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On Two Triads of Triangles Associated With the Perpendicular Bisectors of the Sides of a Triangle

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Abstract. We discover some properties of the triangle centers related to the two triads of triangles associated with the perpendicular bisectors of the sides of a triangle.

1. Introduction

Given a triangle $\mathbf{T} := ABC$, let the perpendicular bisector of the side BC intersect the sidelines AC and AB at B_c and C_b respectively. Define C_a, A_c, A_b, B_a similarly. In this paper we study the two triads of triangles
(i) $\mathbf{T}_a := AA_bA_c$, $\mathbf{T}_b := B_aBB_c$, $\mathbf{T}_c := C_aC_bC$ (see Figure 1a) and
(ii) $\mathbf{T}'_a := AB_aC_a$, $\mathbf{T}'_b := A_bBC_b$, $\mathbf{T}'_c := A_cB_cC$ (see Figure 1b).

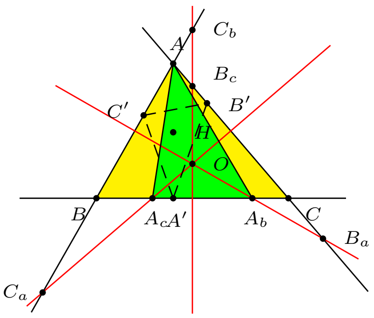

Figure 1a. $\mathbf{T}_a$ and orthic triangle

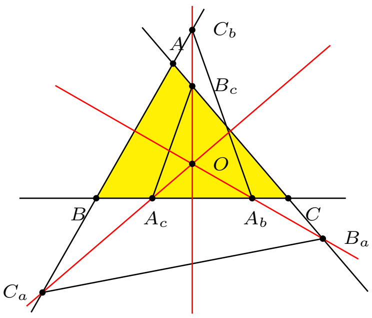

Figure 1b. The triangles $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$

Homogeneous barycentric coordinates are used throughout this work. With the usual notations in triangle geometry, a, b, c for the lengths of the sides BC, CA, AB, and

$$S_A = \frac{b^2+c^2-a^2}{2}, \quad S_B = \frac{c^2+a^2-b^2}{2}, \qquad S_C = \frac{a^2+b^2-c^2}{2},$$

the points on the perpendicular bisectors are

$$\begin{array}{ll} A_b = (0 : -S_A + S_B : S_A + S_B), & A_c = (0 : S_C + S_A : S_C - S_A); \\ B_c = (S_B + S_C : 0 : -S_B + S_C), & B_a = (S_A - S_B : 0 : S_A + S_B); \\ C_a = (-S_C + S_A : S_C + S_A : 0), & C_b = (S_B + S_C : S_B - S_C : 0). \end{array}$$

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Denote by $\mathbf{T}'$ the orthic triangle of ABC. Its vertices are

$$A' = (0 : S_C : S_B), \qquad B' = (S_C : 0 : S_A), \qquad C' = (S_B : S_A : 0).$$

The sidelengths a', b', c' of $B'C'$, $C'A'$, $A'B'$ of the orthic triangle are given by

$$a'^2 = \frac{a^2 S_{AA}}{b^2c^2}, \qquad b'^2 = \frac{b^2 S_{BB}}{c^2a^2}, \qquad c'^2 = \frac{c^2 S_{CC}}{a^2b^2}.$$

From these,

$$a'^2 : b'^2 : c'^2 = a^4 S_{AA} : \ b^4 S_{BB} : \ c^4 S_{CC}.$$

Corresponding to S_A, S_B, S_C, we have

$$\begin{aligned}
S'_A :=& \ \frac{1}{2}(b'^2 + c'^2 - a'^2) = \frac{S_{ABC}}{a^2b^2c^2} \cdot \frac{S^2 - S_{AA}}{S_A},\\
S'_B :=& \ \frac{1}{2}(c'^2 + a'^2 - b'^2) = \frac{S_{ABC}}{a^2b^2c^2} \cdot \frac{S^2 - S_{BB}}{S_B},\\
S'_C :=& \ \frac{1}{2}(a'^2 + b'^2 - c'^2) = \frac{S_{ABC}}{a^2b^2c^2} \cdot \frac{S^2 - S_{CC}}{S_C},
\end{aligned}$$

and

$$S'_A : S'_B : S'_C = S_{BC}(S^2 - S_{AA}) : S_{CA}(S^2 - S_{BB}) : S_{AB}(S^2 - S_{CC}).$$

Lemma 1. (a) *The triangles AA_bA_c, B_aBB_c, and C_aC_bC are all similar to* $\mathbf{T}'$.
(b) *The triangles AB_aC_a, A_bBC_b, and A_cB_cC are all similar to* $\mathbf{T}$.

