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A Gallery of Conics by Five Elements

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Abstract. This paper is a review on conics defined by five elements, i.e., either lines to which the conic is tangent or points through which the conic passes. The gallery contains all cases combining a number (n) of points and a number $(5-n)$ of lines and also allowing coincidences of some points with some lines. Points and/or lines at infinity are handled as particular cases.

1. Introduction

In the following we review the construction of conics by five elements: points and lines, briefly denoted by $(\alpha P \beta T)$, with $\alpha + \beta = 5$. In these it is required to construct a conic passing through α given points and tangent to β given lines. The six constructions, resulting by giving α, β the values 0 to 5 and considering the data to be in general position, are considered the most important ([18, p. 387]), and are to be found almost on every book about conics. It seems that constructions for which some coincidences are allowed have attracted less attention, though they are related to many interesting theorems of the geometry of conics. Adding to the six main cases those with the projectively different possible coincidences we land to 12 main constructions figuring on the first column of the classifying table below. The six added cases can be considered as limiting cases of the others, in which a point tends to coincide with another or with a line. The twelve main cases are the projectively inequivalent to which every other case can be reduced by means of a projectivity of the plane. There are, though, interesting classical theorems for particular euclidean inequivalent cases worth studying, as, for example, the much studied case of parabolas tangent to four lines (case $(0P5T_1)$ in §7.2). In this review the frame is that of euclidean geometry and consequently a further distinction of *ordinary* from points and lines *at infinity* is taken into account. All of the (50) constructions classified below are known and can be found in the one or the other book on conics (e.g. [7, pp. 136]), but nowhere come to discussion all together, so far I know. In a few cases (e.g. §11.1, §3.2) I have added a proof, which seems to me interesting and have not found elsewhere.

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	1	2	3	4	5	6
2	$(5P0T)$	$(5P_10T)$	$(5P_20T)$			
3	$(4P1T)$	$(4P1T_1)$	$(4P_11T)$	$(4P_21T)$		
4	$(3P2T)$	$(3P2T_1)$	$(3P_12T)$	$(3P_22T)$		
5	$(2P3T)$	$(2P3T_1)$	$(2P_13T)$	$(2P_23T)$		
6	$(1P4T)$	$(1P4T_1)$	$(1P_14T)$			
7	$(0P5T)$	$(0P5T_1)$				
8	$(4P1T)_1$	$(4P_11T_1)$	$(4P_11T)_1$	$(4P_11T)_i$	$(4P_21T)_1$	$(4P_21T)_i$
9	$(3P2T)_1$	$(3P2T_1)_1$	$(3P_12T_1)$	$(3P_12T)_i$	$(3P_22T)_1$	$(3P_22T)_i$
10	$(3P2T)_2$	$(3P_12T_1)_1$	$(3P_12T)_{1i}$	$(3P_22T)_{2i}$		
11	$(2P3T)_1$	$(2P3T_1)_1$	$(2P_13T_1)$	$(2P_13T)_i$	$(2P_23T)_i$	
12	$(2P3T)_2$	$(2P3T_1)_2$	$(2P_13T_1)_1$	$(2P_13T)_{1i}$	$(2P_23T)_{2i}$	
13	$(1P4T)_1$	$(1P4T_1)_1$	$(1P_14T_1)$	$(1P_14T)_i$		

The above table serves as the table of contents of this paper, the row labels are the section numbers and the column labels the subsection numbers. The column with label 1 lists the symbols of the twelve projectively inequivalent cases. Each row of the table comprises the cases, which are projectively equivalent to the one of the first column. The notation used is a slight modification of the one introduced by Chasles ([3, p. 304]). The symbol P_n means that n of the given points are at infinity and T_1 means that one of the tangent lines is the line at infinity, later meaning that the conic, to be constructed, is a parabola. The indices, which adhere to the right parenthesis are optional. When absent, it means that the configuration is in *general position*, i.e. none of the ordinary points is coincident with an ordinary line. When present it means that one or two of the points are correspondingly coincident with one or two tangents. The indices $i, 2i$ mean that one/two ordinary lines are correspondingly coincident with one/two points at infinity. The index $1i$ means that an ordinary point coincides with a line and also a point at infinity coincides with the point at infinity of an ordinary line.

Except for the coincidence suggested by the corresponding symbol, the other data are assumed to be in general positions in the projective sense. For example, the symbol $(3P_12T)_i$ stands for the construction of a hyperbola given two points, an asymptote and a tangent. These four elements are assumed in general position, implying that no further coincidences are present, that the intersection of the asymptote and the tangent are not collinear to the two points, that the line of the two points is not parallel to the tangent or the asymptote, etc. The statements on the number of solutions or non existence in each case, presuppose such a restriction. Throughout the text, *existence* is meant in the *real* plane.

The following notation is also used: t_X denotes the tangent at X, $Y = X(A, B)$ denotes the harmonic conjugate of X with respect to (A, B), $X = (a, b)$ denotes the intersection of lines a, b. Points at infinity are occasionally denoted by $[A]$, the same symbol indicating also the direction determined by that point at infinity. For a line e the symbol $[e]$ denotes its point at infinity. For a point at infinity A the symbol XA denotes the parallel from X to the direction determined by A.

Degenerate conics, consisting of a product of two lines a, b, are represented by $a \cdot b$. Regarding hyperbolas, *asymptotics* are distinguished from *asymptotes*. The first term gives the direction only, the second denotes the precise line.

Regarding the organization of the material, there follow three preliminary sections on the background, which comprise: (a) Involutions (§1.1), (b) Pencils and Families of conics (§1.2), (c) The great theorems (§1.3). Then follow fifty sections handling the inequivalent euclidean constructions. The sections are divided in twelve groups, each group headed by the problem to which all other of the group are projectively equivalent.

1.1. *Involutions.* *Involutions* are homographies of projective lines with the property $f^2 = I \;\Leftrightarrow\; f^{-1} = f$. Using proper coordinates, involutions are described by functions of the form

$$y = \frac{ax+b}{cx-a} \quad \Leftrightarrow \quad x = \frac{ay+b}{cy-a},$$

whose graphs are rectangular hyperbolas symmetric with respect to the diagonal line $y = x$. Such functions are completely determined by prescribing their values

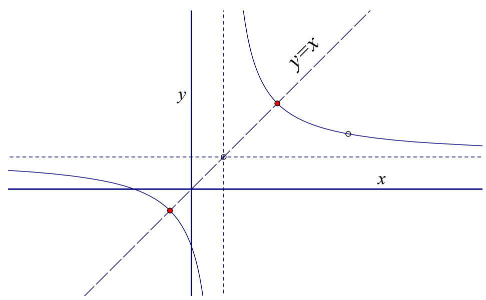

Figure 1. Graph of an involution with existing fixed points

on two elements $(X, f(X))$, $(Y, f(Y))$ and they have either two fixed points or none. Figure 1 shows a case in which there are two fixed points. When the hyperbola has the two branches totally contained in the two sides of the line $y = x$, the corresponding function has no fixed points. An important property of a pair $(X, f(X))$, of related points of an involution, frequently used below, is that it consists of harmonic conjugates with respect to the fixed points of f, when these exist ([7, p. 100], [22, I, p.102], [17, p. 167], [9, p. 35], [2, I, p. 137]).

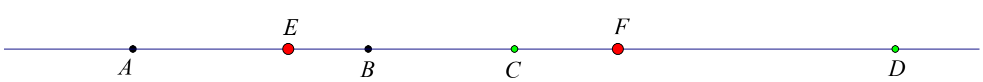

Figure 2. Common harmonics (E, F) of (A, B) and (C, D)

The practical issue of finding the fixed points of involutions is related to the idea of *common harmonics* of two pairs (A, B), (C, D) of collinear points. This is another pair (E, F) of points, which are, as the name suggests, simultaneously harmonic conjugate with respect to (A, B) and with respect to (C, D) (see Figure 2). If such a pair (E, F) exists, then it is easily seen that every circle d passing

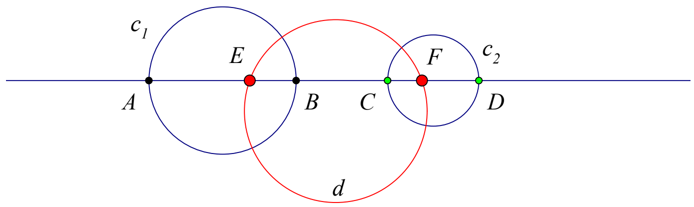

Figure 3. Construction of the common harmonics (E, F) of (A, B) and (C, D)

through E, F is orthogonal to the circles c_1, c_2 with corresponding diameters AB and CD (see Figure 3). Thus, in order to find the common harmonics geometrically, set two circles c_1, c_2 on diameters, respectively, AB and CD and draw a circle d simultaneously orthogonal to c_1 and c_2 (see Figure 4). In case one of the

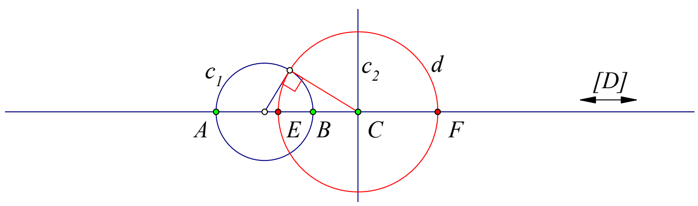

Figure 4. Common harmonics (E, F) of (A, B) and $(C, [D])$

points, D say, is a point at infinity then c_2 is the line orthogonal to AB at C and we can take the circle d to be the one centered at C and orthogonal to c_1, points E, F lying then symmetric with respect to C (see Figure 4).

The common harmonics are precisely the *limiting points* of the *coaxal system* (pencil) of circles determined by c_1 and c_2 ([16, p. 118]). They exist, precisely when the circles c_1, c_2 are non-intersecting.

1.2. *Pencils of conics.* Pencils of conics are *lines* in the five-dimensional projective space of conics. This is reflected in the generation of a pencil as the set $\mathcal{D}$ of linear combinations $c = \alpha \cdot c_1 + \beta \cdot c_2$, where $c_1 = 0$, $c_2 = 0$ are the equations of any two particular members of the pencil. Then $c = 0$ represents the equation of the general member of the pencil for arbitrary real values of α, β with $|\alpha| + |\beta| \neq 0$. The basic pencil, called of *type-I*, is that of all conics passing through four points (see Figure 5). There are five projectively inequivalent pencils, characterized by the fact that all their members intersect by two at the same points, real or imaginary, with the same multiplicities. These are referred to as types I to V pencils ([22, I, p. 128]) and can be considered as limiting cases of the type I pencil. Type II, for instance, results by fixing line $e = AD$ and letting D coincide with A. The resulting pencil

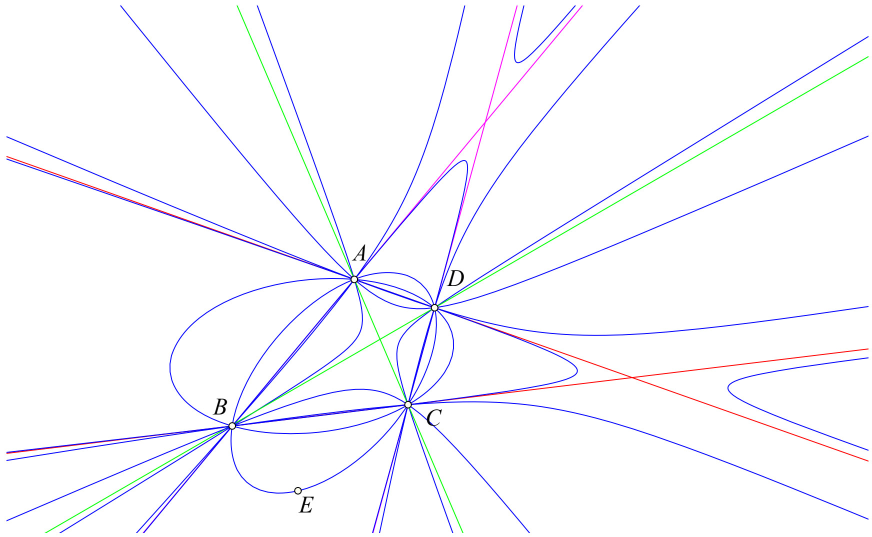

Figure 5. The pencil of conics through A, B, C, D

(seen in §9.1) consists of all conics passing through A, B, C and tangent to e at A. Another type of pencil is obtained from a type I pencil by fixing lines $a = AB$ and $c = CD$ and letting point B converge along a to A and D converge along c to C. The resulting pencil, referred to as type IV pencil (seen in §10.1), consists of all conics tangent to lines a, c correspondingly at their points A and C.

Pencils of conics contain *degenerate members*, at most three ([2, II, p. 124]). In Figure 5 the degenerate members are visible, consisting of the pairs of lines $AB \cdot CD$, $AD \cdot BC$ and $AC \cdot BD$. In the analytic description of pencils $c = \alpha \cdot c_1 + \beta \cdot c_2$, the conics c_1, c_2 can be degenerate members, and this is often convenient and extensively used below.

To every type of pencil corresponds an analogous *range* of conics or *tangential* pencil of conics ([2, II, p. 199], with a notation slightly different from that of Veblen). For example to type I pencils corresponds the range of type I^* of conics, which are tangent to four lines in general positions (see §7.1). Ranges are pencils of conics in the dual projective plane P^* consisting of all lines of the projective plane P. To each pencil of type X corresponds its dual range X^* with properties resulting from those of X by duality.

A particular property of pencils, together with its dual for ranges, is of interest for our subject. For instance, in the case of a pencil $\mathcal{D}$ of conics through points A, B, C, D, it is known ([2, II, p. 197]) that the polars of a fixed point X with respect to all members of the pencil pass through a point Y. This defines a quadratic transformation $Y = f(X)$, which in the coordinates with respect to the projective base $\{E, F, G, A\}$, with $E = (AB, CD)$, $F = (AD, BC)$, $G = (AC, BD)$, is represented by

$$x' = \frac{1}{x}, \quad y' = \frac{1}{y}, \quad z' = \frac{1}{z}.$$

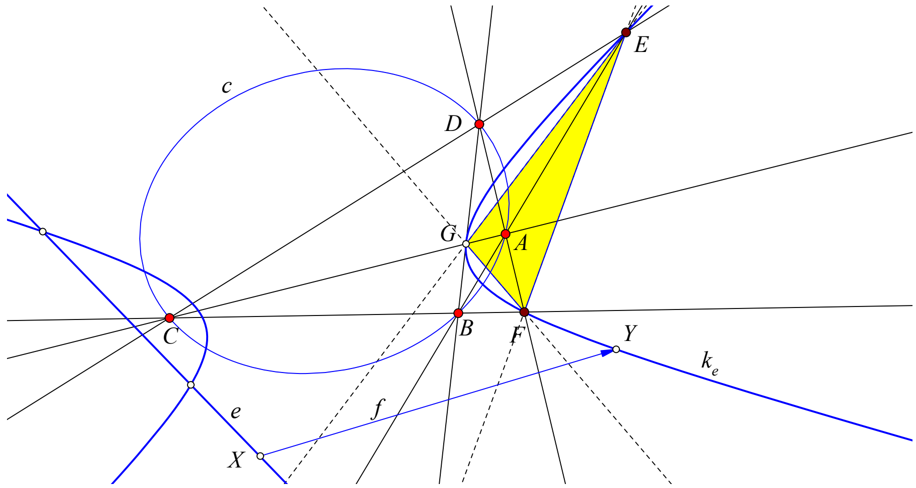

Figure 6. The eleven points conic k_e of A, B, C, D and line e

The image of a line e under this transformation is a conic k_e circumscribing triangle EFG and passing through eight additional points, therefore called an *eleven point conic* ([1, p. 97], [9, p. 66]). Six of the points are the harmonic conjugates $W = V(X, Y)$ of the intersection point $V = (XY, e)$, where X, Y are taken from $\{A, B, C, D\}$. The two remaining points, if real, are the intersection points of k_e with line e and simultaneously the contact points of two members of the pencil $\mathcal{D}$, which are tangent to e (a case handled in §8.1).

The dual to the previous property relates to the range $\mathcal{D}^*$ of conics k tangent to four lines a, b, c, d (see Figure 7). According to this, the poles of a line h with respect to the members of $\mathcal{D}^*$ lie on a line h' and the transformation $h' = F^*(h)$ is a quadratic one of the same nature as the previous one, differing only in that it operates on the dual projective plane P^*. Line h' can be found by a simple criterion, resulting by considering the triangle of diagonals (efg in Figure 7). Lines h, h' intersect each side s of this triangle at points X, Y, which are harmonic conjugate with respect to (U, V), where U, V are the vertices of the quadrilateral lying on s. The images h' under F^* of all lines h passing through a fixed point Q are the tangents of a conic k_Q inscribed in the triangle efg and tangent to eight additional lines, therefore called an *eleven tangents conic*. Six of these lines are the harmonic conjugates $Q(s, s')$ of Q with respect to all pairs (s, s') taken from $\{a, b, c, d\}$. The two remaining tangents, if real, are the tangents through Q of the members of $\mathcal{D}^*$ passing through Q (a case handled in §6.1).

Roughly described, a standard method of constructing a conic by five elements is to find a pencil or range satisfying four of the given conditions, and then use the fifth condition to locate the particular member(s) of the pencil satisfying it. In the case of type I pencils, any fifth point E, different from A, B, C, D, determines exactly one conic of the pencil containing it (a case handled in §2.1).

Pencils of conics, passing through two different points Q, R can be transformed to pencils of circles by a complex projective map, which sends Q, R to the *circular*

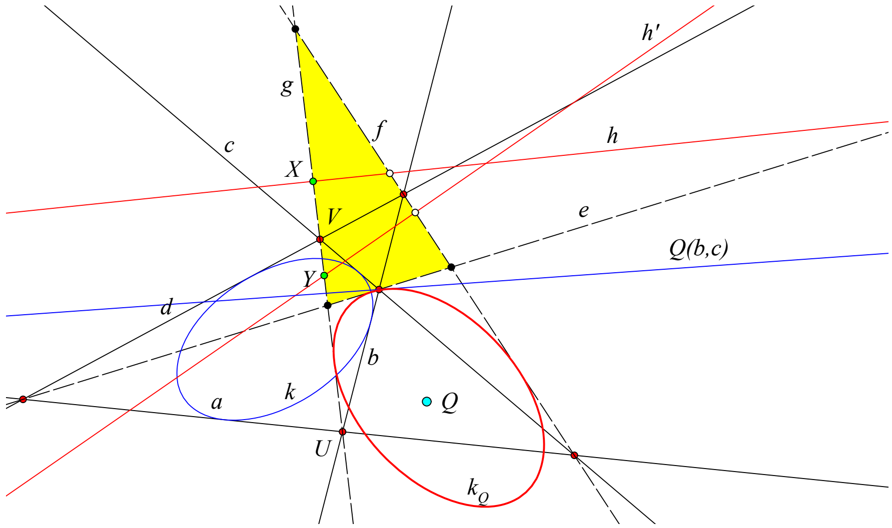

Figure 7. The eleven tangents conic k_Q of a, b, c, d and point Q

points at infinity I, J ([9, p. 71]). By such a map a type I pencil can be transformed to a pencil of intersecting circles and all related conic construction problems reduce to corresponding Apollonius circle construction problems ([6]). This method is concisely expounded in [10].

1.3. *The great theorems.* The basic tool in the context of the present subject is Desargues' theorem, for various types of pencils and ranges, as neatly described in [22, p. 128]. The theorem asserts that a pencil intersects on a fixed line e, through its members, pairs of points (X, Y) *in involution*, later meaning, that there is an involution f on e, such that $Y = f(X) \Leftrightarrow X = f(Y)$. The interesting fact is that f is completely determined by the intersections of e with the *degenerate members* of the pencil, which are products of lines. Line e is assumed to be different from the lines contained in degenerate members of the pencil.

In its dual form, Desargues' theorem, referred to also as Plücker's theorem ([4, p. 25], [2, II, p. 202]) states, that the pairs of tangents (x, y) from a fixed point Q to the members of a range, are *in involution*, later meaning, that there is an involution f^* on the pencil Q^* of lines through Q, such that $y = f^*(x) \Leftrightarrow x = f^*(y)$. This involution is determined again by the *degenerate members* of the range, which are pairs of points. In the case of ranges of conics tangent to four lines, for instance, the degenerate members are pairs of intersection points of the four lines and the involution on Q^* is determined by two pairs of lines joining Q to two such pairs of points ([22, I, p. 129], [9, p. 50], [2, II, p. 197]). Point Q is assumed to be different from the points of degenerate members.

