to intersect the line  $AB$  in a segment  $X_1X_2$ .

(2) On the perpendicular to  $MC$  at  $M$ , choose a point  $X_0$  such that  $MX_0 = X_1X_2$ .

(3) Let  $X$  be a point on  $AB$  such that  $CX = CX_0$ . The segment  $CX$  has length given by (3) above.

(4) Construct a parallel through  $X$  to a tangent from  $O_1$  to the Archimedean circle with center  $O_2$ , to intersect  $CD$  at  $Q$ . This is the unique point  $Q$  in Theorem 2.

**4. Archimedean circle with a diameter parallel to  $AB$  and endpoints on  $(O_1)$  and  $(O_2)$**

We consider the possibility of an Archimedean circle with a diameter parallel to  $AB$  having its endpoints one on each of the semicircles  $(O_1)$  and  $(O_2)$ . If the diameter is at a distance  $d$  from  $AB$ , its endpoints are among the points

$$\begin{aligned} X_- &= (-a - \sqrt{a^2 - d^2}, d), & X_+ &= (-a + \sqrt{a^2 - d^2}, d); \\ Y_- &= (b - \sqrt{b^2 - d^2}, d), & Y_+ &= (b + \sqrt{b^2 - d^2}, d). \end{aligned}$$

The differences between the lengths of the various segments and the diameter of an Archimedean circle are

$$\begin{aligned} X_-Y_+ - 2t &= a + b - 2t + \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2}, \\ X_-Y_- - 2t &= a + b - 2t + \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2}, \\ X_+Y_+ - 2t &= a + b - 2t - \sqrt{a^2 - d^2} + \sqrt{b^2 - d^2}, \\ X_+Y_- - 2t &= a + b - 2t - \sqrt{a^2 - d^2} - \sqrt{b^2 - d^2}. \end{aligned}$$

The condition

$$(X_-Y_+ - 2t)(X_-Y_- - 2t)(X_+Y_+ - 2t)(X_+Y_- - 2t) = 0$$

simplifies into

$$4t(a - t)(b - t)(a + b - t) - (a + b - 2t)^2d^2 = 0. \tag{4}$$

This clearly has a unique positive root  $d$ .

**Theorem 3.** *There is a unique Archimedean circle with a diameter parallel to  $AB$ , having endpoints one on each of the semicircles  $(O_1)$  and  $(O_2)$ .*

Now, by Heron’s formula,  $t(a - t)(b - t)(a + b - t)$  is the square of the area of a triangle with sides  $a$ ,  $b$ , and  $a + b - 2t$ . From (4),  $d$  is the altitude of the triangle on the side  $a + b - 2t$ . This leads to the following simple construction.

**Construction.** Let the Archimedean circle with center  $O_1$  intersect  $O_1C$  at  $X$  and that with center  $O_2$  intersect  $CO_2$  at  $Y$ . Construct a point  $Z$  (on the same side of the arbelos) such that  $XZ = a$  and  $YZ = b$ . The parallel to  $AB$  through  $Z$  is the line which intersects  $(O_1)$  and  $(O_2)$  at two points at a distance  $2t$  apart.

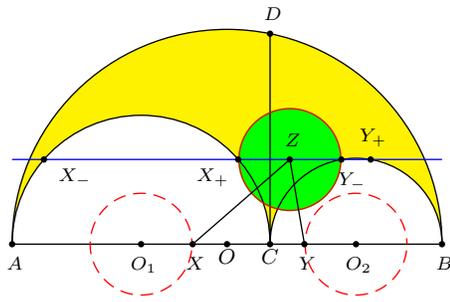


Figure 5a:  $a^3 < a^2b + ab^2 + b^3$

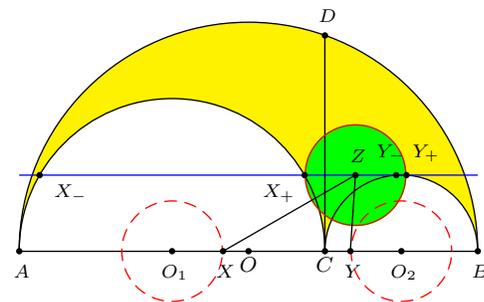


Figure 5b:  $a^3 > a^2b + ab^2 + b^3$

The point  $Z$  is indeed the center of the Archimedean circle in question.

Assume  $a \geq b$  without loss of generality. Note that  $O_1X_+ = XZ = a$ , and they are parallel since

$$\sin X_+O_1C = \frac{d}{a} = \sin ZXY.$$

This means that  $O_1XZX_+$  is a parallelogram, and  $ZX_+ = XO_1 = t$ . The circle, center  $Z$ , passing through  $X_+$  is Archimedean. The other end of the the diameter is  $Y_-$  or  $Y_+$  according as  $a^3$  is less than or greater than  $a^2b + ab^2 + b^3$  (see Figures 5a and 5b). This follows from the simple fact

$$\sqrt{b^2 - d^2} = \begin{cases} \frac{-a^3b + a^2b^2 + ab^3 + b^4}{(a+b)(a^2+b^2)}, & \text{if } a^3 < a^2b + ab^2 + b^3, \\ \frac{a^3b - a^2b^2 - ab^3 - b^4}{(a+b)(a^2+b^2)}, & \text{if } a^3 > a^2b + ab^2 + b^3. \end{cases}$$

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