2. A common point of circumcircles

Theorem 2. *The circumcircles of triangles in the two triads $(\mathbf{T}_a, \mathbf{T}_b, \mathbf{T}_c)$ and $(\mathbf{T}'_a, \mathbf{T}'_b, \mathbf{T}'_c)$ all contain the Euler reflection point*

$$E = \left(\frac{a^2}{b^2 - c^2} : \ \frac{b^2}{c^2 - a^2} : \ \frac{c^2}{a^2 - b^2}\right)$$

of the circumcircle of $\mathbf{T}$.

Proof. We compute the coordinates of E with respect to these triangles, and show from these coordinates that E lies on the circumcircle of each. With respect to $\mathbf{T}$,

$$E = ((S_B{+}S_C)(S_C{-}S_A)(S_A{-}S_B) : (S_B{-}S_C)(S_C{+}S_A)(S_A{-}S_B) : (S_B{-}S_C)(S_C{-}S_A)(S_A{+}S_B))$$

with coordinate sum $\sigma = a^2S_{AA} + b^2S_{BB} + c^2S_{CC} - 6S_{ABC}$.

The coordinates of E in triangle $\mathbf{T}_a = AA_bA_c$ are

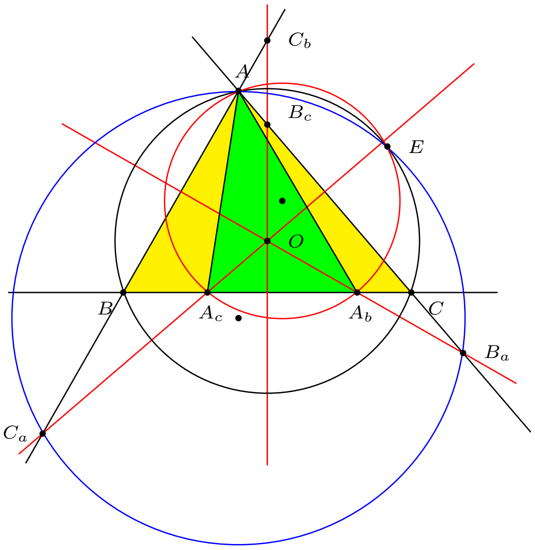

Figure 2

$$
\begin{aligned}
&\text{Area}(EA_bA_c) : \text{area}(AEA_c) : \text{area}(AA_bE)\\
=&\frac{1}{4\sigma S_{BC}}\begin{vmatrix}(S_B+S_C)(S_C-S_A)(S_A-S_B) & (S_B-S_C)(S_C+S_A)(S_A-S_B) & (S_B-S_C)(S_C-S_A)(S_A+S_B)\\ 0 & -(S_A-S_B) & S_A+S_B\\ 0 & S_C+S_A & S_C-S_A\end{vmatrix}\\
:&\frac{1}{2\sigma S_C}\begin{vmatrix}1 & 0 & 0\\ (S_B+S_C)(S_C-S_A)(S_A-S_B) & (S_B-S_C)(S_C+S_A)(S_A-S_B) & (S_B-S_C)(S_C-S_A)(S_A+S_B)\\ 0 & S_C+S_A & S_C-S_A\end{vmatrix}\\
:&\frac{1}{2\sigma S_B}\begin{vmatrix}1 & 0 & 0\\ 0 & -(S_A-S_B) & S_A+S_B\\ (S_B+S_C)(S_C-S_A)(S_A-S_B) & (S_B-S_C)(S_C+S_A)(S_A-S_B) & (S_B-S_C)(S_C-S_A)(S_A+S_B)\end{vmatrix}\\
=&\frac{S_A(S_C-S_A)(S_A-S_B)(S_B+S_C)^2}{2\sigma S_{BC}} : \frac{S_B(S_B-S_C)(S_C-S_A)(S_C+S_A)}{\sigma S_C} : \frac{S_C(S_A-S_B)(S_B-S_C)(S_A+S_B)}{\sigma S_B}\\
=&\frac{S_A(S_B+S_C)^2}{2(S_B-S_C)} : \frac{S_{BB}(S_C+S_A)}{S_A-S_B} : \frac{S_{CC}(S_A+S_B)}{S_C-S_A}\\
=&\frac{S_{AA}(S_B+S_C)^2}{2S_A(S_B-S_C)} : \frac{S_{BB}(S_C+S_A)^2}{(S_C+S_A)(S_A-S_B)} : \frac{S_{CC}(S_A+S_B)^2}{(S_A+S_B)(S_C-S_A)}\\
=&\frac{a'^2}{2S_A(S_B-S_C)} : \frac{b'^2}{(S_C+S_A)(S_A-S_B)} : \frac{c'^2}{(S_A+S_B)(S_C-S_A)}
\end{aligned}
$$