The somewhat difficult to visualize involution on Q^* can be represented by intersecting the rays through Q with a fixed line $e \not\ni Q$. In this way the involution f^* on Q^* defines an involution f on e and the fixed points (rays) of f^* trace on e the fixed points of f. Quite typically for our subject, the requested conics are intimately related to the fixed elements of such involutions ([20, p. 69], [19, p. 365], [2, p. 198, II]).

Pascal's theorem, that the opposite sides of a hexagon, inscribed in a conic, intersect on a line, is used in the present context in order to find additional points on the requested conics. The theorem is used also in its various versions for inscribed pentagons, quadrangles and triangles ([22, I, p. 111], [17, p. 156], [2, II, p. 176]).

Brianchon's theorem, which is dual to Pascal's, asserts that the lines through opposite vertices of a circumscribed to a conic hexagon pass through a fixed point. Again the theorem and its versions for pentagons, quadrangles and triangles is used in order to find additional points on the requested conics.

2. Five points

2.1. *Conic through five points* $(5P0T)$. Construct a conic passing through five points A, B, C, D, E. This is the basic construction, to which, all other constructions may be reduced. Analytically this can be done easily by considering the equations of two line-pairs defined by the five points ([18, p. 232]). Let, for example, the line-pairs $(AB, CD), (AC, BD)$ be given correspondingly by equations $(f = 0, g = 0), (h = 0, j = 0)$. Then the equation

$$k_{\lambda,\mu} = \lambda \cdot (f \cdot g) + \mu \cdot (h \cdot j) = 0,$$

for variable λ and μ, represents the pencil $\mathcal{D}$ of all conics passing through A, B, C, D. The requirement, for such a conic, to pass through E, leads to an equation for λ, μ:

$$k_{\lambda,\mu}(E) = 0,$$

from which λ, μ are determined up to a multiplicative constant, and through this a unique conic is defined as required.

Geometrically, one can use *Pascal's theorem* to produce, from the given five, arbitrary many other points P lying on the conic. Figure 8 illustrates the way this

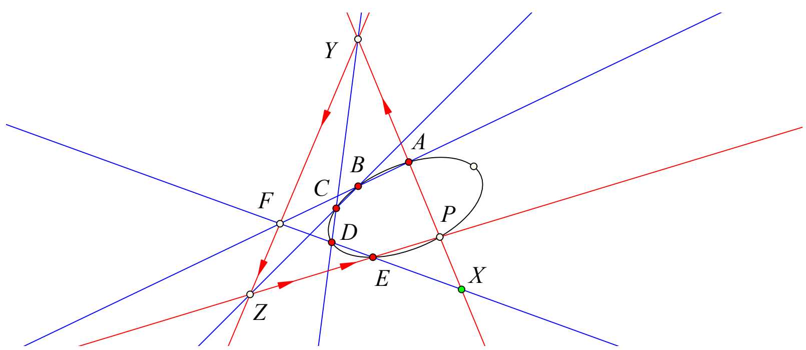

Figure 8. Pascal's theorem producing more points on the conic

is done. Start with an arbitrary point X on line DE and define the intersection point $Y = (CD, AX)$. Join this point to the intersection point $F = (AB, DE)$ and extend the line to find the intersection point $Z = (YF, BC)$. By Pascal's theorem, the intersection point $P = (ZE, AX)$ is on the conic passing through A, B, C, D and E. The books of Russell [17, p. 229] and, for more cases Yiu [24, p. 144], contain many useful constructions, which determine various elements of the conic, such as the intersections with a line, the center, the axes, the foci etc. out of the five given points.

2.2. *Conic through five points, one at infinity* $(5P_10T)$. Construct a conic passing through five points $A, B, C, D, [E]$ in general position. The conic is a hyperbola and in some cases a parabola. Additional points can be constructed as in the

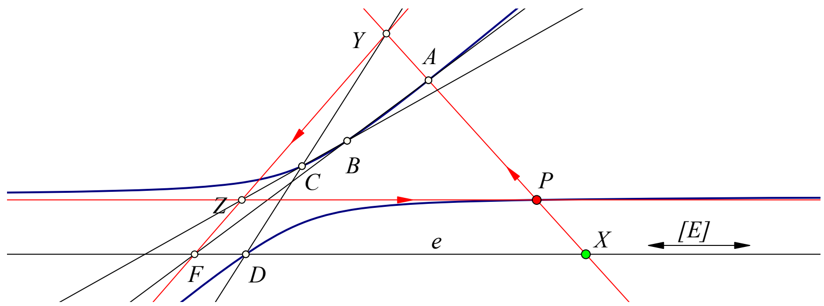

Figure 9. Finding additional points on the conic through $A, B, C, D, [E]$

previous section. Start with a point X on line $e = DE$, find the intersections $Y = (AX, CD)$, and $Z = (FY, BC)$, where $F = (AB, DE)$ (see Figure 9). Point $P = (ZE, XA)$ is on the requested conic, which can be constructed to pass through the five points A, B, C, D, P. There is always a unique solution.

Remark. In general the conic is a hyperbola, and $[E]$ represents the direction of one of its asymptotes. Fixing points A, B, C, D, there are either none or two directions $[E]$, determined by the four points, for which the conic passing through $A, B, C, D, [E]$ is a parabola. This is the case $(4P1T_1)$ of §3.2.

2.3. *Hyperbola from asymptotics and three points* $(5P_20T)$. Construct a conic passing through five points $A, B, C, [D], [E]$, thus a hyperbola with asymptotic directions given by $[D], [E]$. The pentagon $ABCDE$ is infinite with DE the line

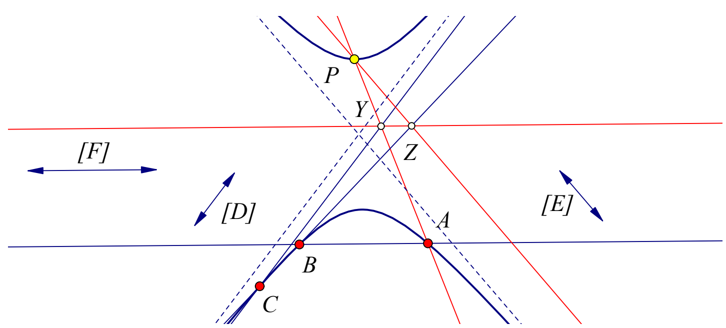

Figure 10. Pascal's theorem with D, E at infinity

at infinity and its intersection F with AB is also at infinity. An arbitrary ray of the pencil A^* of lines through A, and its intersection Y with CD defines the point $Z = (YF, BC)$ and $P = (ZE, AY)$. Last is a point on the requested conic (see Figure 10). There is always one solution. Figure 11 shows a related pencil consist-

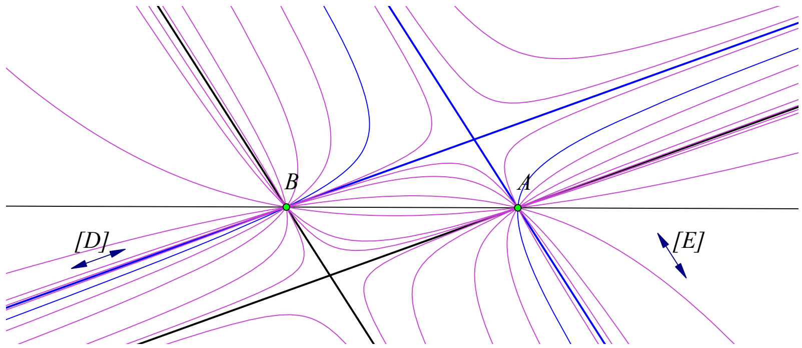

Figure 11. The pencil of conics passing through $A, B, [D], [E]$

ing of all conics passing through $A, B, [D], [E]$, i.e. all hyperbolas with asymptotic directions $[D], [E]$ and passing through the points A, B.

3. Four points and one tangent

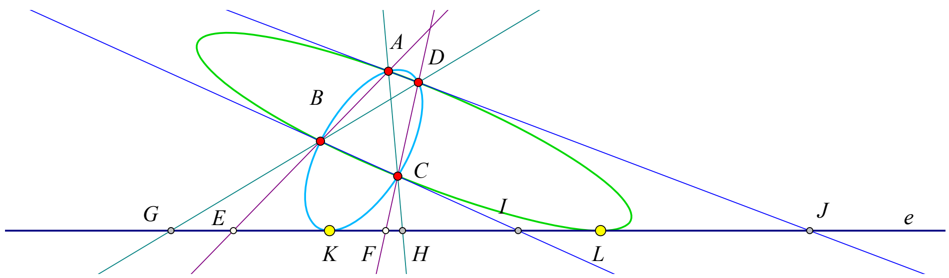

Figure 12. The two conics through A, B, C, D tangent to line e

3.1. *Conic by four points and one tangent* $(4P1T)$. Construct a conic passing through four points A, B, C, D and tangent to one line e. By Desargues' theorem (see §1.3), each member of the pencil $\mathcal{D}$ of conics through A, B, C, D (seen in §1.1) intersects line e to a pair of points in involution. The contact points K, L of the requested conic with e are the fixed points of this involution. Two pairs of points, defining the involution, are the intersections of e with two degenerate members c_1, c_2 of the pencil $\mathcal{D}$, consisting of the pairs of lines $c_1 = AB \cdot CD$ and $c_2 = AD \cdot BC$, intersecting the line respectively in (E, F) and (I, J) (see Figure 12). The fixed points of the involution are the common harmonics (K, L) of the pairs $(E, F), (I, J)$.

An alternative construction for this case relates to the *eleven points conic* of four points and a line (see §1.2), consisting of the poles of line e with respect to all conics of the pencil $\mathcal{D}$. This conic intersects line e precisely at its contact points

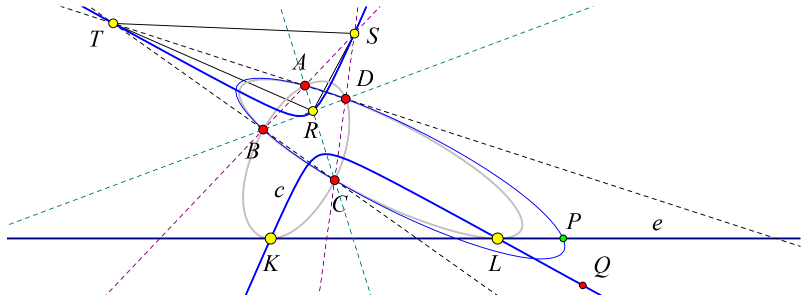

Figure 13. Eleven point conic of $ABCD$ and line e

K, L with the requested conics.

Next two figures display the domains of existence of solutions for variable D, assuming given the positions of A, B, C and line e.

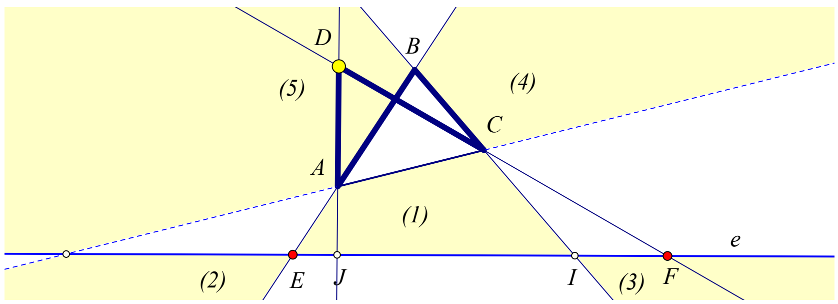

Figure 14. Domains of existence for variable D, e non-intersecting ABC

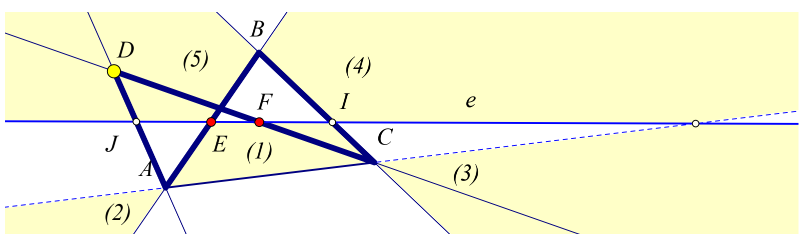

Figure 15. Domains of existence for variable D, e intersecting ABC

3.2. *Parabola through four points* $(4P1T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, passing through four points A, B, C, D. The only difference from the previous case is that line e is now at infinity. The involution on e can be represented on the pencil O^* of lines through the arbitrary but fixed point O ([4, p. 40], [20, p. 69], [17, p. 180]). In fact, draw from O parallels to the lines joining all possible pairs of the four points A, B, C, D. They result in three

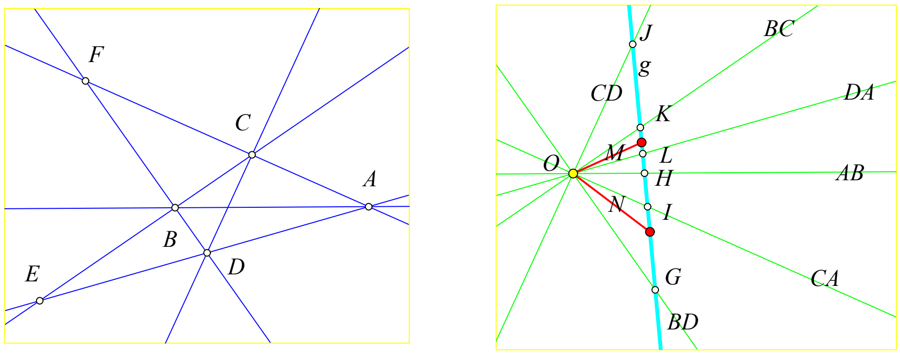

Figure 16. Common harmonic directions

pairs of lines $(BD, CA), (AB, CD), (DA, BC)$, which are *in involution*. This, by intersecting the pencil with an arbitrary line g defines an involution in g (see Figure 16). The corresponding pairs of points $(G, H), (I, J), (K, L)$ of g are in

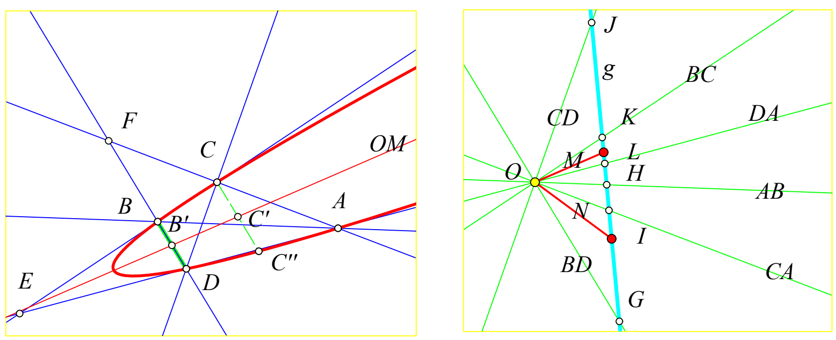

Figure 17. Parabola through A, B, C, D with axis-direction OM

involution. The common harmonics M, N of these pairs, if they exist, define two directions OM and ON, i.e. two points at infinity, which represent the directions of the axis of the requested parabolas, passing through A, B, C, D. Thus, there are either two or none parabola passing through four points A, B, C, D in general position. Figure 17 shows how the construction of one of these two parabolas can be done. Use is made of one of the chords BD of the requested parabola. From the middle B' of BD we draw a parallel h to the direction OM. Then we project C on C' and extend CC' to its double CC''. The projection is by parallels to BD. By a well known property of parabolas, point C'' will also belong to the parabola under construction. Thus, taking C'' as the fifth point we define the parabola as the conic passing through A, B, C, D and C''. An analogous construction can be carried out for the second parabola, whose axis is parallel to the direction ON. The parabolas exist if none of the four given points is contained in the triangle of the other three.

Besides this construction of the two parabolas, which is considered the standard one, there is another approaching the problem from a different point of view. For

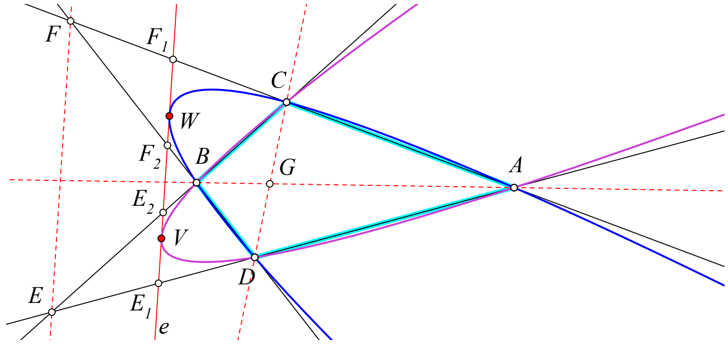

Figure 18. The two parabolas through A, B, C, D

this, consider the line e parallel to the diagonal EF of the quadrangle $ACBD$ at half the distance of $G = (AB, CD)$ from EF (see Figure 18). This line is *the common tangent* to the two requested parabolas. This follows, for example, by considering the conic passing through the eight contact points of the common tangents of two conics ([18, p. 345]) or the properties of the so-called *harmonic locus* of two conics, specialized for two parabolas ([14, II, p. 121]). The contact points V, W of the two parabolas with e are the common harmonics of the point-pairs $(E_1, E_2), (F_1, F_2)$, where $E_1 = (AD, e), E_2 = (BC, e), F_1 = (AC, e), F_2 = (BD, e)$. Once V, W are constructed, the parabolas can be defined to pass through the corresponding fivetuples of points.

Notice that every conic of the pencil $\mathcal{D}$ has a pair of conjugate diameters parallel to the axes of the two parabolas ([19, p. 292]).

3.3. *Conic by 3 points, 1 at infinity, 1 tangent* $(4P_1 1T)$. Construct a conic passing through four points $A, B, C, [D]$, and tangent to line e. The conic is either a hyperbola with one asymptotic direction $[D]$ or a parabola with axis parallel to $[D]$. Using the method of §3.1 we construct the fixed points D_1, D_2 of the involution,

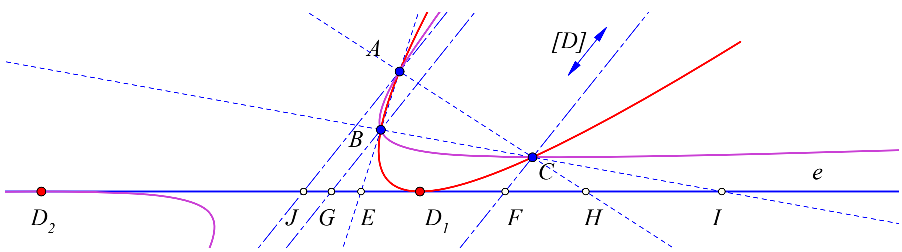

Figure 19. The two conics through $A, B, C, [D]$ tangent to e

defined on line e by its intersections with the line pairs (AB, CD), (AC, BD), (AD, BC). The common harmonics D_1, D_2 of the point-pairs $(E, F), (G, H)$ are the contact points with line e (see Figure 19). Adding one of the D_i to the three points A, B, C we can, as in §8.1, construct a fifth point and pass a conic through the five points. Fixing A, B, C and the position of line e, there are some directions

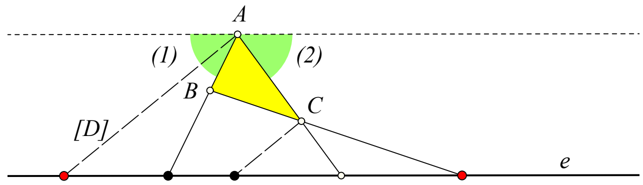

Figure 20. Angular domains of existence

$[D]$ for which no solutions exist. Figure 20 shows a case, in which e does not intersect the interior of ABC and the two angular domains for the direction AD for which there are solutions of the problem.

In general, the conics are two hyperbolas with one asymptotic direction determined by the point at infinity D, or a pair of a hyperbola, as before, and a parabola with axis direction $[D]$. If the line e does not intersect the interior of the triangle ABC, then, for four particular directions $[D]$, there are corresponding parabolas passing through A, B, C, axis direction $[D]$ and tangent to e. The directions $[D]$, for which this happens, can be determined from the triangle ABC and the line e. This is the case $(3P2T_1)$, handled in §4.2. The problem is related to the pencil of conics through $A, B, C, [D]$. This is a specialization of the one in §1.1, resulting from it by sending D at infinity (see Figure 21). In this pencil, all members ex-

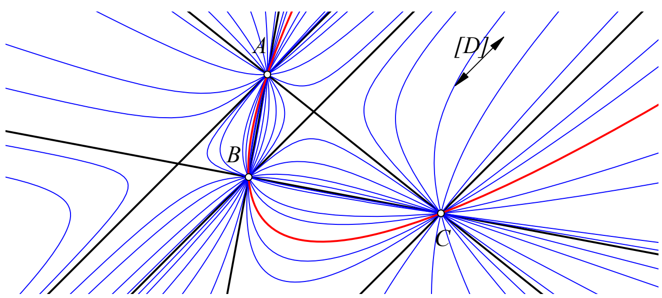

Figure 21. The pencil of conics through $A, B, C, [D]$

cept one are hyperbolas with one asymptotic direction $[D]$. The one exceptional member is the parabola constructed in §8.2.