This is the isogonal conjugate (in triangle $\mathbf{T}_a = AA_bA_c$) of the point

$$(2S_A(S_B-S_C) : (S_C+S_A)(S_A-S_B) : (S_A+S_B)(S_C-S_A)),$$

which is an infinite point since the coordinate sum

$$2S_A(S_B-S_C) + (S_C+S_A)(S_A-S_B) + (S_A+S_B)(S_C-S_A) = 0.$$

This shows that E is on the circumcircle of triangle $\mathbf{T}_a$. Similarly, E is also on the circumcircles of the triangles $\mathbf{T_b}$ and $\mathbf{T}_c$, with coordinates

$$\left(\frac{a'^2}{(S_B+S_C)(S_A-S_B)}:\frac{b'^2}{2S_B(S_C-S_A)}:\frac{c'^2}{(S_A+S_B)(S_B-S_C)}\right)$$

and

$$\left(\frac{a'^2}{(S_B+S_C)(S_C-S_A)}:\frac{b'^2}{(S_C+S_A)(S_B-S_C)}:\frac{c'^2}{2S_C(S_A-S_B)}\right)$$

respectively. A similar calculation shows that E is also on the circumcircles of triangles $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$, with coordinates

$$\left(\frac{a^2}{2S_A(S_B-S_C)}:\frac{b^2}{(S_C+S_A)(S_A-S_B)}:\frac{c^2}{(S_A+S_B)(S_C-S_A)}\right),$$
$$\left(\frac{a^2}{(S_B+S_C)(S_A-S_B)}:\frac{b^2}{2S_B(S_C-S_A)}:\frac{c^2}{(S_A+S_B)(S_B-S_C)}\right),$$
$$\left(\frac{a^2}{(S_B+S_C)(S_C-S_A)}:\frac{b^2}{(S_C+S_A)(S_B-S_C)}:\frac{c^2}{2S_C(S_A-S_B)}\right)$$

respectively in these triangles. □

3. Counterparts of a point in the triad $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$

Let P be a point with homogeneous barycentric coordinates $(x:y:z)$ with respect to triangle $\mathbf{T}=ABC$. The counterparts of P in the triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ are the points A_P, B_P, C_P which have the same coordinates $(x:y:z)$ in these triangles. In homogeneous barycentric coordinates,

$$\begin{aligned}
A_P &= (2S_{BC}x:\ S_C(-S_A+S_B)y+S_B(S_C+S_A)z:\ S_C(S_A+S_B)y+S_B(S_C-S_A)z),\\
B_P &= (S_A(S_B+S_C)z+S_C(S_A-S_B)x:\ 2S_{CA}y:\ S_A(-S_B+S_C)z+S_C(S_A+S_B)x),\\
C_P &= (S_B(-S_C+S_A)x+S_A(S_B+S_C)y:\ S_B(S_C+S_A)x+S_A(S_B-S_C)y:\ 2S_{AB}z)
\end{aligned}$$

Denote by $\mathbf{T}(P)$ the triangle $A_PB_PC_P$. Basic properties of $\mathbf{T}(P)$ can be found in [1].