3.4. *Hyperbola by 2 points, 2 asymptotics, 1 tangent* $(4P_21T)$. Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line e. This is a hyperbola passing through the points A, B, having directions of asymptotes $[C], [D]$ and being tangent to line e. This can be reduced to the case $(5P_20T)$ of §2.3 by locating the contact point S of e with the conic. This is done as in §3.1: Find the intersections $(H, G), (E, F)$ with e of line-pairs $(BD, AC), (BC, AD)$ respectively (see Figure 22). The contact points of the conics with e are the common harmonics S, S' of these two pairs of points.

An alternative solution results by using the *eleven points conic* of $A, B, [C], [D]$ and e as in §3.1, defined as the locus k of poles P of line e with respect to the

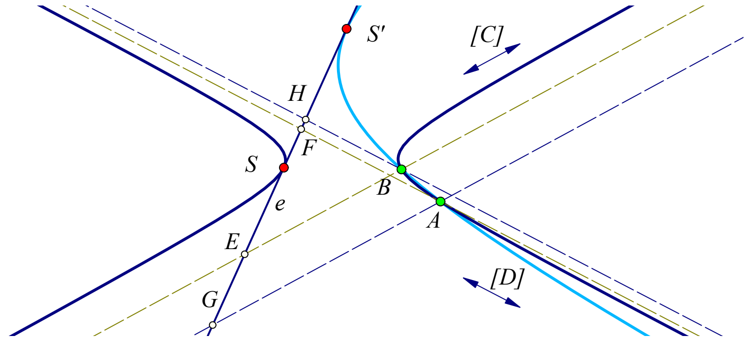

Figure 22. The two hyperbolas through $A, B, [C], [D]$ tangent to e

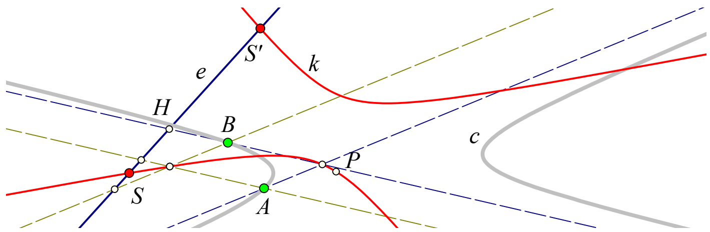

Figure 23. The eleven points conic of $a, b, [C], [D]$ and line e

conics c passing through $A, B, [C], [D]$ (their pencil shown in §2.3). This conic intersects e precisely at the points S, S' (see Figure 23).

Fixing $A, B, [C], [D]$ the lines e for which there are solutions are those defining non-separating segments EF, GH, where $E = (BC, e)$, $F = (AD, e)$, $G =$

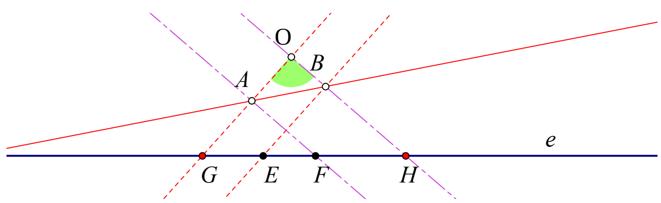

Figure 24. Directions for which exist solutions to $(4P_2 1T)$

(AC, e), $H = (BD, e)$ (see Figure 24). These are all lines except those, which separate points A, B and their parallels from $O = (AG, BH)$ fall outside the angular domain AOB.

4. Three points and two tangents

4.1. *Conic by three points and two tangents* $(3P2T)$. Construct a conic passing through three points A, B, C and tangent to two lines d, e. The construction can be reduced to that of a conic passing through five points (§2.1) by locating the points of tangency G, H of the two tangents. This can be done by finding the two

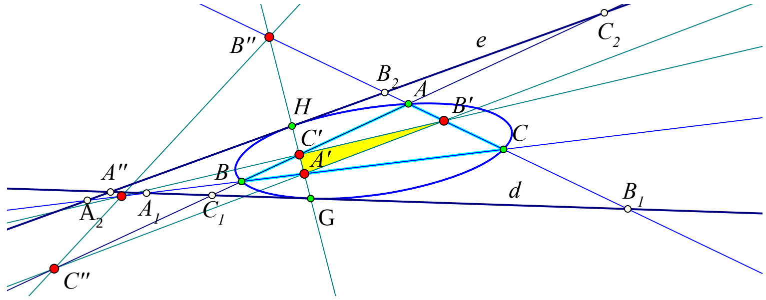

Figure 25. Conic through A, B, C, tangent to d, e

intersection points A', C' of line GH respectively with the known lines BC and AB. The key fact here is, that in all cases of existence of solutions, there is a cevian triangle $A'B'C'$ with respect to ABC, with corresponding tripolar $A''B''C''$, such that the contact points of each one of the requested conics are the intersections of lines e, d with some side of this triangle or its tripolar ([11], [23], [15]). In Figure 25, for example, appears one of the requested conics, tangent to d, e, respectively,

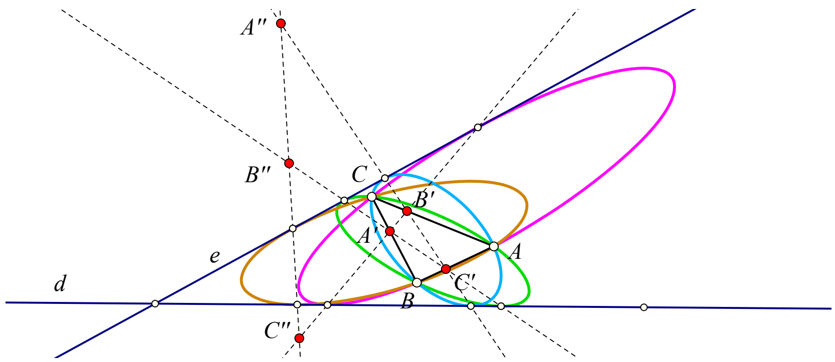

Figure 26. The four conics through A, B, C, tangent to d, e

at the points G, H, which are on the side $A'C'$ of a certain cevian triangle of ABC. The conic is led to pass through the five points A, B, C, G and H. Analogously are constructed three other conics. The determination of the cevian triangle $A'B'C'$ is done again through the construct of common harmonics. For example, points B', B'' are the common harmonics of the point pairs (AC, B_1B_2). These represent the fixed points of an involution, defined, by Desargues' theorem ([17, p. 204]), on line AC, by the intersections with members of the pencil $\mathcal{D}$ of all conics, which are tangent to d, e respectively at G, H (seen in §10.1). Regarding the existence, there are four solutions in the case none of the lines d, e passes through the interior of the triangle ABC (see Figure 26), or both of them intersect the interior of the same couple of sides of this triangle. In all other cases there are no solutions.

An intuitive way to answer, *why four*, offers Figure 27, displaying a cone and a plane ε on which the cone is projected. Plane ε is the one containing the lines e, d

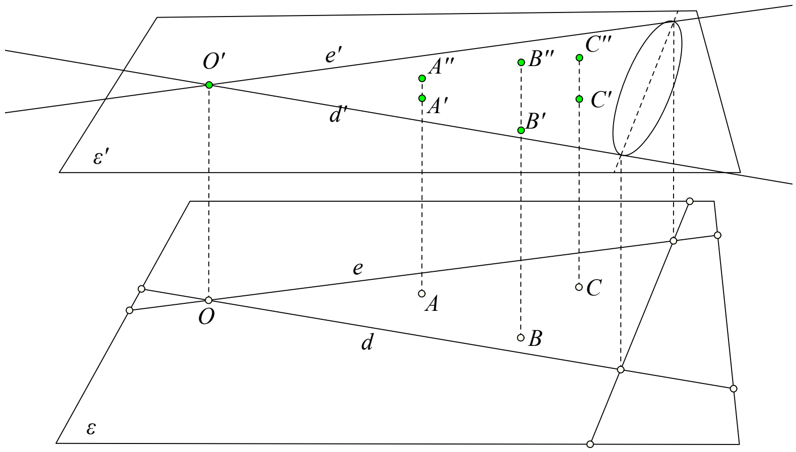

Figure 27. Spacial interpretation of the four solutions

intersecting at O. Plane ε' is parallel to ε from an arbitrary point O' projecting orthogonally to O. The circular cone is constructed so that the parallels d', e' from O' are generators and its axis is contained in ε'. The lines orthogonal to ε at the three points A, B, C intersect the cone respectively at pairs of points (A', A''), (B', B''), (C', C''). The plane through (A', B', C') intersects the cone along a conic, which projects to one of the conics solving the construction problem. The same is true with the triples of points on the conic (A'', B', C'), (A', B'', C'), (A', B', C''). They define respectively, a plane, a conic on the cone, and its projection on ε, representing a solution of the construction problem. The other possible triples of points (e.g such as the triple (A'', B'', C')), because of the reflective symmetry of the cone with respect to the plane ε', deliver conics on the cone, which are reflections of the previous four (e.g. (A'', B'', C') is symmetric to (A', B', C'')), hence by the projection falling onto the same four solutions of the construction problem.

Remark. To handle this, most interesting case and rich in structure of our constructions, one could consider the set of conics passing through three points A, B, C

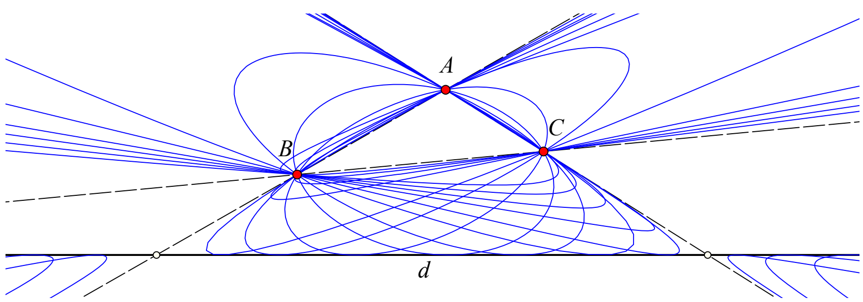

Figure 28. The set of conics through A, B, C tangent to line d

and tangent to one line d, as seen in Figure 28, and attempt to find the members of this set satisfying the fifth condition of tangency with line e. Unfortunately this set of conics is not a pencil, and Desargues' theorem does not apply to it to produce the solutions as usual. The same is true for the set of conics passing

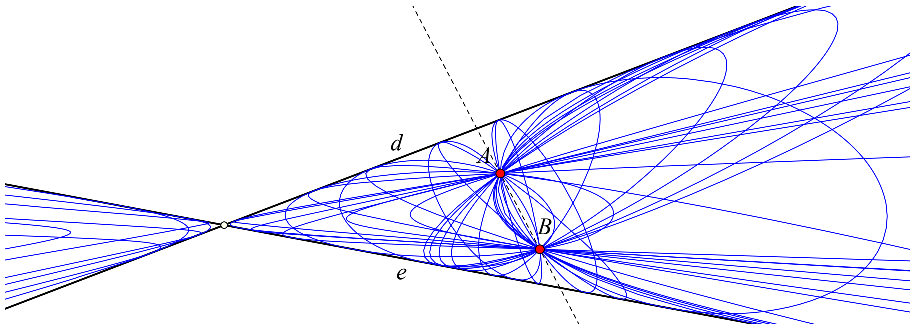

Figure 29. The set of conics through A, B tangent to lines d, e

through two points A, B and tangent to two lines d, e seen in Figure 29. This set of conics admits also a spacial interpretation, as the set of projections of intersections of a cone with all planes passing through points X, Y on the cone, where $X \in \{A', A''\}, Y \in \{B', B''\}$, the points of the two sets projecting respectively on A and B.

4.2. *Parabola by three points and a tangent* $(3P2T_1)$. Construct a conic passing through three points A, B, C and tangent to line d and the line at infinity, thus a parabola. For this, projectively equivalent to the previous, case, the process of de-

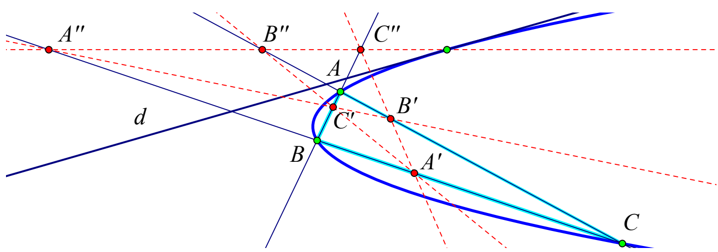

Figure 30. Parabola through A, B, C, tangent to d

termination of the cevian triangle and the tripolar, used there, is even simpler, since line e is now at infinity. The segments $A'A'', B'B'', C'C''$ of common harmonics, determined on each side of ABC, are now bisected by the tangent d and the chords of contact points with lines d, e are parallel to the axes of the parabolas (see Figure 30). Figure 31 displays all four parabolas passing through A, B, C and tangent to line d. There are four solutions if line d does not intersect the interior of triangle ABC and no solution if it does.

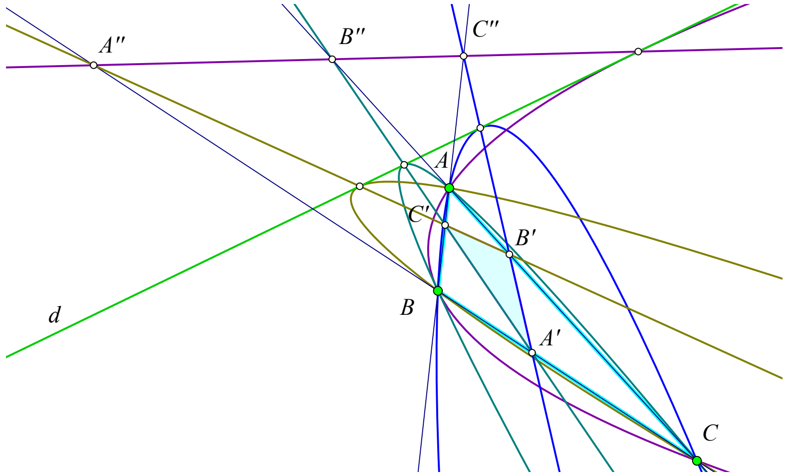

Figure 31. The four parabolas through A, B, C, tangent to d

4.3. *Conic by 2 points, 1 infinity, 2 tangents* $(3P_12T)$. Construct a conic passing through points $A, B, [C]$ and tangent to lines d, e. Triangle ABC is infinite with two sides parallel to the direction $[C]$. Points (C', C'') are the common harmonics of (A, B) and of the pair of intersections of AB with lines d, e. Analogously are defined the pairs of points (B', B'') on AC and (A', A'') on BC. $A'B'C'$ is a cevian triangle of ABC and points A'', B'', C'' are on the corresponding tripolar.

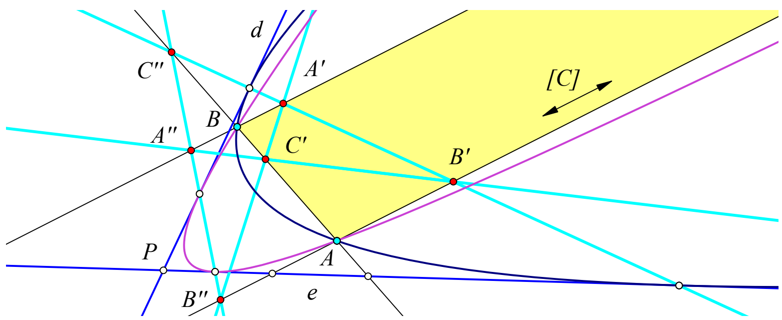

Figure 32. Two (of the four) hyperbolas through $A, B, [C]$ and tangent to lines a, b

Each one of the requested conics passes through A, B, C and is tangent to d, e at their intersection points with one side of the cevian triangle or the tripolar. In Figure 32 only two, out of the four, conics are shown. Additional points on the conics can be found by taking harmonic conjugates with respect to the polar of $P = (d, e)$. The construction of conics can be also completed by using the remark in $(3P2T)_2$ of §10.1. In general the conics are hyperbolas with one asymptotic direction parallel to $[C]$. Fixing A, B and d, e , there are four directions $[C]$ for which the corresponding conic is a parabola. These are determined in $(2P3T_1)$ of §5.2. There are four solutions if the lines d, e either do not intersect the interior of ABC or they intersect the interior of the same pair of sides of this triangle.

4.4. *Hyperbola 1 point 2 asymptotics 2 tangents* $(3P_22T)$. Construct a conic, passing through three points $A, [B], [C]$ and tangent to two lines d, e. This is a hyperbola with asymptotic directions $[B], [C]$. There are again four solutions determined

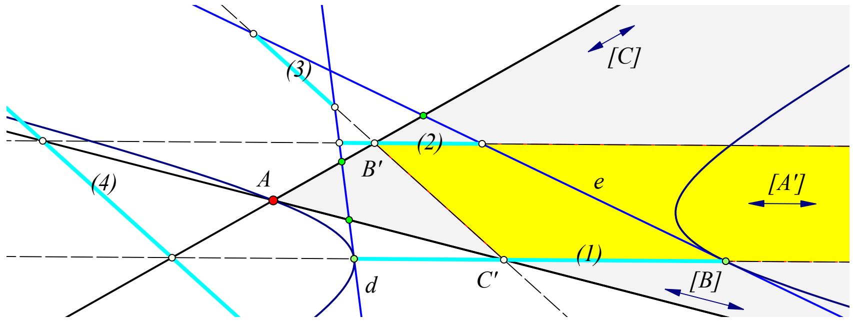

Figure 33. Hyperbola through $A, [B], [C]$ and tangent to lines d, e

by the sides and corresponding tripolar of a cevian triangle $A'B'C'$ of ABC. The triangle ABC has one side-line at infinity. The corresponding cevian triangle is determined as in §4.1, but now one of its vertices is at infinity. The four chords joining contact points of the same conic with lines d, e shown in Figure 33 are denoted by $(1), (2), (3), (4)$. The conic touching d, e at the endpoints of chord (1) is drawn. There are four solutions if the lines d, e do not intersect the interior of ABC or both intersect the interior of the same couple of sides of the triangle. In all other cases there are no solutions.

5. Two points and three tangents

5.1. *Conic by two points and three tangents* $(2P3T)$. Construct a conic passing through two points D, E and tangent to three lines a, b, c. The structure of the solution rests upon the dual theorem of Desargues ([17, p. 215], [9, p. 51], [19, p. 229]) and can be described as follows ([4, p. 58], [11], [23]). The three lines in general positions define a triangle ABC (A opposite to a etc.) and the two given points D, E determine a third point F, with the following properties. A, B, C, F define a *projective base* ([2, p. 95, I]) and in the coordinates with respect to this base the quadratic transformation ([21, p. 127])

$$f : (x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right), \text{ maps } f(D) = E.$$

There are four conics with the prescribed properties (see Figure 34). Each one of them is tangent to the three sides of the triangle ABC and to two additional lines. The pairs of additional lines corresponding to the four conics are (FD, FE), (F_1D, F_1E), (F_2D, F_2E), (F_3D, F_3E), where F_1, F_2, F_3 the harmonic associates ([24, p. 100]) of F. Figure 35 shows the domains for which there are solutions for the $(2P3T)$ problem. The two points D, E must lie, both, in the domain with the same label. Otherwise there are no solutions.

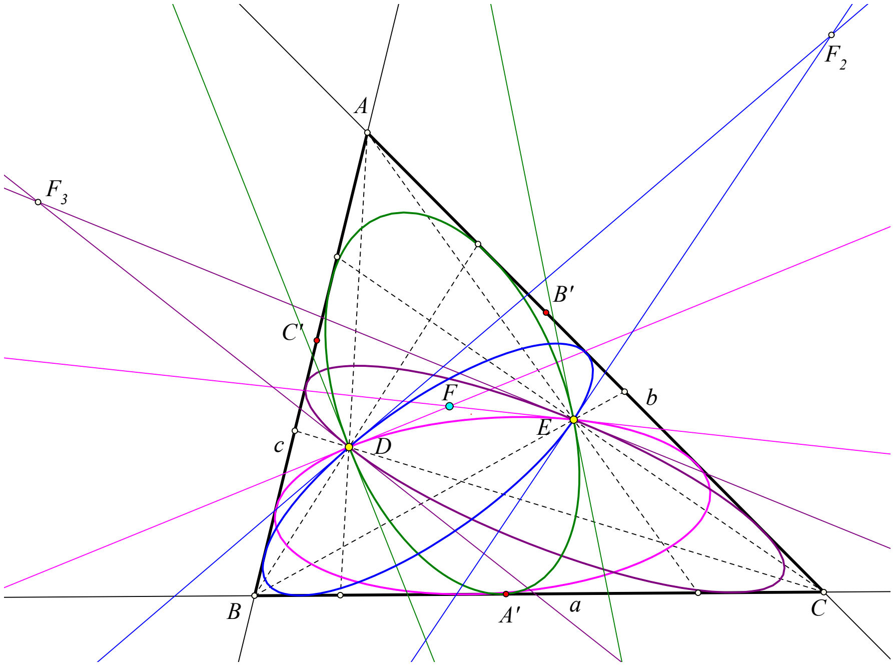

Figure 34. The four conics tangent to a, b, c and passing through D, E

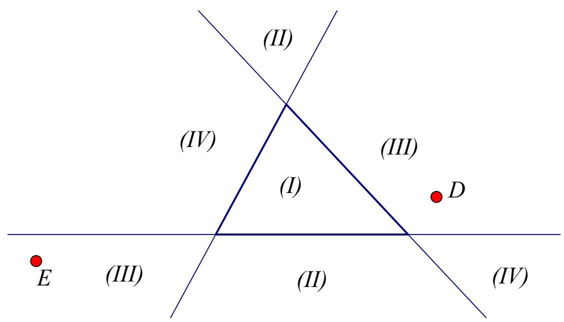

Figure 35. Domains of existence of $2P3T$ conics

This case is the *dual* of the previous one and is reducible to that by taking poles and polars with respect to a fixed conic. For example, taking poles and polars with respect to the conic k with *perspector* F ([24, p. 115]), for each conic c tangent to the sides of ABC and passing through D, E, we obtain a conic c' passing through the vertices of the cevian $A'B'C'$ of F and tangent to d, e, which are the polars of D, E with respect to k (see Figure 36). By this *polarity* ([22, p. 263, I]) the tangents t_D, t_E to c at D, E map to the contact points D_1, E_1 of c' with lines d, e and the intersection point $F = (t_D, t_E)$ maps to line $f = D_1E_1$.

Using a polarity, as before, we could reduce the cases to the half; but it is not the purpose of the present review to produce a least number of pictures.

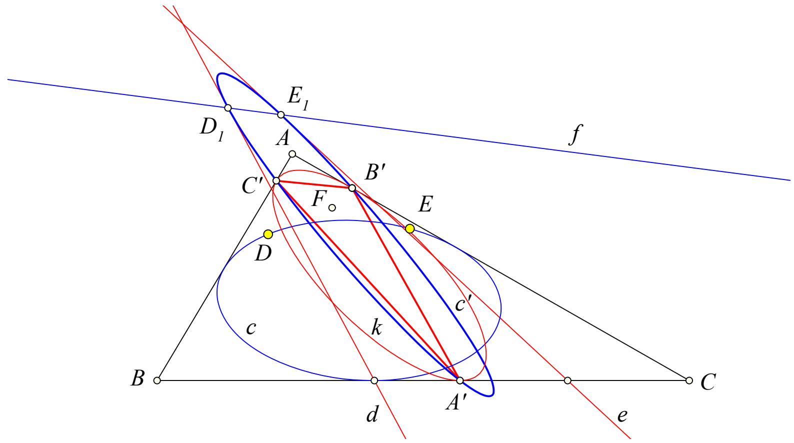

Figure 36. Reduction to the dual by the polarity with respect to k

Remark. In this case, as noticed also in the previous one, some difficulty in handling the construction lies on the fact that the set of conics tangent to a, b, c and passing through D is not a proper pencil of conics (see Figure 37). This set of con-

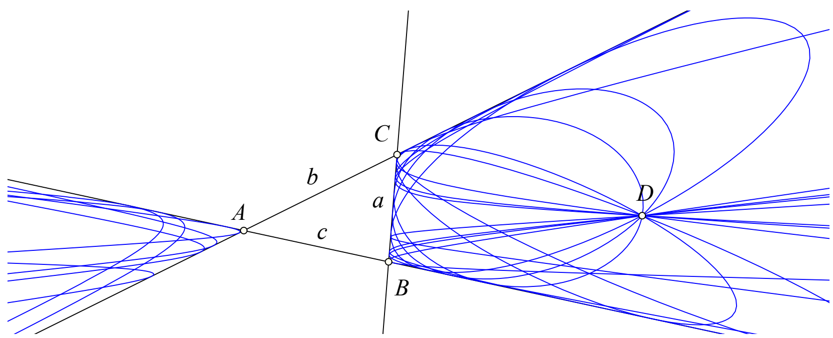

Figure 37. Conics tangent to a, b, c, passing through D

ics admits a spacial interpretation, as in §4.1, representing the conics as projections of intersections of a cone with planes. The cone is constructed as in that section, and the planes intersecting the cone are defined by pairs (X, α) of points and lines. Point $X \in \{D', D''\}$, the two points of the cone being those which project on D. Line α is a tangent to the conic defined on the cone by the plane orthogonal to the plane ε of b, c and intersecting it along line a.

5.2. *Parabola by 2 points, 2 tangents* $(2P3T_1)$. Construct a conic passing through two points A, B and tangent to two lines c, d and the line at infinity, thus, a parabola. Figure 38 shows the process of determination of the point F and its harmonic associates F_1, F_2, F_3 stepping on the previous section. Points B_1, B_2 are the common harmonics of the point-pairs (A', B') and $(G, [d])$, where A', B' are the parallel to c projections of A, B on d and $G = (c, d)$. Analogously are defined on c the common harmonics C_1, C_2 of two similar point-pairs on c. Point F is the

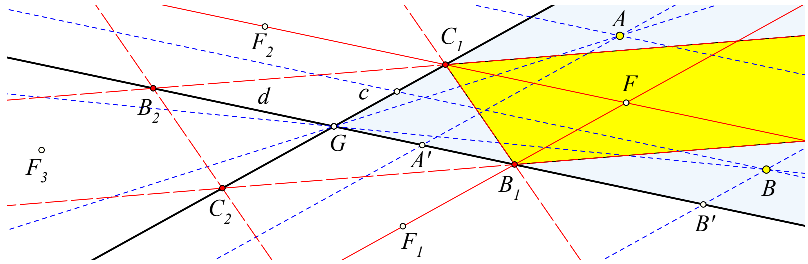

Figure 38. Searching for parabolas through A, B and tangent to c, d

intersection of the parallels from B_1, C_1 respectively to c, d. Figure 39 shows the

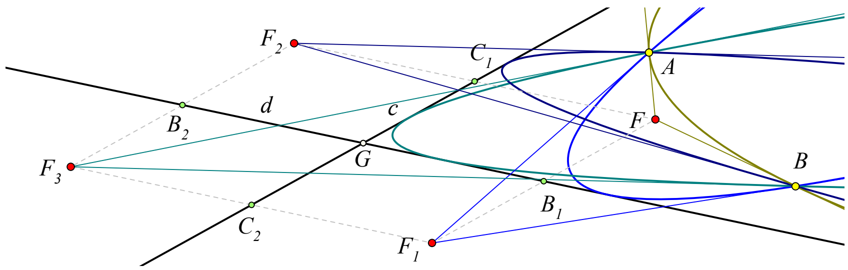

Figure 39. The four parabolas through A, B and tangent to c, d

four parabolas solving the problem. Each of them is tangent to the two given lines c, d and the pair of lines F_iA, F_iB, where F_i are either F or one of its harmonic conjugates (with respect to the triangle with infinite sides $G[c][d]$). There are four solutions if points A, B are in the same or opposite angular domains of lines c, d. Otherwise there are no solutions.

Remark. When line FF_3 passes through the middle M of AB, the two corresponding parabolas, tangent respectively to the line-pairs $(FA, FB), (F_3A, F_3B)$, are homothetic with respect to G and solve problem $(3P_12T_1)$ of §9.3, line FF_3 being then parallel to the axis of the two parabolas and also being harmonic conjugate of AB with respect to lines (c, d).

5.3. *Conic by 1 point, 1 infinity, 3 tangents* $(2P_13T)$. Construct a conic passing through two points $A, [B]$ and tangent to three lines c, d, e. The construction can be done by adapting the one of §5.1. By that method, we first find the cevians of A, B with respect to the triangle $A'B'C'$, whose sides are c, d, e. Then we find the common harmonics A_0, A_1 on line c of the point-pair consisting of the traces of the cevians from A, B and (B', C') . Analogously are defined points B_0, B_1 on d and C_0, C_1 on e (see Figure 40). The six points thus determined define a cevian triangle with perspector F and the corresponding tripolar. Then we define the three harmonic associates F_1, F_2, F_3 of F. Each one of the requested conics is tangent to the three lines c, d, e and also tangent to the two lines F_iA, F_iB, joining some of the points F_i with A and B. There are four solutions if A is on the exterior of

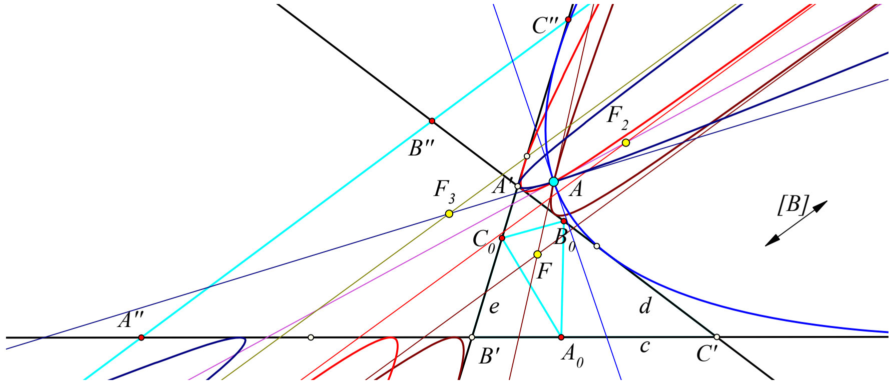

Figure 40. The four conics tangent to c, d, e passing through $A, [B]$

the triangle $A'B'C'$ and in the angular domain containing the parallel to $[B]$ from a vertex. Otherwise there are no solutions. The conics are in general hyperbolas. Fixing the lines c, d, e and point A, there are two directions $[B]$ for which the conics are parabolas. This is handled in $(1P4T_1)$ of §6.2.

5.4. *Hyperbola by 2 asymptotics and 3 tangents* $(2P_23T)$. Construct a conic passing through two points $[A], [B]$ and tangent to three lines a, b, c. This is a hyperbola with asymptotic directions $[A], [B]$ and tangent to three lines. Proceeding as in

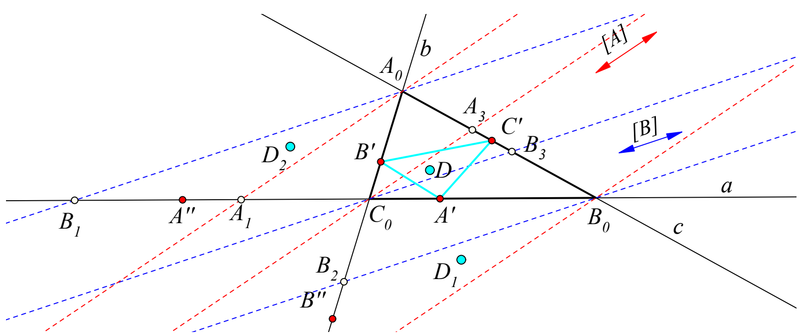

Figure 41. Cevian triangle and perspector in the case $(2P_23T)$

§5.1, we draw parallels to those directions from each vertex of the triangle $A_0B_0C_0$ with side-lines a, b, c. These parallels define on each side two points A_1, B_1 on a, A_2, B_2 on b etc. The common harmonics (A', A'') of point pairs (A_1, B_1) and (C_0, B_0) and the corresponding common harmonics for the other sides, define the cevian triangle $A'B'C'$, its perspector D and the corresponding harmonic associates D_1, D_2, D_3 (see Figure 41). Each of the requested hyperbolas is constructed as a conic tangent to the three lines a, b, c plus the two lines joining D_i to the points at infinity A, B, i.e. the parallels from D_i to $[A]$ and $[B]$ (see Figure 42). Given the directions of lines a, b, c, there are four solutions if drawing parallels to these directions and to $[A], [B]$, later are not separated by the first. Otherwise there are no solutions.

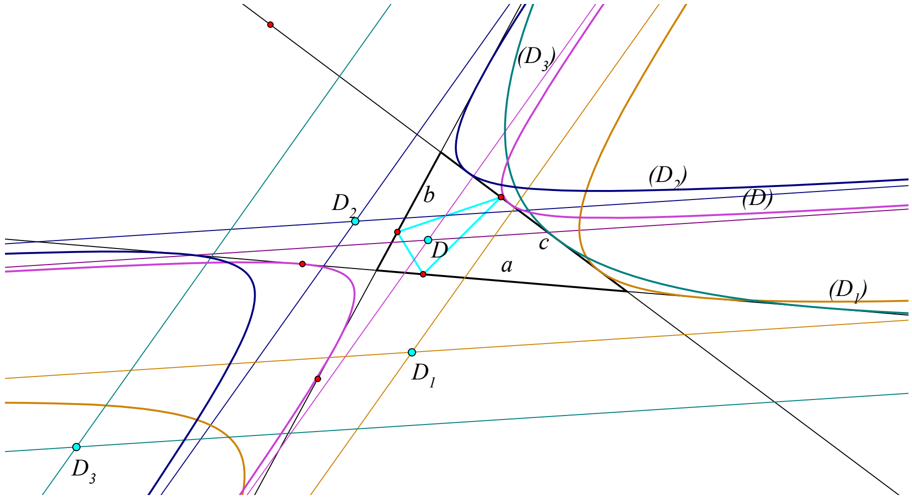

Figure 42. The four hyperbolas tangent to a, b, c with two given asymptotic directions

6. One point and four tangents

6.1. *Conic by 1 point and 4 tangents* $(1P4T)$. Construct a conic tangent to four given lines a, b, c, d and passing through a given point A. One way to the construction is to reduce the problem to its dual $(4P1T)$ of §3.1. In fact, if IJK is

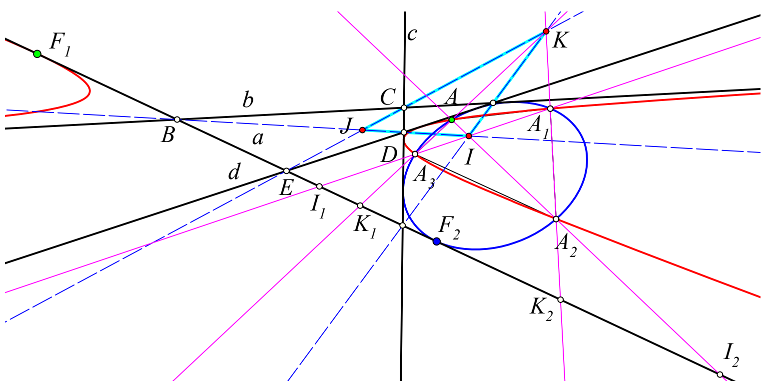

Figure 43. The two conics tangent to a, b, c, d, passing through A

the diagonal triangle of the complete quadrilateral whose sides are a, b, c, d (see Figure 43), then the harmonic associates A_1, A_2, A_3 of A with respect to IJK are also points of the conic. Thus, one can apply the recipe of §3.1 by taking these four points and one of the four given lines.

Another way to define the conics is by using Desargues' theorem in its dual form ([4, p. 57]) in order to locate a fifth tangent to the conic, namely the one passing through A. In fact, according to that theorem, the tangents from an arbitrary point A to the conics of the one-parameter pencil of conics $\mathcal{D}$, which are tangent to four lines a, b, c, d, define an involution on the pencil A^* of all lines passing through that point. The tangents to the members of that pencil, which pass through A are the fixed elements of this involution. By considering the intersection of each line through A with line a, we represent this involution by a corresponding involution of points of a. The fixed elements of the involution in A^* correspond

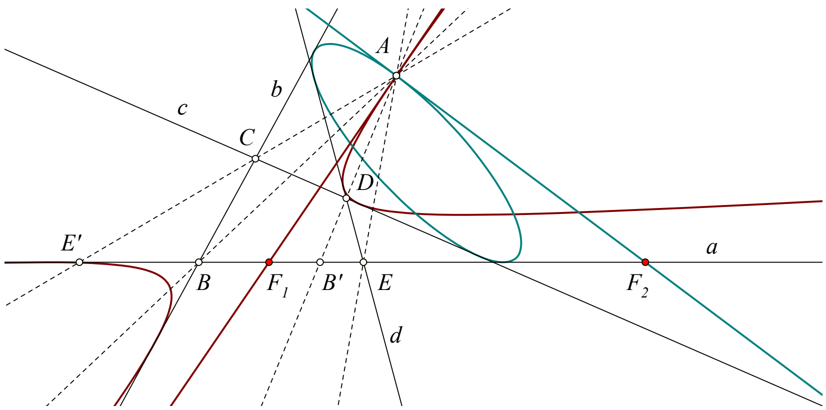

Figure 44. The two conics tangent to a, b, c, d, passing through A

to the fixed points of the corresponding involution on a. It is easy to see that two particular pairs of points in involution on a are the pairs (B, B') and (E, E'), where $B = (b, a), E = (d, a), B' = (AD, a), E' = (AC, a)$. The common harmonics F_1, F_2 of these two pairs define the two requested tangents, which in turn define the two conics (see Figure 44).

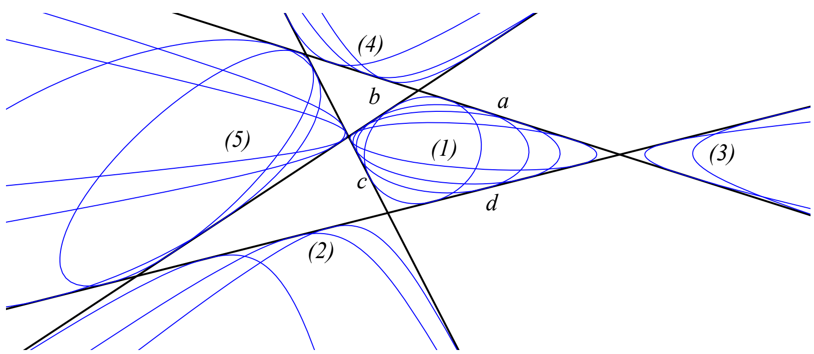

Figure 45. The five domains of existence of solutions

There are two solutions if point A is in one of the five domains $(1) - (5)$ shown in Figure 45. Otherwise there are no solutions. The reason for this is, as is visible in the figure, that the pencil $\mathcal{D}$ of conics tangent to four given lines a, b, c, d does not cover all connected domains defined by the four lines ([2, p. 200, II]). Noticable in the figure is also the fact that for every point in these five domains there are two conics of the pencil passing through the point.

A third method to construct the requested conics is to use the dual of the *eleven points conic* of §3.1, which is the *eleven tangents conic*, defined by four lines a, b, c, d and a point A ([1, p. 97]). This conic is the envelope k of the polars of A with

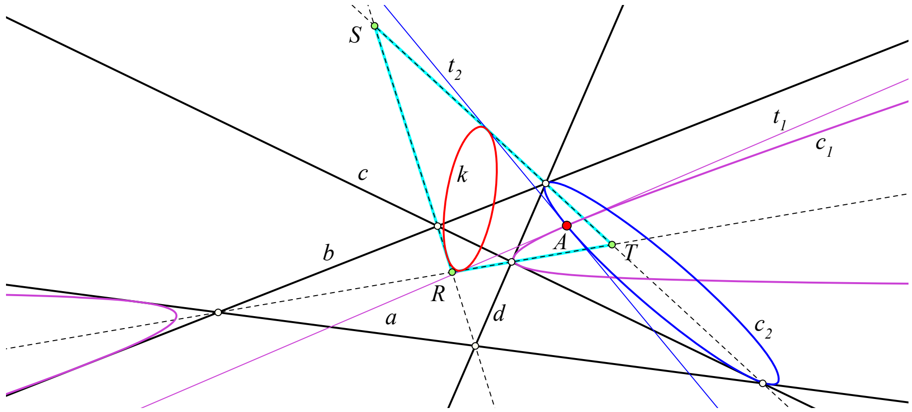

Figure 46. The two conics tangent to a, b, c, d, passing through A

respect to all conics tangent to the four given lines. This conic is tangent to the 3 sides of the diagonal triangle RST of the quadrilateral of the four lines. It is also tangent to the 6 polars of A with respect to all line-pairs of the quadrilateral, and is also tangent to the two tangents t_1, t_2 to the requested conics at A (see Figure 46). Conic k can be constructed by the methods of §7.1 and then t_1, t_2 can be found by drawing the tangents to k from A. The two requested conics can be defined by applying again the methods of the next section and determining the conic tangent to five lines $a, b, c, d, t_i,\ (i = 1, 2)$.