Theorem 3 (Bui)**.** *The triangle* $\mathbf{T}(P)$ *is*
(a) *oppositely similar to* $\mathbf{T}$*,*
(b) *orthologic to* $\mathbf{T}$*,*
(c) *perspective with* $\mathbf{T}$ *if and only if* P *lies on the Jerabek hyperbola. In this case, the perspector traverses the Euler line.*

If $P=(x:y:z)$, the perpendiculars from A to B_PC_P, B to C_PA_P, C to A_PB_P concur at

$$Q=\left(\frac{1}{-b^2c^2S_{BC}x+c^2S_AS_{CC}y+b^2S_AS_{BB}z}:\cdots:\cdots\right),$$

which lies on the circumcircle of $\mathbf{T}$ (see Figure 3).

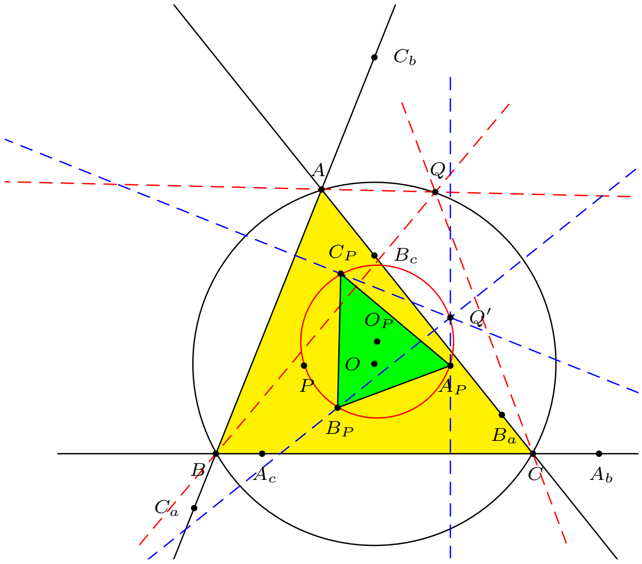

Figure 3.

On the other hand, the perpendiculars from A_P to BC, B_P to CA, C_P to AB concur at

$$\begin{aligned} Q' = \; & (S_{BC}(a^2(S_{AA}-S_{BC})-S_A(S_{BB}+S_{CC})x+c^2a^2S_AS_{CC}y+a^2b^2S_AS_{BB}z \\ & :\cdots:\cdots). \end{aligned}$$

According to [1], the coordinates of Q' with respect to $\mathbf{T}(P)$ are the same as those of Q with respect to $\mathbf{T}$. It follows that Q' is a point on the circumcircle of $\mathbf{T}(P)$.

Proposition 4. *The triangle $\mathbf{T}(P)$ has orthocenter O.*

Proof. The sum of the coordinates of A_P given at the beginning of the present section is $S_{BC}(x+y+z)$; similarly for B_P and C_P. The orthocenter of $\mathbf{T}(P)$ has coordinates

$$\begin{aligned} & (2S_{BC}x,\ S_C(-S_A+S_B)y+S_B(S_C+S_A)z,\ S_C(S_A+S_B)y+S_B(S_C-S_A)z), \\ & +\ (S_A(S_B+S_C)z+S_C(S_A-S_B)x,\ 2S_{CA}y,\ S_A(-S_B+S_C)z+S_C(S_A+S_B)x), \\ & +\ (S_B(-S_C+S_A)x+S_A(S_B+S_C)y,\ S_B(S_C+S_A)x+S_A(S_B-S_C)y,\ 2S_{AB}z) \\ = \; & (x+y+z)(S_A(S_B+S_C),\ S_B(S_C+S_A),\ S_C(S_A+S_B)). \end{aligned}$$

This is the circumcenter O of $\mathbf{T}$. □

Theorem 5. *The points A_P, B_P, C_P and P are concyclic.*

Proof. The coordinates of P with respect to $\mathbf{T}(P) = A_PB_PC_P$ are

$$\begin{aligned}
&\text{Area}(PB_PC_P) : \text{ area}(A_PPC_P) : \text{ area}(A_PB_PP)\\
=&\frac{(b^2S_Bx - a^2S_Ay)(a^2S_Az - c^2S_Cx)}{4S_{ABC}S_A(x+y+z)^2} : \frac{(c^2S_Cy - b^2S_Bz)(b^2S_Bx - a^2S_Ay)}{4S_{ABC}S_B(x+y+z)^2}\\
&: \frac{(a^2S_Az - c^2S_Cx)(c^2S_Cy - b^2S_Bz)}{4S_{ABC}S_C(x+y+z)^2}\\
=&\frac{a^2}{a^2S_A(c^2S_Cy - b^2S_Bz)} : \frac{b^2}{b^2S_B(a^2S_Az - c^2S_Cx)} : \frac{c^2}{c^2S_C(b^2S_Bx - a^2S_Ay)}\\
=&\frac{a^2}{\frac{y}{b^2S_B} - \frac{z}{c^2S_C}} : \frac{b^2}{\frac{z}{c^2S_C} - \frac{x}{a^2S_A}} : \frac{c^2}{\frac{x}{a^2S_A} - \frac{y}{b^2S_B}}.
\end{aligned}$$

This shows that P is the isogonal conjugate (in triangle $\mathbf{T}(P)$) of an infinite point. It is a point on the circumcircle of $\mathbf{T}(P)$. □

Since $\mathbf{T}(P)$ is similar to $\mathbf{T}$, its circumcenter is the point

$$O_P := \frac{a^2S_A \cdot A_P + b^2S_B \cdot B_P + c^2S_C \cdot C_P}{2S^2}$$

in absolute barycentric coordinates. In homogeneous barycentric coordinates with respect to $\mathbf{T}$,

$$\begin{aligned}
O_P = &(S_{BC}(4S^2 \cdot S_A - a^2b^2c^2)x + c^2a^2S_{CCA}y + a^2b^2S_{ABB}z\\
&: b^2c^2S_{BCC}x + S_{CA}(4S^2 \cdot S_B - a^2b^2c^2)y + a^2b^2S_{AAB}z\\
&: b^2c^2S_{BBC}x + c^2a^2S_{CAA}y + S_{AB}(4S^2 \cdot S_C - a^2b^2c^2)z).
\end{aligned}$$

4. Counterparts of a point in the triad $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$

For $P = (x : y : z)$ with respect to $\mathbf{T}$, we also consider its counterparts A'_P, B'_P, C'_P in the triangles $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$. These are the points

$$\begin{aligned}
A'_P &= (2S_Ax - (a^2 - b^2)y - (a^2 - c^2)z : b^2z : c^2y),\\
B'_P &= (a^2z : 2S_By - (b^2 - c^2)z - (b^2 - a^2)x : c^2x),\\
C'_P &= (a^2y : b^2x : 2S_Cz - (c^2 - a^2)x - (c^2 - b^2)y).
\end{aligned}$$

Denote by $\mathbf{T}'(P)$ the triangle $A'_PB'_PC'_P$.

Proposition 6. *For $P = (x : y : z)$, the triangle $\mathbf{T}'(P)$ is*
(a) *oppositely similar to the orthic triangle $\mathbf{T}'$,*
(b) *perspective with $\mathbf{T}$ at the isogonal conjugate of P in $\mathbf{T}$, namely,*

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right).$$

(c) *orthologic to $\mathbf{T}$ if and only if P lies on the Euler line. In this case,*
(i) *the perpendiculars from A, B, C to $B'_PC'_P$, $C'_PA'_P$, $A'_PB'_P$ are concurrent at*

$$X(265) = \left(\frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}}\right),$$

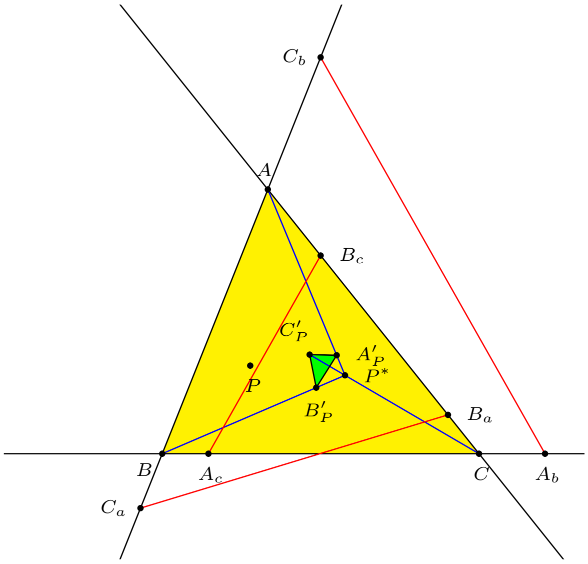

Figure 4.