6.2. *Parabola by 1 point, 3 tangents* $(1P4T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, and also tangent to three lines a, b, c and passing through a point E. Any of the methods of the previous section can be modified

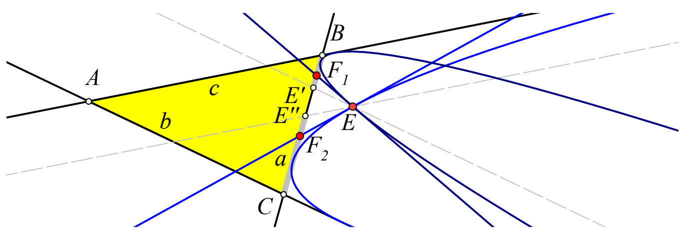

Figure 47. Parabolas tangent to a, b, c, passing through E

to produce the requested parabolas. For example, applying the second method, we draw first parallels to b, c through E intersecting a at E', E'' (see Figure 47). The pairs of lines (EE', EB) and (EE'', EC) through E are related with respect to the involuetion defined by Desargues' theorem. They are tangents from E to degenerate members of the pencil of parabolas tangent to four lines three of which are a, b, c. The common harmonics F_1, F_2 of these pairs define the tangents at E of the requested parabolas passing through E. There are two solutions if point E lies in one angular domain containing the triangle ABC but outside of the triangle. Otherwise there are no solutions.

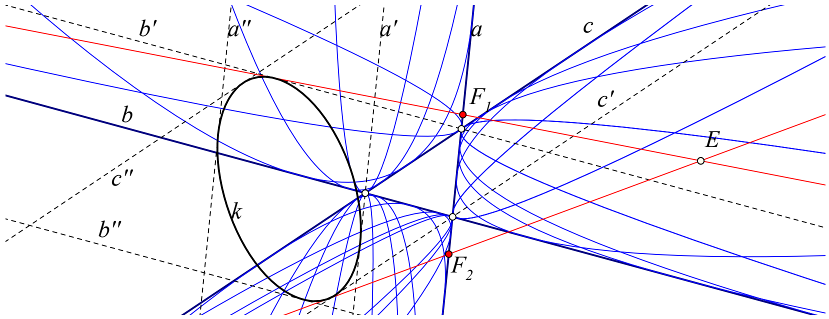

Figure 48. The pencil of parabolas tangent to a, b, c

An alternative solution is obtained by using the *nine tangents conic* of the three lines a, b, c, the line at infinity e and the point E. This is the conic k, defined as envelope of all polars of E with respect to the members of the pencil of parabolas tangent to the three lines a, b, c (see Figure 48). Conic k is tangent to $a', a'', b', b'', c', c''$, where a', a'' are parallel to a, respectively, from point $A = (b, c)$ and the symmetric E_a of E with respect to a, and the other lines are defined analogously. The tangents to the requested parabolas at E are the two tangents from E to k.

6.3. *Conic, 1 infinity, 4 tangents* $(1P_14T)$. Construct a conic tangent to four lines a, b, c, d and passing through a point at infinity $[E]$. Again a solution results by adapting any of the methods of section §6.1. For example, to adapt the sec-

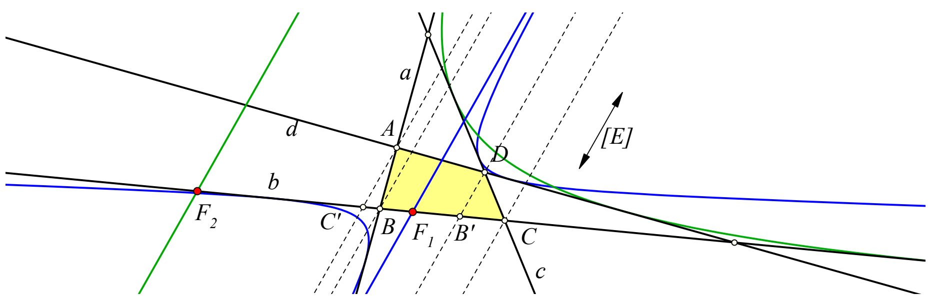

Figure 49. The two conics tangent to a, b, c, d, passing through $[E]$

ond method to the present configuration, define the points B', C' on b, to be the projections of D, A parallel to $[E]$ (see Figure 49). Points F_1, F_2 are the common harmonics of pairs $(B, B'), (C', C)$, and the parallels EF_1, EF_2 define the tangents at E of the requested conics, which are constructed from five tangents. Fixing lines a, b, c, d, there are two solutions when the parallel to $[E]$ from B

falls inside the angle CAB or the parallel from D falls inside the complement of ADC. Otherwise there are no solutions. The above construction assumes that the lines through F_1, F_2 are ordinary and as a consequence the conics are hyperbolas. If the conic is tangent to the line at infinity, thus a parabola, then $[E]$ is uniquely determined from the lines a, b, c, d. This is handled in $(0P5T_1)$ of §7.2.

7. Five tangents

7.1. *Conic by five tangents* $(0P5T)$. Construct a conic tangent to five lines a, b, c, d, e. A first solution is to reduce the construction to its dual of a conic through five points, as in §2.1. For this, use Brianchon's theorem to find the contact points with the sides ([19, p. 225]). In fact, the diagonals BD, CE of the pentagon of the

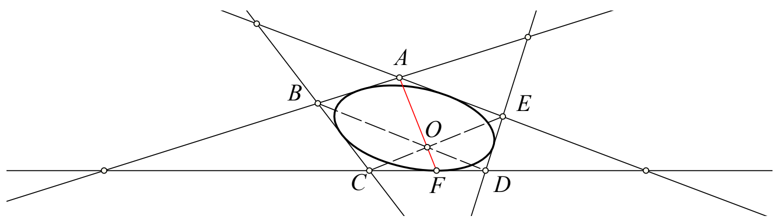

Figure 50. Conic through five tangents, find the contact points

given lines intersect at a point O (see Figure 50), which lies also on the line joining the remaining vertex of the pentagon A to the contact point F of the opposite side. Thus F is constructible from the data. Analogously are found the other contact points with the sides of the pentagon. Using again the theorem of Brianchon in its general form for hexagons circumscribed to a conic, one can construct arbitrary many other tangents to the conic. In fact, take a point F on side AB and define

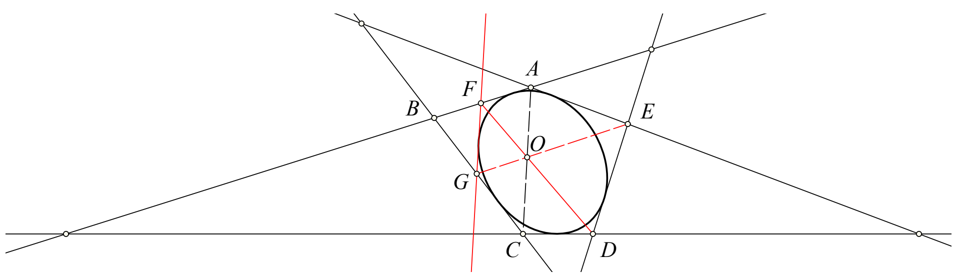

Figure 51. Conic through five tangents, draw arbitrary many tangents FG

the intersection point O of FD and AC (see Figure 51). By Brianchon's theorem the diagonal GE will pass also through O. Hence the position of G can be found by intersecting BC with OE. Thus, moving F on line AB and determining G on BC by the above procedure, we can find arbitrary many tangents FG to the conic. There is always a unique solution.

An image of the pencil of conics tangent to four lines, related to this problem, is contained in §6.1.

7.2. *Parabola by four tangents* $(0P5T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to four given lines a, b, c, d. The case is projectively equivalent to the previous one and the method used there can be adapted to solve the problem. For this, apply Brianchon's theorem to the pentagon $ABCDE$, which

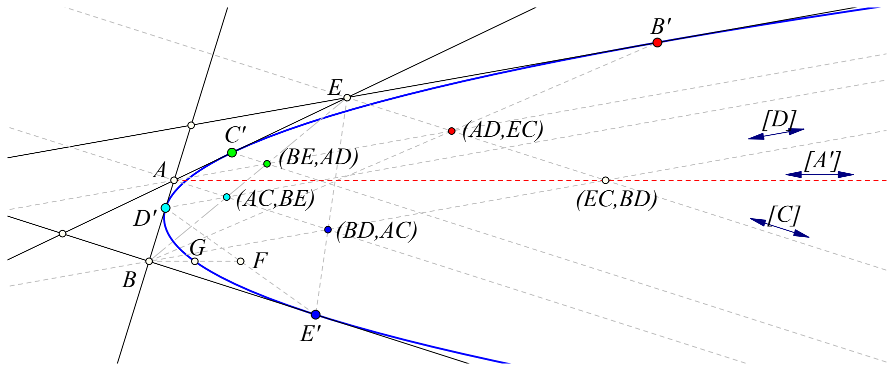

Figure 52. The contact points of the tangent parabola

now has points C, D at infinity (see Figure 52). The contact points of the sides opposite to the vertices are denoted correspondingly A', B', C', D', E'. Point A' is at infinity, and determines the axis of the parabola. By Brianchon's theorem, line AA' passes through the intersection (EC, BD), which is constructible from the data. Analogously are constructible the intersections $(AD, EC$, (BE, AD), (CA, BE), (DB, CA). Join these points correspondingly with the vertices B, C, D, E to find B', C', D', E' through their intersections with the opposite sides of the pentagon. To the four points on the parabola a fifth one G can be defined by taking the middle F of $D'E'$ and the middle G of FB. Thus the parabola can be constructed as a conic passing through the five points B', C', D', E', G ([2, II, p. 212]).

Another way to solve the problem, is through the properties of the created parabola related to the Miquel circles of the quadrilateral of the four given lines ([12, p. 83]). These are the circumcircles of the four triangles formed by the four given lines. A theorem of Miquel asserts that all four circles pass through the same point F (see Figure 53). A theorem of Steiner ([17, p. 161]) completes then the construction, by showing that this point F is the focus of the parabola, while the directrix carries all four orthocenters of the aforementioned triangles ([19, p. 70]). Thus, in order to construct the parabola, it suffices to take the circumcircles and the orthocenters of two such triangles and define F and their orthocenters H_1, H_2 ([12, p.45], [14, p. 100, II]). The parabola then is constructed from its focus and the directrix H_1H_2. There is always a unique solution.

8. Four points one tangent one coincidence

8.1. *Conic by three points and one tangent-at* $(4P1T)_1$. Construct a conic passing through four points A, B, C, D and tangent to a line e at D. In this case it is easy to find a fifth point on the conic and reduce the construction to that of $(5P0T)$ in

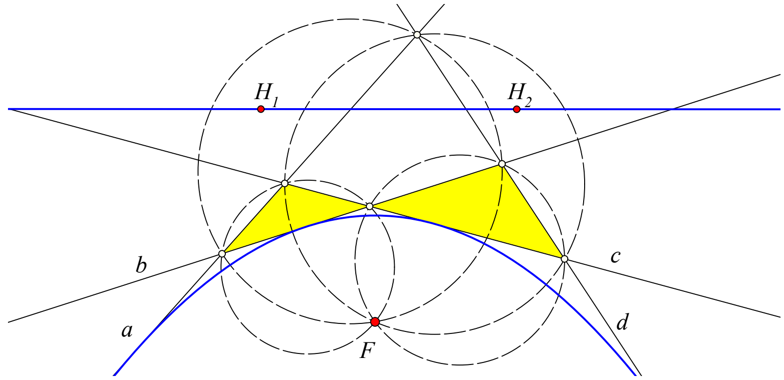

Figure 53. The classical construction of the parabola tangent to four lines

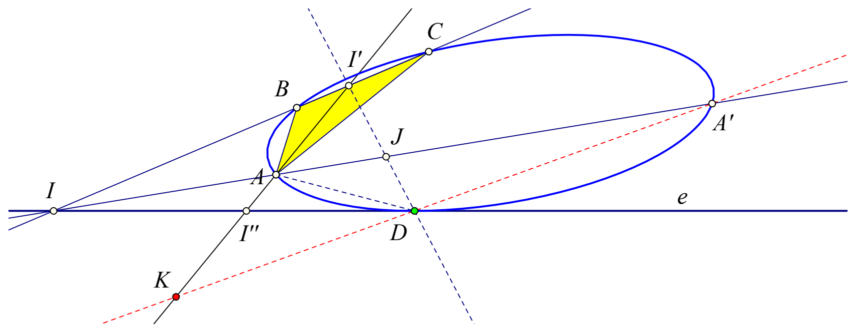

Figure 54. A fifth point A' of the conic through A, B, C tangent to e at D

§2.1. In fact, take $I = (BC, e), I' = I(B, C)$. Then line $I'D$ is the polar of I (see Figure 54). If $J = (I'D, AI)$, then $A' = A(I, J)$ is a point on the conic. There is always a unique solution.

Noticeable in the figure is point $K = (DA', AI')$. It is a fixed point on line AI', since the cross ratio $(A, K, I', I'') = (A, A', J, I) = -1$. Hence points D, A', and through them, the various conics passing through A, B, C and tangent to e, are defined by turning a line about K and considering its intersections with the fixed lines e and IA. Figure 55 shows the structure resulting by making the same construction with respect to the other sides of ABC. In this point E is the tripolar of line e and K_A, K_B, K_C are its harmonic associates with respect to ABC. Each point $D \in e$ defines three other points of the conic passing through A, B, C and tangent to e at D. These points are $A' = (DK_A, K_BK_C), B' = (DK_B, K_CK_A), C' = (DK_C, K_AK_B)$. It is easily seen that points K_A, K_B, K_C are the harmonic associates of D with respect to $A'B'C'$.

8.2. *Parabola by three points and axis-direction* $(4P_11T_1)$. Construct a conic tangent to the line at infinity, hence a parabola, passing through four points $A, B, C, [D]$. The last point, at infinity, defines the direction of the axis of the parabola. The construction is carried out by locating two more points B'', C'' on the parabola and

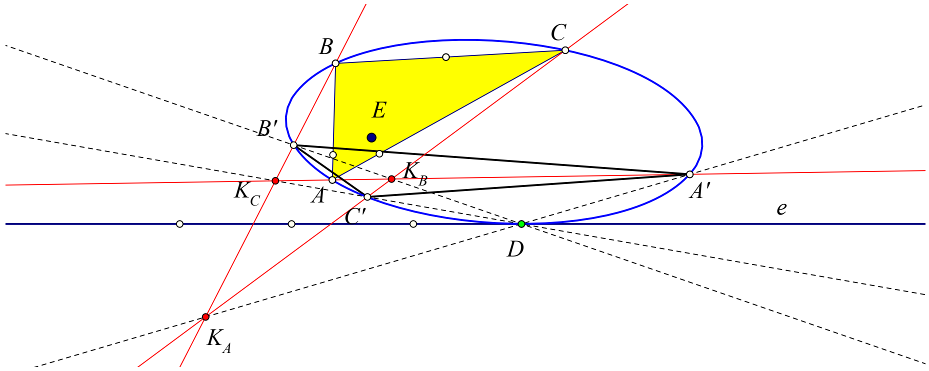

Figure 55. Conic through A, B, C tangent to e at D. Three additional points A', B', C'

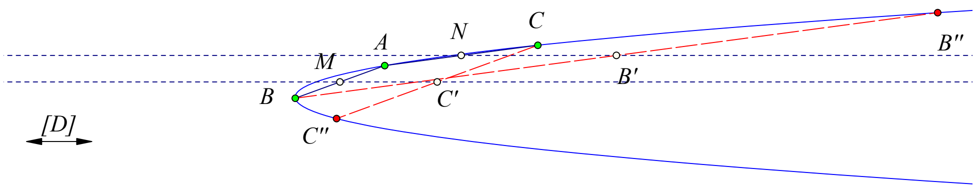

Figure 56. Parabola through A, B, C and given axis-direction $[D]$

passing a conic through A, B, C, B'', C'' (see Figure 56). The construction of the additional points is the same with that of the previous section.

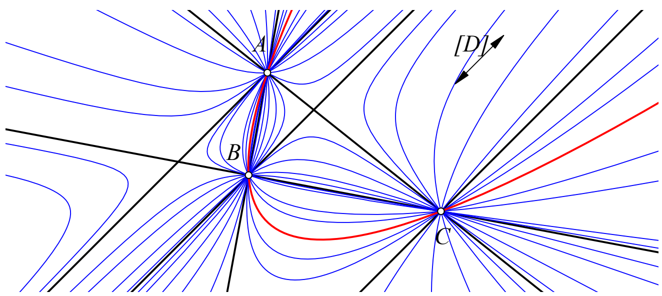

Figure 57. The pencil of conics through $A, B, C, [D]$

The pencil of conics involved is a specialization of the one in §1.1, resulting from it by sending D at infinity (see Figure 57). The construction shows that in this pencil, all members except one are hyperbolas with one asymptotic direction $[D]$. The one exceptional member is the requested parabola.

8.3. *Conic by 2 points, 1 at infinity, 1 tangent-at* $(4P_1 1T)_1$. Construct a conic passing through four points $A, B, C, [D]$ and tangent to a line e at C. Additional points on the conic can be found as in §8.1. In Figure 58 points L, K are two such

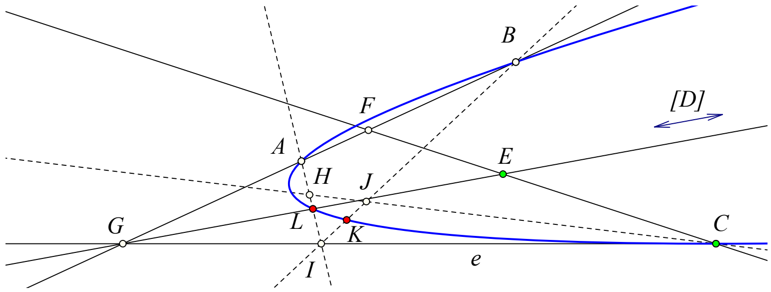

Figure 58. Conic through $A, B, [D]$ tangent to e at C

additional points. They are constructed as conjugates of B and D after constructing the polars CF of $G = (e, AB)$ and CH of $I = (e, AL)$. There is always one solution, which, in general, is a hyperbola. The pencil involved is the one of conics through $A, B, C, [D]$, seen in §8.2. Line e can be considered to turn about point C, defining in each of the obtained locations a corresponding member of that pencil. There is a single line through C, for which the corresponding conic is a parabola with axis $[D]$. In all other cases the conic is a hyperbola with an asymptotic direction $[D]$.

Fixing A, B, C, e, and varying $[D]$ we obtain, in general, hyperbolas. There are, though, two special directions $[D]$, determined in terms of A, B, C, e, for which the resulting conic is a parabola. This is handled in $(3P2T_1)_1$ of §9.2.

8.4. *Hyperbola by 3 points, 1 asymptote* $(4P_11T)_i$. Construct a conic passing through four points $A, B, C, [D]$ and tangent to a line e at $[D] \in e$. This is equivalent with the construction of a hyperbola passing through the points A, B, C and

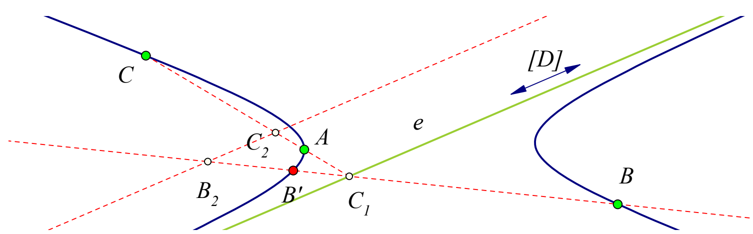

Figure 59. Hyperbola through A, B, C and given asymptote d

having the asymptote line e. Adapting the method of §8.1, we can find two additional points B', C' and construct the requested conic as a $(5PT0)$ conic. For example, to define B', take successively $C_1 = (AC, e), C_2 = C_1(A, C)$. The parallel to e from C_2 is the polar of C_1. Take then the intersection B_2 of that line with BC_1 and $B' = B(C_1, B_2)$, which is a point on the conic. Analogously is defined point C'. There is always one solution. The pencil of conics involved is the one shown in $(4P_11T_1)$ of 8.2. A line parallel to $[D]$ determines a unique member of the pencil having this line as an asymptote.