(ii) *if* $OP : PH = t : 1-t$, *the perpendiculars from* A'_P, B'_P, C'_P *to* BC, CA, AB *are concurrent at* Q *where* $OQ : QH = 1-t : t$.

Remark. $X(265)$ is the reflection conjugate of O. Equivalently, it is the reflection of O in the Jerabek center $X(125)$.

Proposition 7. *The points* A'_P, B'_P, C'_P, P *are concyclic if and only if* P *lies on the circumconic which is the isogonal conjugate* (*in* $\mathbf{T}$) *of the perpendicular bisector of the segment* OH.

Proof. With respect to $\mathbf{T}'(P)$, the point P has coordinates

$$\begin{aligned}
&x' : y' : z' \\
=&\ \text{Area}(PB'_PC'_P) : \ \text{area}(A'_PPC'_P) : \ \text{area}(A'_PB'_PP) \\
=&\ \frac{2a^2S_Ax^2 + a^4yz - (S^2+2S_{BC}-S_{CC})zx - (S^2+2S_{BC}-S_{BB})xy}{-4S_{BC}(x+y+z)^2} \\
&: \frac{2b^2S_By^2 + b^4zx - (S^2+2S_{CA}-S_{AA})xy - (S^2+2S_{CA}-S_{CC})yz}{-4S_{CA}(x+y+z)^2} \\
&: \frac{2c^2S_Cz^2 + c^4xy - (S^2+2S_{AB}-S_{BB})yz - (S^2+2S_{AB}-S_{AA})zx}{-4S_{AB}(x+y+z)^2} \\
=&\ S_A(2a^2S_Ax^2 + a^4yz - (S^2+2S_{BC}-S_{CC})zx - (S^2+2S_{BC}-S_{BB})xy) \\
&: S_B(2b^2S_By^2 + b^4zx - (S^2+2S_{CA}-S_{AA})xy - (S^2+2S_{CA}-S_{CC})yz) \\
&: S_C(2c^2S_Cz^2 + c^4xy - (S^2+2S_{AB}-S_{BB})yz - (S^2+2S_{AB}-S_{AA})zx)
\end{aligned}$$

The point P lies on the circumcircle of $\mathbf{T}'(P)$ (which is similar to $\mathbf{T}'$) if and only if

$$\begin{aligned} 0 =&\ a'^2y'z' + b'^2z'x' + c'^2x'y' \\ =&\ a^4S_{AA}y'z' + b^4S_{BB}z'x' + c^4S_{CC}x'y' \\ =&\ 2S_{ABC}\cdot S^2(x+y+z)^2|HP|^2\left(\sum_{\text{cyclic}} a^4(S^2-3S_{AA})yz\right). \end{aligned}$$

There are two possibilities.

(i) $|HP| = 0 \implies P = H$. In this case, $A'_H = B'_H = C'_H = O$.

(ii) P lies on the circumconic $\sum_{\text{cyclic}} a^4(S^2 - 3S_{AA})yz = 0$, which is the isogonal conjugate (in $\mathbf{T}$) of the perpendicular bisector of the segment OH. □

5. Triangle centers of $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ and $\mathbf{T}'_a$, $\mathbf{T}'_b$, $\mathbf{T}'_c$

Consider the circumcenter $O = (a^2S_A : b^2S_B : c^2S_C)$. Note that A_O, B_O, C_O are not the circumcenters of the triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ respectively. Indeed, these three points coincide with O: $A_O = B_O = C_O = O$. Instead, the circumcenters of triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$ are the points

$$\begin{aligned} O_a =&\ (a^4(S^2-S_{AA}) : b^2(2S^2\cdot S_B - S_C(S^2-S_{BB})) : c^2(2S^2\cdot S_C - S_B(S^2-S_{CC}))), \\ O_b =&\ (a^2(2S^2\cdot S_A - S_C(S^2-S_{AA})) : b^4(S^2-S_{BB}) : c^2(2S^2\cdot S_C - S_A(S^2-S_{CC}))), \\ O_c =&\ (a^2(2S^2\cdot S_A - S_B(S^2-S_{AA})) : b^2(2S^2\cdot S_B - S_A(S^2-S_{BB})) : c^4(S^2-S_{CC})). \end{aligned}$$

These form the vertices of $\mathbf{T}(P)$ for

$$P = X(1147) = (a^4S_A(S^2-S_{AA}) : b^4S_B(S^2-S_{BB}) : c^4S_C(S^2-S_{CC})),$$

which, according to the ENCYCLOPEDIA OF TRIANGLE CENTERS [2], is the midpoint of O and $X(155)$, the orthocenter of the tangential triangle (see Figure 5). The three circumcenters are concyclic with $X(1147)$ (see Theorem 5). The center of the circle containing them is $X(156)$, the nine-point center of the tangential triangle

More generally, let P be a triangle center of ABC, with coordinates expressed in terms of a, b, c. The same triangle center P_a of $\mathbf{T}_a = AA_bA_c$ is the point with coordinates in which a, b, c are replaced by a', b', c' respectively (likewise S_A, S_B,

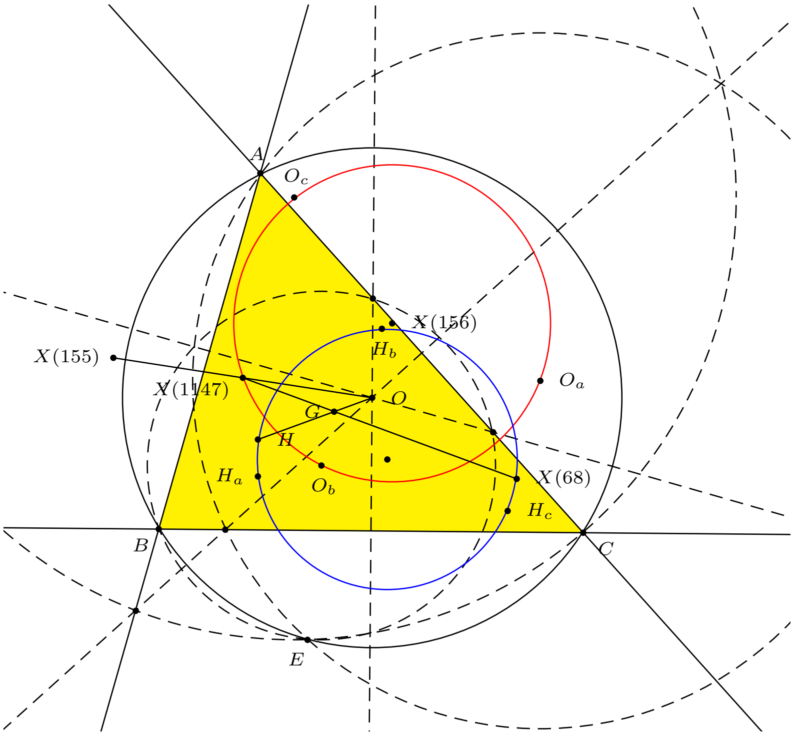

Figure 5.

S_C by S'_A, S'_B, S'_c respectively). For example, for orthocenters,

$$
\begin{aligned}
H_a &= \left(\frac{1}{S'_A} : \frac{1}{S'_B} : \frac{1}{S'_C}\right) \\
&= \left(\frac{S_A}{S^2 - S_{AA}} : \frac{S_B}{S^2 - S_{BB}} : \frac{S_C}{S^2 - S_{CC}}\right) \text{ with respect to } \mathbf{T}_a \\
&= \frac{S_A}{S^2 - S_{AA}} \cdot (1, 0, 0) + \frac{S_B}{S^2 - S_{BB}} \cdot \frac{(0, -S_A + S_B, S_A + S_B)}{2S_B} \\
&\qquad + \frac{S_C}{S^2 - S_{CC}} \cdot \frac{(0, S_C + S_A, S_C - S_A)}{2S_C} \text{ in absolute barycentric coordinates} \\
&= ((S^2 - S_{BB})(S^2 - S_{CC}) : a^2 S_C(S^2 - S_{AA}) : a^2 S_B(S^2 - S_{AA}))
\end{aligned}
$$

with respect to $\mathbf{T}$.