8.5. *Hyperbola by 1 point 2 asymptotics 1 tangent at* $(4P_2 1T)_1$. Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line e at $B \in e$. The conic is a hyperbola with given asymptotic directions $[C], [D]$, passing through a point A and tangent at a point B to a given line e. In this case, as we did in §8.1,

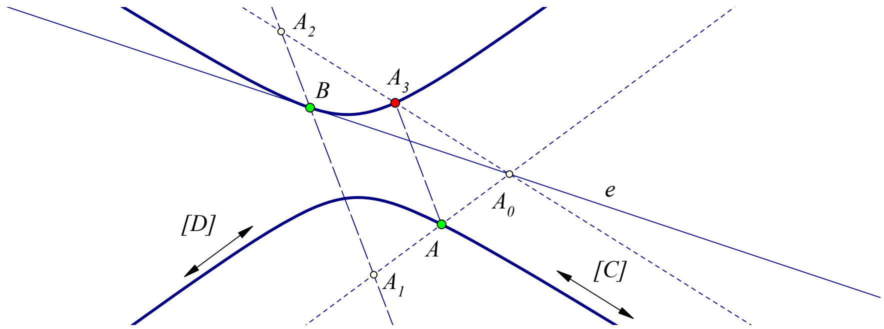

Figure 60. Hyperbola through $A, B, [C], [D]$ tangent to $e \ni B$

we can find additional points on the conic. For this let A_0 be the intersection point with e of the parallel to $[D]$ through A. The symmetric A_1 of A_0 with respect to A defines line BA_1 which is the polar of A_0. The parallel to $[C]$ from A_0 intersects A_1B at A_2 and the middle A_3 of A_0A_2 is on the conic. Repeating the construction with A_3 in place of A and continuing this way, we can construct arbitrary many points on the conic. There is always one solution.

8.6. *Hyperbola by 2 points 1 asymptote 1 asymptotic* $(4P_2 1T)_i$. Construct a conic passing through four points $A, B, [C], [D]$ and tangent to a line e at $[D]$. The requested conic is a hyperbola passing through two points A, B having one asymptotic direction $[C]$ and an asymptote $e \ni [D]$. By a well known property of the hyperbola ([4, p. 42]), the segments AA', BB' intercepted by the asymptotes on the secant AB are equal, hence, knowing AA', we locate B' on AB and given the direction of the other asymptote $[C]$ we determine it completely and find its

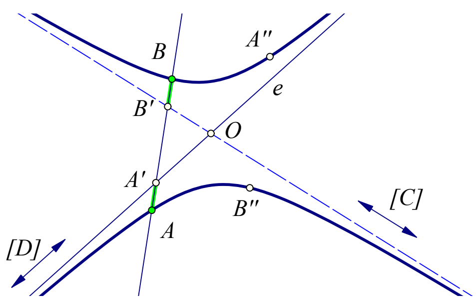

Figure 61. Hyperbola through $A, B, [C], [D]$ and asymptote $e \ni [D]$

intersection point O with the given asymptote e, which is the center of the hyperbola (see Figure 61). Two additional points A'', B'' are immediately constructed, by taking the symmetrics of A, B with respect to O. Arbitrarily many points on the conic can be then constructed by the method of the previous section. There is always a unique solution.

9. Three points two tangents one coincidence

9.1. *Conic by 2 points, 1 tangent, 1 tangent-at* $(3P2T)_1$. Construct a conic passing through three points A, B, C and tangent to two lines $d \ni A, e$. Using again the power of Desargues' theorem, we find first the contact points of the requested

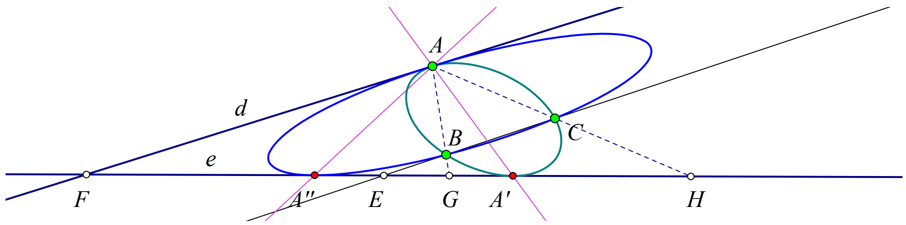

Figure 62. The two conics through A, B, C, tangent to e and also to d at A

conics with line e. These are the common harmonics A', A'' of the point-pairs $(E, F), (G, H)$, where $F = (d, e), E = (e, BC), G = (e, AB), H = (e, AC)$ (see Figure 62). Note that these pairs are defined as intersections of e with two degenerate members of the pencil $\mathcal{D}$ of conics tangent to d at A and passing through B, C. The first pair is the intersection with the degenerate conic of two lines $d \cdot BC$ and the second with the degenerate conic of the two lines $AB \cdot AC$. After locating the contact point, a fifth point on the conic can be obtained by using the polar of F, which is AA' or AA'' and taking the conjugate of B or C. There are no solutions if only one of the lines d, e separates points B, C. Otherwise there are two solutions. Figure 63 displays a pencil $\mathcal{D}$ of conics tangent to line d at a fixed point A and

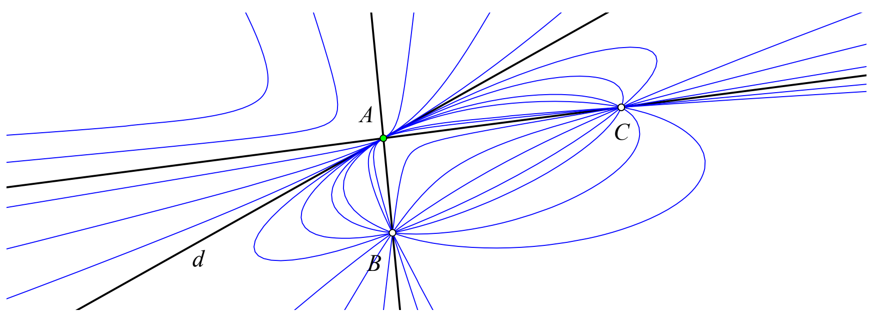

Figure 63. The pencil of conics tangent to d at A, passing through B, C

passing through two points B, C. It is visible there that from every point of the plane passes a unique member of the pencil and that for every line of the plane not separating B, C there are two members tangent to that line ([2, II, p. 193]).

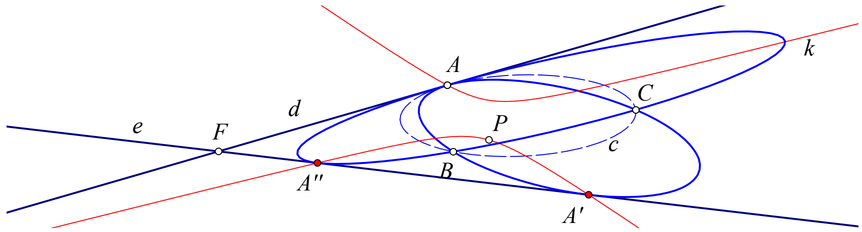

Figure 64. The two conics through A, B, C, tangent to e and also to d at A

For an alternative method, as in §3.1, we consider the poles P of line e with respect to all members c of $\mathcal{D}$. Their locus is the eleven points conic k intersecting the line e at the contact points A', A'' of the requested conics (see Figure 64).

9.2. *Parabola by 2 points, 1 tangent-at* $(3P2T_1)_1$. Construct a conic passing through three points A, B, C, tangent to line d at A and tangent to the line at infinity e, thus a parabola. The involution on e, induced by its intersections with the members of

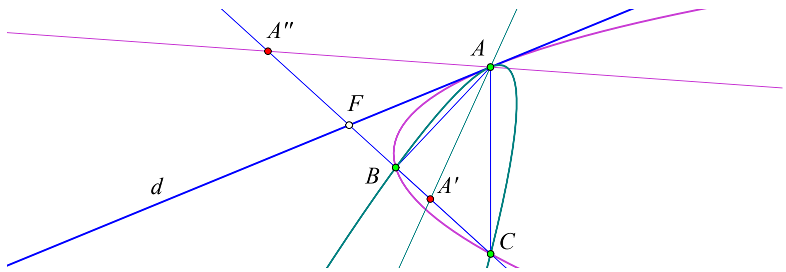

Figure 65. The two parabolas through B, C, tangent to d at A

the pencil $\mathcal{D}$ of conics tangent to d at A and passing through B, C, induces an involution on the pencil A^* of lines through A and, through the intersections of these lines with d, induces also an involution on d. Two, related by this involution point-pairs on d are (B, C) and $(F, [BC])$, where $F = (BC, d)$ and $[BC]$ the point at infinity of BC. The common harmonics A', A'' of these two pairs define, by joining them with A, the directions of the axes of the parabolas (they pass from the contact point at infinity) (see Figure 65). Two additional points on each parabola can be defined by projecting B, C parallel to d on the parallel to the corresponding axis through A and doubling the resulting segments. There are two solutions if the line d does not intersect the interior of segment BC and no solution if it does.

A different way to think about this problem is the following (see Figure 66). All conics c tangent to line d at A and passing through B, C have their centers on a conic k, passing through A and $D = (d, BC)$ and also through the middles P, Q, R of segments AB, AC, BC ([19, p. 299]). This is the *eleven points conic* of $\mathcal{D}$ with respect to the line at infinity. If this conic is a hyperbola, then its points at infinity are the centers of the two requested parabolas. To find the parabolas in this case draw from the middle Q of AC parallels $C'Q, C''Q$ to the asymptotes

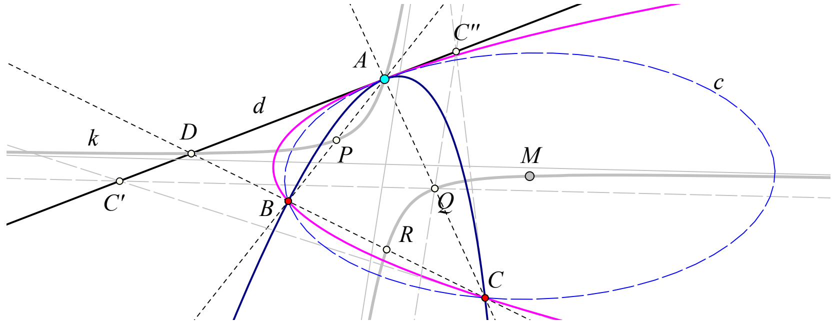

Figure 66. The two parabolas through B, C, tangent to d at A

intersecting d ad C', C'' respectively. One of the requested parabolas is tangent to lines $C'A, C'C$ at A, C correspondingly and passes through B. This is the case $(3P2T)_2$ of §10.1. Analogously can be constructed the other parabola, starting with C'' instead of C'. Figure 67 shows the pencil of all conics passing through

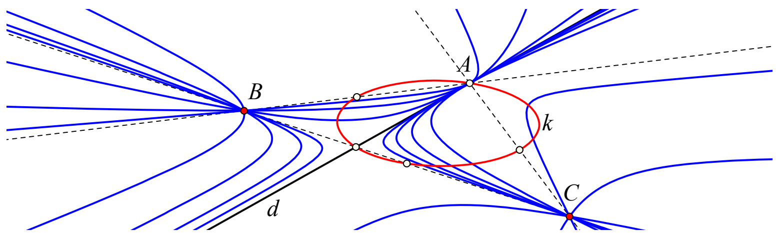

Figure 67. The pencil of conics tangent to d at A and passing through B, C

B, C and tangent to d at A, but having B, C on both sides of d. All conics are hyperbolas and the conic k of their centers is an ellipse. This is the reason of non-existence of solutions in this case.

9.3. *Parabola by 2 points, 1 tangent, axis-direction* $(3P_1 2T_1)$. Construct a conic passing through two points $A, B, [C]$, tangent to line d and also tangent to the line at infinity e, hence a parabola. The point at infinity $[C]$ determines the direction of

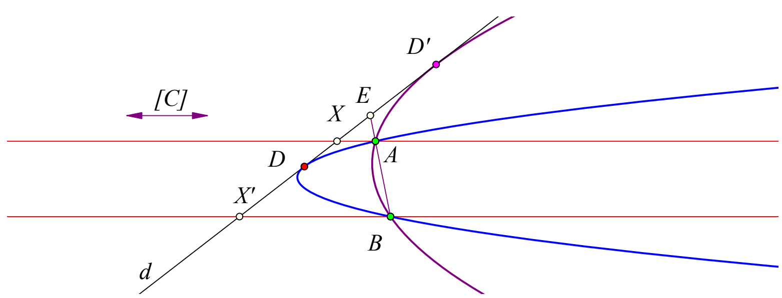

Figure 68. The two parabolas through A, B, tangent to d and axis parallel to $[C]$

the parabola's axis. By Desargues' theorem, the pencil $\mathcal{D}$ of all parabolas passing through $A, B, [C]$, i.e. passing through A, B and having axis direction $[C]$, define through their intersection points X, X' with line d an involution. The fixed points D, D' of this involution are contact points of the requested parabolas. There are two obvious, degenerate, parabolas passing through $A, B, [C]$ defining two pairs of points in involution. One pair consists of the intersections (X, X') with d of the two parallels to $[C]$ from A and B (see Figure 68). The other pair consists of $(E, [d])$, defined by line AB and the line at infinity, where $E = (AB, d)$. The fixed points D, D' of the involution are the common harmonics of these point-pairs. Once the contact points D, D' are known, the requested parabolas are easily constructible by the method of §10.2. If A, B are on the same side of a then there are two solutions, otherwise there is no solution.

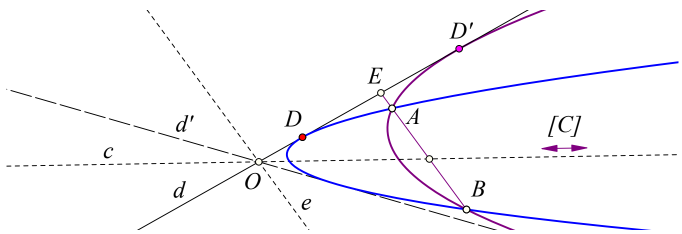

Figure 69. Parabolas through A, B, tangent to d and axis parallel to $[C]$

Another computational solution of the problem results by using the homothety relating the two parabolas. In fact, the intersection point O of d with the parallel c to $[C]$ from the middle of AB is the center of a homothety, mapping one of the parabolas to the other. The other common tangent d' to the two parabolas from O, can be constructed from the given data, since it is the harmonic conjugate of d with respect to the line pair (c, e), where e is the parallel to AB from O. The computations are straightforward and I omit them. See the remark in $(2P3T_1)$ of §5.2, which relates that problem to the present one.

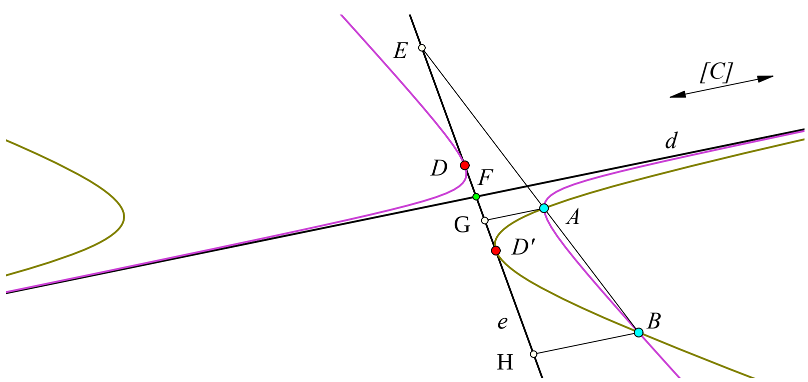

Figure 70. Hyperbola through A, B, asymptote d and tangent to e

9.4. *Hyperbola by 2 points, 1 tangent, 1 asymptote* $(3P_12T)_i$. Construct a conic passing through points $A, B, [C]$ and tangent to line d at $[C]$ and to line e. Thus, d is an asymptote and the conic is a hyperbola. The hyperbolas touch line e at D, D', which are the common harmonics of pairs of points $(E, F), (G, H)$, with $E = (e, AB), F = (d, e), G = (AC, e), H = (BC, e)$ (see Figure 70). Additional points can be found by considering conjugate points with respect to the polar of F. There are two hyperbolas, if A, B are in one and the same angular domain out of the four defined by lines d, e, or they lie on opposite angular domains. Otherwise there are no solutions. Figure 71 shows the pencil $\mathcal{D}$ of hyperbolas through A, B

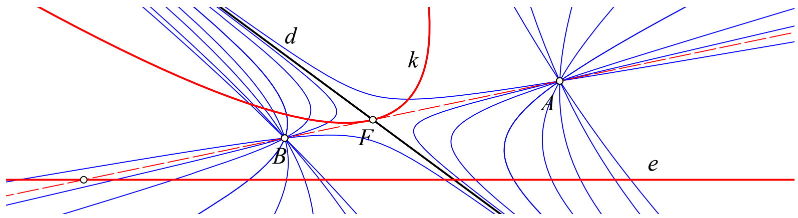

Figure 71. The pencil of hyperbolas through A, B with asymptote d

with one asymptote line d. It shows also a line e, for which there are no tangent members of the pencil. Conic k is the locus of poles of the line d with respect to the members of the pencil (the eleven points conic of $\mathcal{D}$ with respect to e). It is a hyperbola with one asymptote parallel to d, passing through $F = (d, AB)$.

9.5. *Hyperbola by 2 asymptotics, 1 tangent-at* $(3P_22T)_1$. Construct a conic passing through three points $A, [B], [C]$, tangent to $d \ni A$ and tangent also to line e. The conic is a hyperbola with asymptotic directions $[B], [C]$. In analogy to the

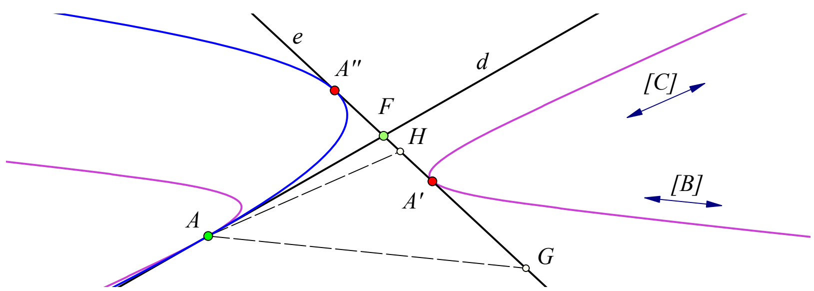

Figure 72. The two hyperbolas through $A, [B], [C]$, tangent to d at A and to line e

method of §9.1, project first A on e parallel to $[B], [C]$ to find respectively points G, H. The contact points A', A'' of the conics with line e are the common harmonics A', A'' of the point-pairs (G, H) and $(F, [e])$, where $F = (d, e)$. Once the two contact points of lines d, e are known, the methods of $(4P_21T)_1$ in §8.5 can be used to complete the construction. There are two solutions if the parallels to $[B], [C]$ from F fall in the same angular domain of lines (d, e). Otherwise there

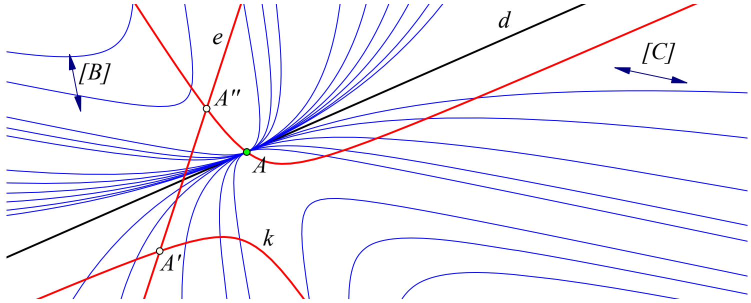

Figure 73. The pencil of hyperbolas with asyptotics $[B], [C]$ tangent to d at A

are no solutions. Figure 73 shows the pencil $\mathcal{D}$ of conics tangent to d at A and asymptotic directions $[B], [C]$. Shown is also the hyperbola k, defined as the locus of poles of a fixed line e with respect to the members of the pencil (the eleven points conic of $\mathcal{D}$ and e). The intersection points A', A'' of k with e are the contact points of requested conics with line e.

9.6. *Conic by 1 asymptote 1 asymptotic 1 point 1 tangent* $(3P_22T)_i$. Construct a conic passing through three points $A, [B], [C]$, tangent to $d \ni [B]$ and tangent to e. This is a hyperbola with an asymptote d, an asymptotic direction $[C]$, passing through point A and tangent to line e. The following construction method is a variation of the one given in §9.1. The pencil of conics $\mathcal{D}$, used in the theorem of Desargues, consists now of all conics tangent to d at $[B]$ and passing through A and $[C]$. This is the pencil of hyperbolas having their centers on line d, one asymptote

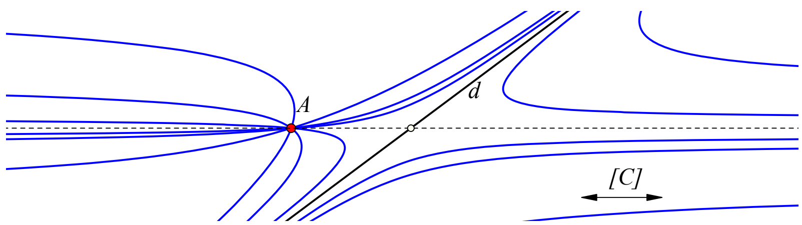

Figure 74. The pencil of hyperbolas through A, asymptote d, asymptotic $[C]$

d, the other parallel to $[C]$ and passing through A (see Figure 74). Two degenerate members of this pencil consist of (a) the product of line AB and the line at infinity, (b) the product of lines $d \cdot AC$. These two members define on e respectively the point-pairs $(G, [e]), (E, F)$, where $G = (e, AB), E = (e, AC)$ and $F = (d, e)$. The contact points D, D' of the requested hyperbolas with line e are the common harmonics of these two point-pairs. They lie on d symmetrically with respect to G (see Figure 75). Once the contact points with line e are known, the methods of $(3P2T)_2$ in §10.1 can be applied to complete the construction of the conics. Fixing the positions of A, d, e, there are two solutions if $[C]$ defines $E = (AC, e)$, such that EF is not separated by G. Otherwise there are no solutions. The points D, D' can be found also as intersections of line e with the conic k which is the locus of

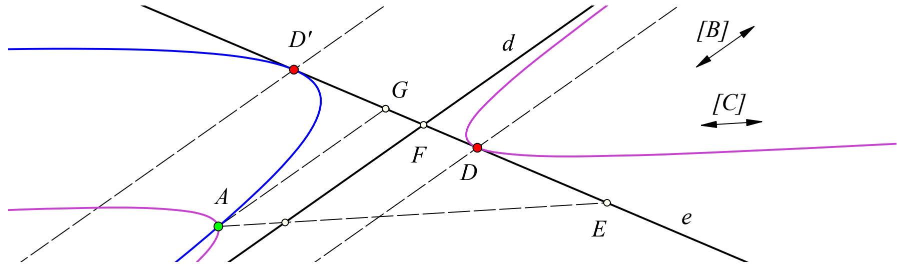

Figure 75. The two hyperbolas through A, asymptote d, asymptotic $[C]$, tangent e

poles of e with respect to the members of the pencil (the eleven points conic). In this case k is a hyperbola with one asymptote parallel to d.

10. Three points two tangents two coincidences

10.1. *Conic by two tangents-at and a point* $(3P2T)_2$. Construct a conic passing through three points A, B, C and tangent to two lines d, e at $A \in d$ and $B \in e$. In this case it is easy to find additional points and pass the conic through five points. Line AB is the polar of $F = (d, e)$ and the conjugate $D = C(F, J)$, where $J = (FC, AB)$ is on the conic. The conjugate $I = J(A, B)$ is the pole of FC and more points can be constructed as shown in Figure 75. The problem has always one solution.

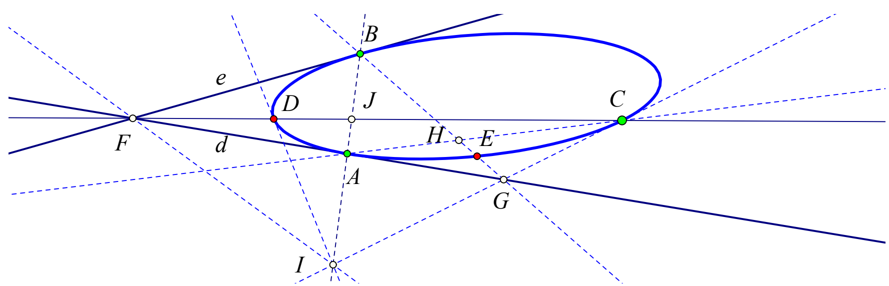

Figure 76. Conic through C, tangent to d, e at A, B

Remark. In this case the simplicity of the analytic solution is worth noticing. Representing lines d, e with two equations respectively $f = 0, g = 0$, and line AB with $h = 0$, the general equation of the conic passing through A, B and tangent there to lines d, e is given by a quadratic equation ([18, p. 234])

$$j = \lambda \cdot (f \cdot g) + \mu \cdot h^2 = 0,$$

where λ and μ are arbitrary constants. The requirement for the conic to pass through C, namely, $j(C) = 0$, determines, the constants λ, μ up to a multiplicative factor, and through these determines a unique conic. The conics resulting for variable λ, μ build the *bitangent* pencil ([2, II, p. 187]), used also in §4.1.

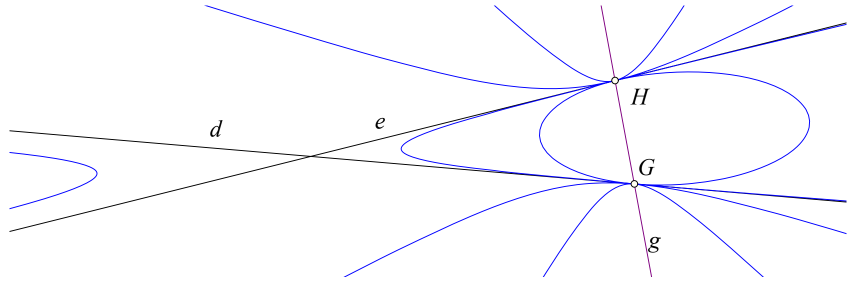

Figure 77. A bitangent pencil of conics $\kappa(d \cdot e) + \lambda(g^2)$

Figure 77 presents such a kind of (type IV) pencil. All conics of the pencil are tangent to the lines d, e at their intersections with line g. The two conics $c_1 = d \cdot e$ and $c_2 = g^2$ are degenerate members of the pencil.

10.2. *Parabola by 1 point, 1 tangent-at, axis-direction* $(3P_1 2T_1)_1$. Construct a conic passing through three points $A, B, [C]$, tangent to line d at A and also tangent to the line at infinity e. Thus, the conic is a parabola with axis parallel to the direction $[C]$. Project B parallel to d on AC to M and extend BM to the double to find B', which is on the parabola (this map $B \mapsto B'$ is the *affine reflection* with *axis* AC and *conjugate direction* d ([5, p. 203])). Define N to be the symmetric

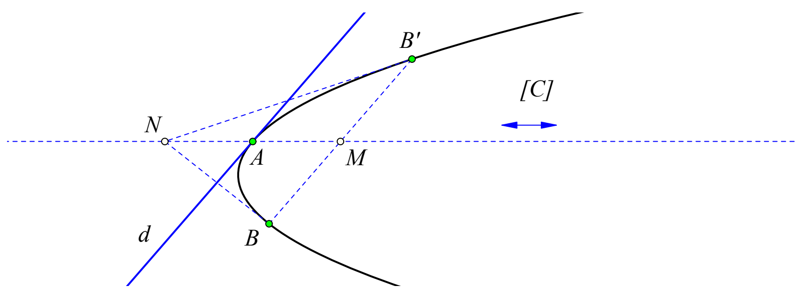

Figure 78. Parabola through A, B, tangent to d and axis parallel to $[C]$

of M with respect to A (see Figure 78). Since BN is tangent at B to the parabola, we can construct arbitrary many points of the parabola by repeating this procedure. There is always one solution.

10.3. *Hyperbola 1 point 1 tangent-at 1 asymptote* $(3P_1 2T)_{1i}$. Construct a conic passing through points $A, B, [C]$ and tangent to lines $d \ni [C], e \ni B$. This is a hyperbola with asymptote d. A slight variation of the solution in §10.1, leading to the determination of the other asymptote and the center of the hyperbola, is as follows. Let $F = (d, e)$. Then the symmetric F_1 of F with respect to B is on the other asymptote d' of the hyperbola (see Figure 79). Draw also the parallel to d from A intersecting e at D. The symmetric D' of D with respect to A defines the polar BD' of D. The intersection point $E = (BD', d)$ is the pol of line AD. Hence AE is the tangent at A and the symmetric E_1 of E with respect to A defines

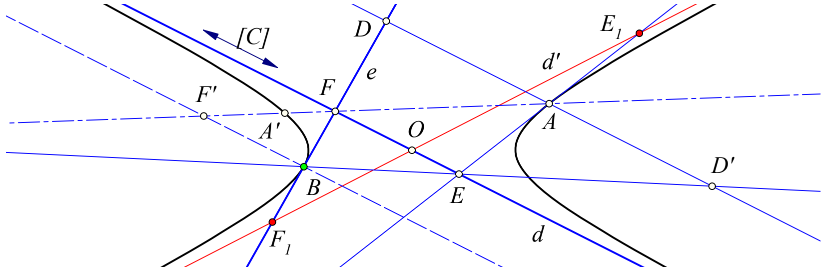

Figure 79. Hyperbola through A, asymptote d and tangent to e at B

a point on the other asymptote d' of the hyperbola. Thus, the other asymptote d' can be constructed to pass from the two points F_1, E_1. The intersection point O of the two asymptotes defines the center of the hyperbola and by the symmetry with respect to O we can find more points on the conic. An additional point on the conic is also A', constructed by first drawing the parallel from B to d. This parallel is the polar of F and if F' is its intersection point with AF, then the harmonic conjugate of A with respect to F, F' is on the conic. There is always one solution.

10.4. *Hyperbola from two asymptotes and a point* $(3P_2 2T)_{2i}$. Construct of a conic passing through three points $A, [B], [C]$ and tangent to two lines $d \ni B, e \ni C$. This is a hyperbola with asymptotes the lines d, e passing through a point A. This

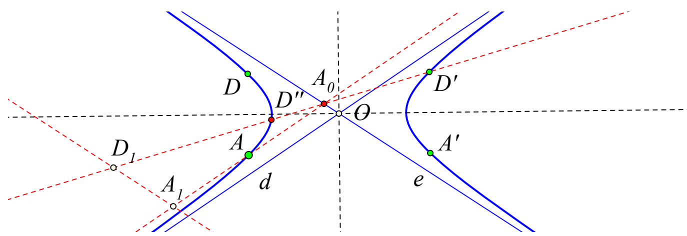

Figure 80. Hyperbola with given asymptotes d, e and passing through A

can be done by determining the successive symmetrics D, D', A' of A with respect to the axes and a fifth additional point D'' easily constructible from the data (see Figure 80). In fact, draw a parallel to the asymptote d intersecting the other asymptote in A_0. The polar of A_0 is the line parallel to e, such that its intersection A_1 with AA_0 is the symmetric of A_0 with respect to A. Consider the intersection D_1 of that polar with line A_0D'. The harmonic conjugate D'' of D' with respect to (A_0, D_1) is on the conic and coincides with the middle of D_1D'. Note the A_0 divides $D'D''$ in ratio $(2:1)$. There is always a unique solution.

11. Two points three tangents one coincidence

11.1. *Conic by 1 point, 1 tangent-at, 2 tangents* $(2P3T)_1$. Construct a conic passing through two points D, E and tangent to three lines $a \ni D, b, c$. The solution can be given by applying a special case of the dual of Desargues' theorem, referred to also as Plücker's theorem ([4, p. 25], [2, p. 202, II]). This case concerns the one-parameter pencil $\mathcal{D}$ of all conics, which are tangent to line a at D and also tangent to two lines b, c (see Figure 84 below in this section). If E is another, arbi-

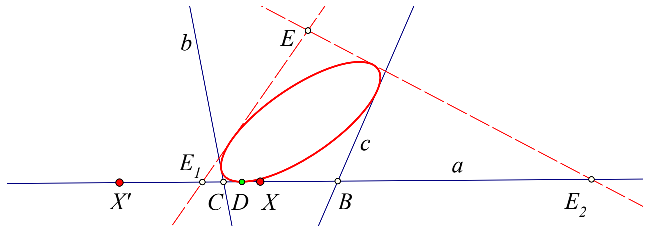

Figure 81. Conics tangent a, b, c, passing through $D \in a, E$

trary, but fixed point, not lying on any of a, b, c, Desargues' theorem asserts, that the pairs of tangents to these conics from E define an involution on the pencil E^* of lines through E. By intersecting the rays of this pencil with a line, such as a, we can represent this involution through one which permutes the points of that line. Thus, the tangents from E to an arbitrary conic of that pencil intersect line a at a pair of points (E_1, E_2), related by this involution (see Figure 81). The requested conics are those, which pass through E and their tangents at E pass through the fixed points X, X' of this involution. In order to construct these points it suffices to find two easily constructible pairs of points in involution on a. One such pair consists of the points $C = (a, b), B = (a, c)$. Another pair is found by drawing

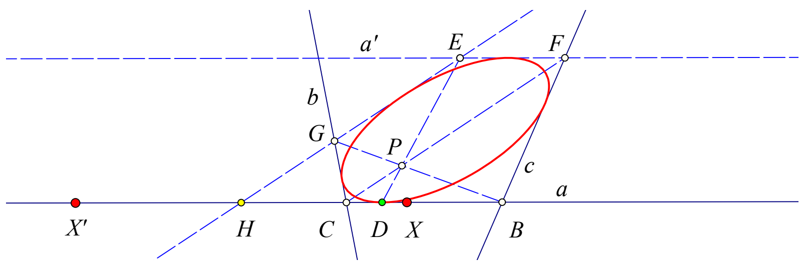

Figure 82. A particular conic tangent to a, b, c, passing through $D \in a$

the parallel a' to a through E, intersecting c at F (see Figure 82). The second tangent from E to the conic inscribed in the quadrilateral with sides a', b, a, c and passing through D, can be found by applying Brianchon's theorem to the pentagon $CBFEG$. This theorem guarantees that lines DE, CF, BG pass through a common point P. Thus, P is constructed by intersecting DE with CF and G is found as the intersection $G = (PB, b)$. In this case the two tangents from E are EG, EF and consequently H corresponds by the involution to the point at infinity of line a. It follows that the fixed points X, X' of the involution are common harmonics of pairs $(H, [a])$ and (B, C). Once the tangents at E are found, each one of the two conics can be constructed by locating one more point on it and applying the recipe of §10.1 using Brianchon's theorem. There are two solutions if points D, E

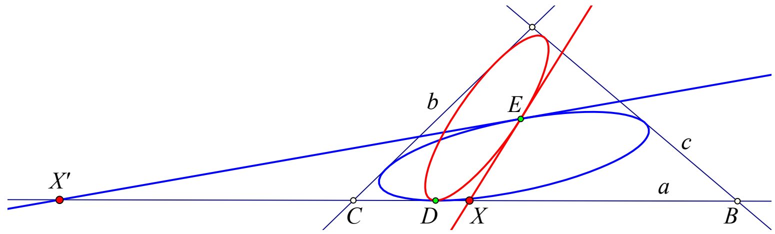

Figure 83. The two conics tangent to a, b, c, passing through $D \in a$, E

are in the same angular domain defined by lines b, c or they are in opposite angular domains. In all other cases there are no solutions. This is visible also in Figure 84 ,

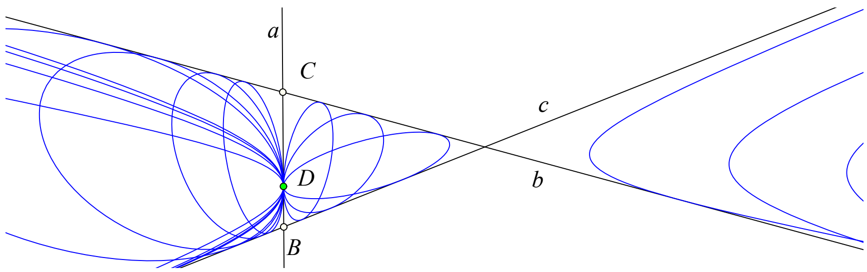

Figure 84. A pencil of conics tangent to a, b, c and passing through $D \in a$

which displays a pencil of conics tangent to a, b, c and passing through $D \in a$. When D is on the exterior of segment BC, then the conics are all located in the angular domain of b, c containing D and its opposite ([2, II, p. 201]).

An alternative solution of the problem is the following. Consider the triangle with sides a, b, c and the conic k passing through its vertices and tangent to EB, EC at B, C respectively, which is a construction of the type $(3P2T)_2$ of

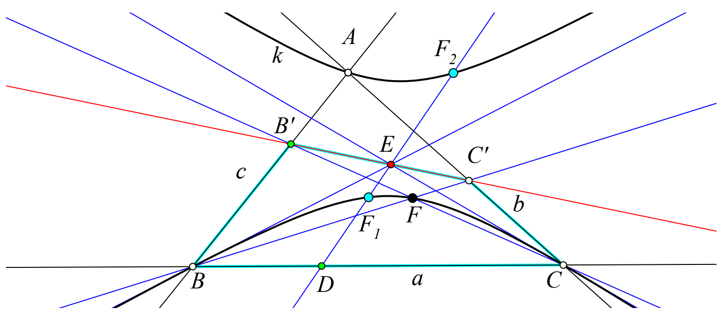

Figure 85. The locus of $F = (B'C, C'B)$ for $B'C'$ passing through E

§10.1 (see Figure 85). It is easy to see, using Maclaurin's theorem ([1, p. 77], [18, p. 230]), that this conic is the geometric locus of points F, which are intersections of diagonals of quadrilaterals $B'BCC'$ with $B'C'$ passing through E. From Brianchon's theorem follows that if $B'C'$ were the tangent at E to our requested conic, then the diagonals $B'C, C'B$ would intersect on line DE. Thus, their intersection point F must coincide with the intersection points F_1, F_2 of line DE with the conic k. Having these two points, the construction of the two tangents at E is immediate and the rest, of the construction of conics, goes as before.

11.2. *Parabola by 1 point, 1 tangent, 1 tangent-at* $(2P3T_1)_1$. Construct a conic passing through two points A, B, tangent to two lines $c \ni A, d$ and tangent also to the line at infinity, thus a parabola. By Desargues' theorem, applied as in §11.1,

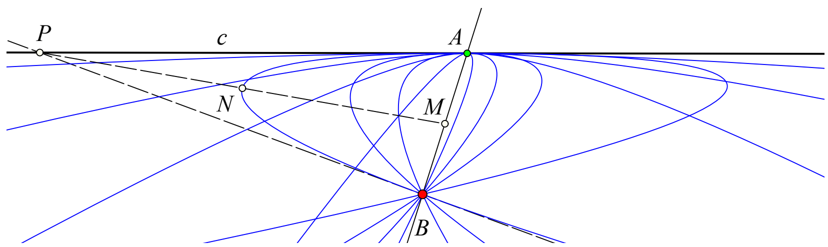

Figure 86. Parabolas through A, B and tangent to c at A

the tangents at B to the members of the pencil $\mathcal{D}$ (see Figure 86) of all conics tangent to c at A, passing through B and also tangent to d and the line at infinity (thus parabolas), define an involution on the pencil B^* of all lines through B. The

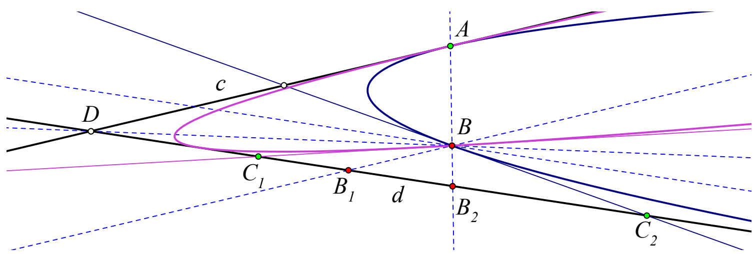

Figure 87. Parabolas through A, B and tangent to $c \ni A, d$

tangents at B to the requested parabolas are the fixed elements of this involution. We can represent this involution through points on the line d, by corresponding to each ray through B its intersection point with d. There are two particular degenerate parabolas of this pencil, coinciding with the lines parallel to c, d through B. The parallel to d defines the pair of corresponding points $(B_2, [d])$, where $B_2 = (AB, d)$. The parallel to c defines the pair of corresponding points (B_1, D), where $B_1 = (B[c], d)$ and $D = (c, d)$. The rays through B representing the fixed elements of the involution in B^* are BC_1, BC_2, where C_1, C_2 are the common harmonics of the pairs $(B_1, D), (B_2, [d])$. Once the tangents through B are located, the parabolas are constructed as in the next section. There are two solutions if points A, B are not separated by line d and no solution if they are.

11.3. *Parabola by 1 point, 2 tangents, axis-direction* $(2P_1 3T_1)$. Construct a conic passing through two points $A, [B]$, tangent to two lines c, d and also tangent to the line at infinity, thus a parabola. Point $[B]$ determines the direction of the axis of

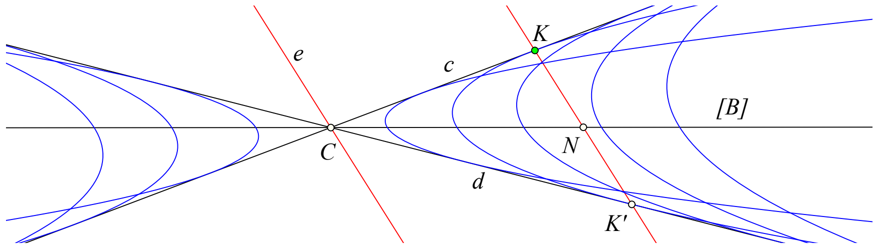

Figure 88. A pencil of parabolas tangent to c, d and axis direction $[B]$

the parabola. The pencil of parabolas tangent to c, d with axis direction $[B]$ can be easily constructed by taking the harmonic conjugate e of CB with respect to c, d. This is namely the direction of chords bisected by CB, where $C = (c, d)$. Having that direction, we can define a second point A' on the requested parabola. This is the result of the affine reflexion on CB parallel to e (see Figure 88). The rest of the construction is thus reducible to that of §9.3, which gives either two solutions or none, if point A is outside of the angular domains determined by c, d, which contain the parallel CB to the given direction $[B]$.

Another, computational, method to locate the two parabolas could be the one using the equation of the parabola with respect to the axes e, CB (see Figure 89).

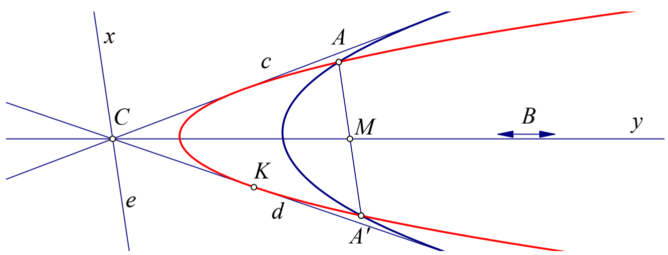

Figure 89. The two parabolas through A, tangent to c, d and axis direction $[B]$

In these axes the equation of the parabola has the form

$$y = \alpha x^2 + \beta,$$

and constants α, $\beta$are easily determined by the data. In fact, the given point A has known coordinates (x_1, y_1) with respect to these axes and the coordinates (x_2, y_2) of the contact point K with c satisfy $y_2 = 2\beta$ and $\frac{y_2}{x_2} = \lambda$, later being a constant determined by the data. It turns out that α, βsatisfy the two equations

$$y_1 = \alpha x_1^2 + \beta, \quad \text{and} \quad \alpha\beta = \frac{\lambda^2}{4},$$

which determine the same solutions under the same conditions as before.

11.4. *Hyperbola by 1 point, 1 asymptote, 2 tangents* $(2P_13T)_i$. Construct a conic passing through two points $A, [B]$, tangent to three lines $c, d, e \ni B$. This is a

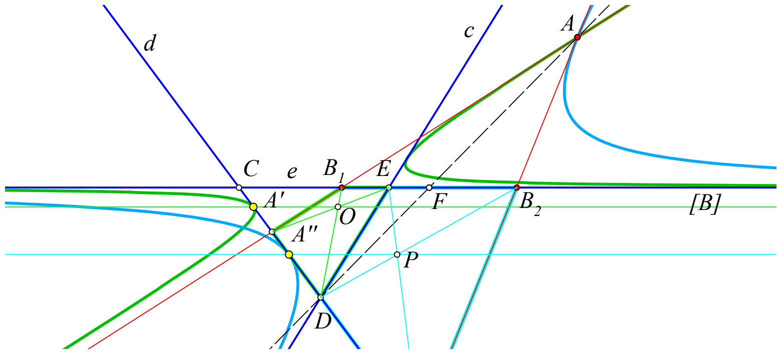

Figure 90. The two hyperbolas through A with asymptote e and tangent to c, d

hyperbola with an asymptote e. By the method of §11.1, the tangents at A of the requested conics are determined by the common harmonics B_1, B_2 of the two pairs of points (C, E) and $(F, [B])$, where $C = (e, d), E = (e, c), D = (c, d), F = (AD, e)$ (see Figure 90). If O is a diagonal point of the quadrilateral formed by the three tangents c, d, e, AB_1, then, by Brianchon's theorem, the intersection point A' of d with the parallel to e through O will be the contact point of d with the conic. The problem reduces then to the construction of the conic tangent at $A' \in d, A \in AB_1$ and passing through A, which is $(3P2T)_2$ of §10.1. Analogous properties hold for the other conic with tangent at A the line AB_2. Fixing the lines c, d, there are two solutions if $A, [B]$ are in the same or opposite angular domains defined by c, d. Otherwise there are no solutions.

11.5. *Hyperbola 1 asymptote 1 asymptotic 2 tangents* $(2P_23T)_i$. Construct a conic passing through two points $[A], [B]$ and tangent to three lines: a at $[A], b, c$. This is a hyperbola with one asymptote a, the other asymptotic direction $[B]$ and two other tangents b, c. The problem is projectively equivalent to $(2P3T)_1$ of §11.1. Here

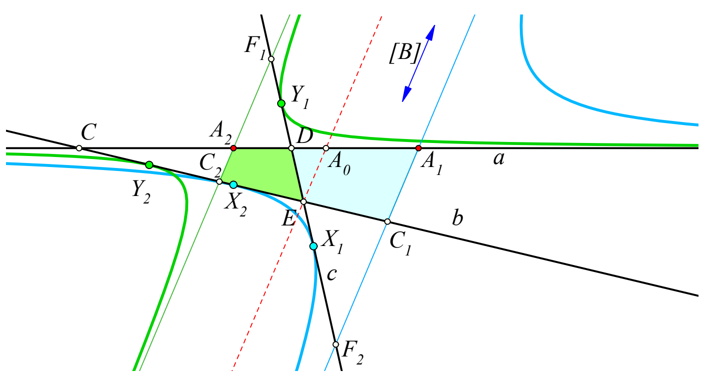

Figure 91. The two hyperbolas with asymptote a, asymptotic $[B]$ and tangents b, c

again the recipe is essentially the one of §11.1, with some simplifications allowing for a faster determination of the conics. In Figure 91, displaying the conics, $A_0 = (a, EB)$ and points A_1, A_2 are the common harmonics of pairs $(C, D), (A_0, [A])$. Parallels to $[B]$ define the two quadrilaterals DA_2C_2E, DA_1C_1E. Each quadrilateral determines a conic tangent to its sides. In this case the contact points of the conics with the sides of the quadrilateral are easily determined. In fact, by the well known property of segments intercepted between asymptotes follows, that the contact points X_1, X_2 are respectively the middles of DF_2, CC_1 and the contact points Y_1, Y_2 are the middles of DF_1, CC_2. There are two solutions if, drawing parallels from a point to b, c and to $[B]$, later does not fall between the two first. Otherwise there are no solutions.

12. Two points three tangents two coincidences

12.1. *Conic by two tangents-at and a tangent* $(2P3T)_2$. Construct a conic tangent to three lines a, b, c, passing through two points A, B with $A \in a$ and $B \in b$. In this case the contact point C of the requested conic with the third line is easily

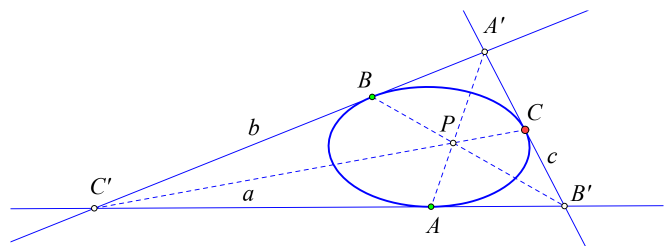

Figure 92. Conic tangent to a, b, c at $A \in a, B \in b$

constructed, since all lines joininig the vertices of the triangle formed by the three lines to the opposite contact point pass through the same point P, the *perspector* of the conic with respect to that triangle $A'B'C'$ (see Figure 92). There is always a unique solution.

12.2. *Parabola by 2 tangents-at* $(2P3T_1)_2$. Construct a conic passing through two points A, B, tangent to two lines $c \ni A, d \ni B$ and also tangent to the line at infinity, thus a parabola. If point $C = (a, b)$, taking the middle D of AB and

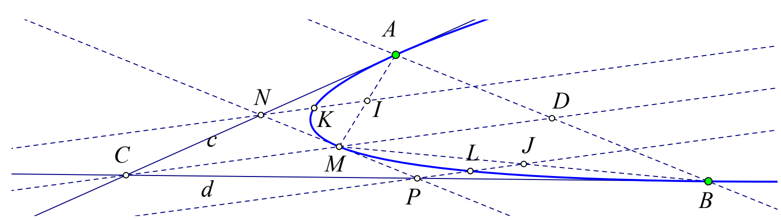

Figure 93. Parabola through A, B and tangent to $a \ni A, b \ni B$

the middle M of CD we construct a new point on the parabola. Analogously are obtained new points K, L from the middles of MA, MB respectively. The parabola is led as a conic through the five points A, B, M, K, L. There is always one solution identified with one first-kind *Artzt parabola* of triangle ABC ([13, p. 518]). Triangle ABC is referred by times as an *Archimedes triangle* ([8, p. 239]).

12.3. *Parabola 1 tangent 1 tangent-at, axis-direction* $(2P_1 3T_1)_1$. Construct a conic passing through two points $A, [B]$, tangent to two lines $c \ni A, d$ and tangent also to the line at infinity, thus, a parabola with axis parallel to $[B]$. One solution is to

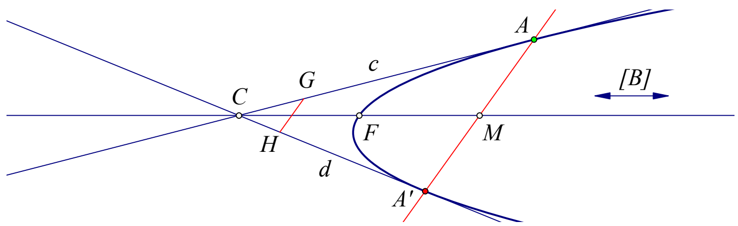

Figure 94. The parabola tangent to d and c at A and axis direction $[B]$

construct, as in the previous section, the direction GH of chords of the parabola, which are bisected by CB. Then, find the point A' on the parabola, such that AA' is parallel to GH and bisected by CB. Point A' is the contact point of the parabola with d and the construction reduces to that of $(2P3T_1)_2$ of §12.2. There is always a unique solution.

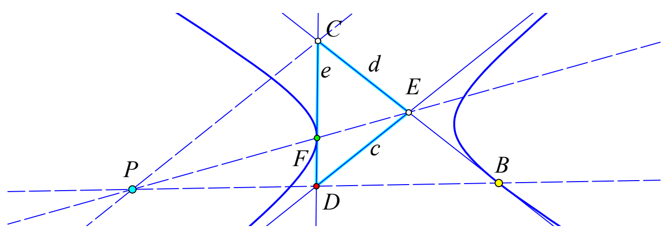

Figure 95. Hyperbola with asymptote c tangent to $d \ni B$ and tangent to e

12.4. *Hyperbola 1 asymptote 1 tangent 1 tangent-at* $(2P_1 3T)_{1i}$. Construct a conic passing through two points $[A], B$ and tangent to three lines $c \ni A, d \ni B, e$. This is a hyperbola with one asymptote c, tangent to d at B and also tangent to e. The triangle CDE with sides the tangents c, d, e is known and the perspector P of the conic, tangent to the sides of this triangle, can be found (see Figure 95). In fact, draw from $C = (d, e)$ parallel to the asymptote c and find its intersection P with BD, where $D = (c, e)$. If $E = (c, d)$, then line PE passes through the contact point F of e with the conic. The case, as the one of §12.1, has always a unique solution.

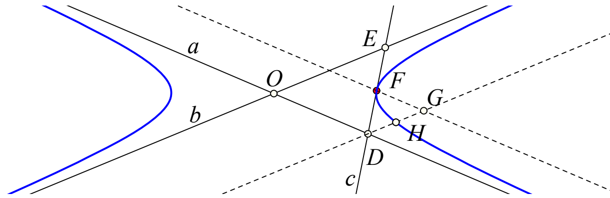

Figure 96. Hyperbola with asymptotes a, b and tangent c

12.5. *Hyperbola 2 asymptotes 1 tangent* $(2P_2 3T)_{2i}$. Construct a conic passing through two points $[A], [B]$ and tangent to three lines $a \ni A, b \ni B, c$. This is a hyperbola with given asymptotes a, b and a tangent c. This is an easy case, since the contact point of the tangent c is the middle F of DE, where $D = (a, c), E = (b, c)$. The parallel to a from F is the polar of D and the parallel to the other asymptote b from D intersects the first parallel at G. The middle H of DG is a point of the hyperbola. An analogous point can be constructed starting with E. Taking the symmetrics with respect to the center $O = (a, b)$ of the hyperbola we have enough points to define the conic through five points. There is always a unique solution.

13. One point four tangents one coincidence

13.1. *Conic by one tangent-at and three tangents* $(1P4T)_1$. Construct a conic tangent to a given line a at a given point D and also tangent to three other lines b, c, e. The basic underlying structure results from Brianchon's theorem. In fact, consider the intersection point O of the diagonals of the quadrilateral $BCGF$ defined by the

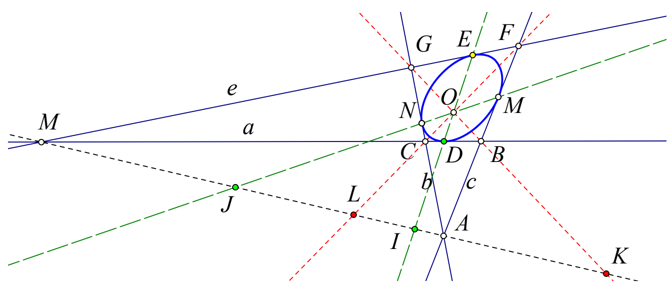

Figure 97. Conic tangent to a, b, c, e, passing through $D \in a$

four lines (see Figure 97). According to Brianchon's theorem, the lines joining opposite contact points DE, MN intersect also at O. Thus, point E is constructible from the given data. Further, if $K = (b, c), M = (a, e), I = (AM, DE)$, the line MN of the other two contact points defines point $J = (MN, AM)$, such that $(KLIJ) = -1$. This allows the determination of J and from this the points M, N, by intersecting line JO with the sides b, c. The problem is thus reducible to $(4P1T)_1$ of §8.1 and has one solution. The pencil involved here is the one of conics tangent to a at D and also tangent to b, c, appearing also in $(2P3T)_1$ of §11.1.

Remark. Besides quadrangle $BCGF$, the complete quadrilateral, defined by lines a, b, c, e, contains also the quadrangles $AGMB$, whose diagonals intersect at L and $CMFA$, whose diagonals intersect at K. It is also easily seen, that the contact points N, E, M are the harmonic associates of D with respect to the diagonal triangle OLK of the complete quadrilateral. Thus the definition of N, E, M from D, does not depend on which one of the three quadrangles (and corresponding intersection of diagonals O, K or L) we select to work with.

13.2. *Parabola by 1 tangent-at, 2 tangents* $(1P4T_1)_1$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to line a at $E \in a$ and tangent to two lines b, c. Here the construction is somewhat simpler than that of the previous case,

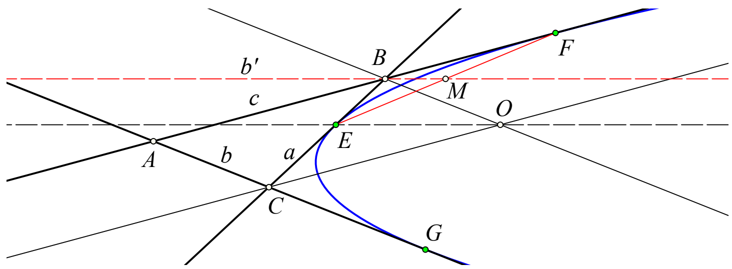

Figure 98. Parabola tangent to a at E and tangent to b, c

because of the nice properties of parabolas. In fact, let O be the intersection of the parallels from B to b and from C to c (see Figure 98). Then the line EO is parallel to the axis of the parabola. Having the direction of the axis, we can construct more points on the parabola using the method of $(2P_13T_1)$ of §11.3. Using this we can find the direction of the chords bisected by the parallel to the axis from B and determine the contact point F with line c. Analogously we can find the contact point G of b, and from these points by similar methods find other arbitrary many points on the parabola. Alternatively, we can use the fact that points $\{F, G, O\}$ are collinear and the line d, carrying them, intersects line BC at the harmonic conjugate $E' = E(B, C)$. There is always a unique solution.

13.3. *Parabola by axis-direction, 3 tangents* $(1P_14T_1)$. Construct a conic tangent to the line at infinity, i.e. a parabola, tangent to three lines a, b, c and passing through $[D]$, i.e. with given axis-direction. This is a case similar to the previous one. Again we construct the intersection point O of the parallels to b, c from the

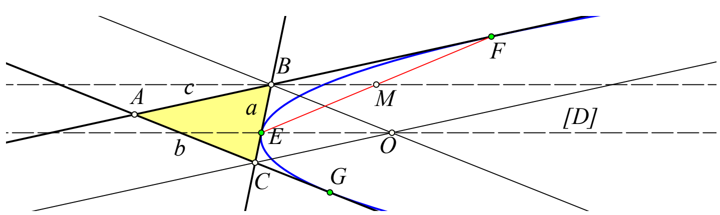

Figure 99. Parabola tangent to a, b, c with given axis-direction

opposite vertices. The line through O, parallel to the given axis-direction, determines now on a the contact point E with the parabola. From there the construction of the other contact points F, G with sides c, b and the completion of the parabola construction is the one described in the previous section. There is always a unique solution.

13.4. *Hyperbola, 1 asymptote, 3 tangents* $(1P_1 4T)_i$. Construct a conic tangent to line e at its point at infinity $[E]$, i.e. a hyperbola with an asymptote e, and tangent to three lines a, b, c. In analogy to §13.1 we can find the contact points of the

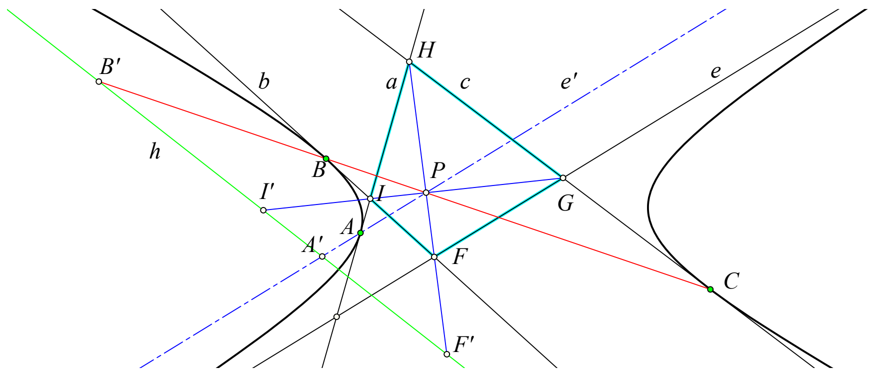

Figure 100. Hyperbola tangent to a, b, c with asymptote e

three tangents with the hyperbola. In fact, consider the intersection point P of the diagonals of quadrilateral $FGHI$, whose all sides are given and are tangents to the hyperbola. By Brianchon's theorem, the line e' parallel to the asymptote e from point P will intersect the side a of the quadrilateral at its contact point A with the hyperbola (see Figure 100). Consider now an arbitrary line h and its intersection points $A' = (e', h), I' = (IG, h), F' = (FH, h)$. The other chord of contact-points BC will intersect line h at the harmonic conjugate B' of A' with respect to (I', F'). Thus, B' is constructible from the given data, and drawing PB' we determine the positions B, C of the contact points on the tangents b, c respectively. Having one asymptote and the contact points on the tangents, we can determine the other asymptote, the center, and, by symmetry to that center, three more points on the conic. The method is described already in $(3P_1 2T)_{1i}$ of §10.3. There is always one solution.

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