The orthocenters H_a, H_b, H_c are the vertices of $\mathbf{T}(Q)$ for

$$Q = X(68) = \left(\frac{S_A}{S^2 - S_{AA}} : \frac{S_B}{S^2 - S_{BB}} : \frac{S_C}{S^2 - S_{CC}}\right).$$

The triangle center $X(68)$ is the superior of $X(1147)$. It lies on the circle containing the three orthocenters (see Figure 5 and Theorem 5). The circle $H_aH_bH_c$ also contains the orthocenter H.

The centroids G_a, G_b, G_c are the points

$$\begin{aligned}
G_a &= (2S_{BC} : 2S_{BC} + S_A(S_B - S_C) : 2S_{BC} - S_A(S_B - S_C)),\\
G_b &= (2S_{CA} - S_B(S_C - S_A) : 2S_{CA} : 2S_{CA} + S_B(S_C - S_A)),\\
G_c &= (2S_{AB} + S_C(S_A - S_B) : 2S_{AB} - S_C(S_A - S_B) : 2S_{AB}).
\end{aligned}$$

In this case, $G_aG_bG_c = \mathbf{T}(G)$.

On the other hand, since the triangles $\mathbf{T}'_a = AB_aC_a$, $\mathbf{T}'_b = A_bBC_b$, $\mathbf{T}'_c = C_aC_bC$ are similar to ABC, we have

$$P'_a = A'_P, \qquad P'_b = B'_P, \qquad P'_c = C'_p.$$

For example, the circumcenters are

$$\begin{aligned}
O'_a = A'_O &= (3a^2S_{AA} - S_A(S_B - S_C)^2 - a^2S_{BC} : b^2c^2S_C : b^2c^2S_B),\\
O'_b = B'_O &= (c^2a^2S_C : 3b^2S_{BB} - S_B(S_C - S_A)^2 - b^2S_{CA} : c^2a^2S_A),\\
O'_c = C'_O &= (a^2b^2S_B : a^2b^2S_A : 3c^2S_{CC} - S_C(S_A - S_B)^2 - c^2S_{AB}).
\end{aligned}$$

$$\begin{aligned}
G'_a &= (2b^2 + 2c^2 - 3a^2 : b^2 : c^2),\\
G'_b &= (a^2 : 2c^2 + 2a^2 - 3b^2 : c^2),\\
G'_c &= (a^2 : b^2 : 2a^2 + 2b^2 - 3c^2).
\end{aligned}$$

$H'_a = H'_b = H'_c = O$.

6. Orthology with $\mathbf{T}$ and pedal triangles

6.1. *The triangle $O_aO_bO_c$.*

Proposition 8. *The triangle $O_aO_bO_c$ is orthologic to ABC.*
(a) *The perpendiculars from O_a to BC, O_b to CA, and O_c to AB concur at the triangle center* [1]

$$Y_{(1)} := \left(a^2(S_{AA}(S_{BB} + S_{BC} + S_{CC}) - S_{BB}S_{CC}) : \cdots : \cdots\right).$$

(b) *The perpendiculars from A to O_bO_c, B to O_cO_a, and C to O_aO_b concur at the triangle center $X(74)$ on the circumcircle of $\mathbf{T}$.*

Remark. The orthology center $Y_{(1)}$ lies on the circle $O_aO_bO_c$.

Proposition 9. *The triangle $O_aO_bO_c$ is orthologic with the pedal triangle of P if and only if P lies on the Euler line. If P lies on this line,*
(a) *the perpendiculars from O_a, O_b, O_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$16S^2 \cdot S_{ABC} \sum_{\text{cyclic}} a^2S_A(S_B - S_C)yz - (x + y + z)(f_ax + f_by + f_cz) = 0,$$

[1]The triangle center $Y_{(1)}$ does not appear in the current edition of [2]. It has (6-9-13)-search number $-5.64011769173\cdots$.

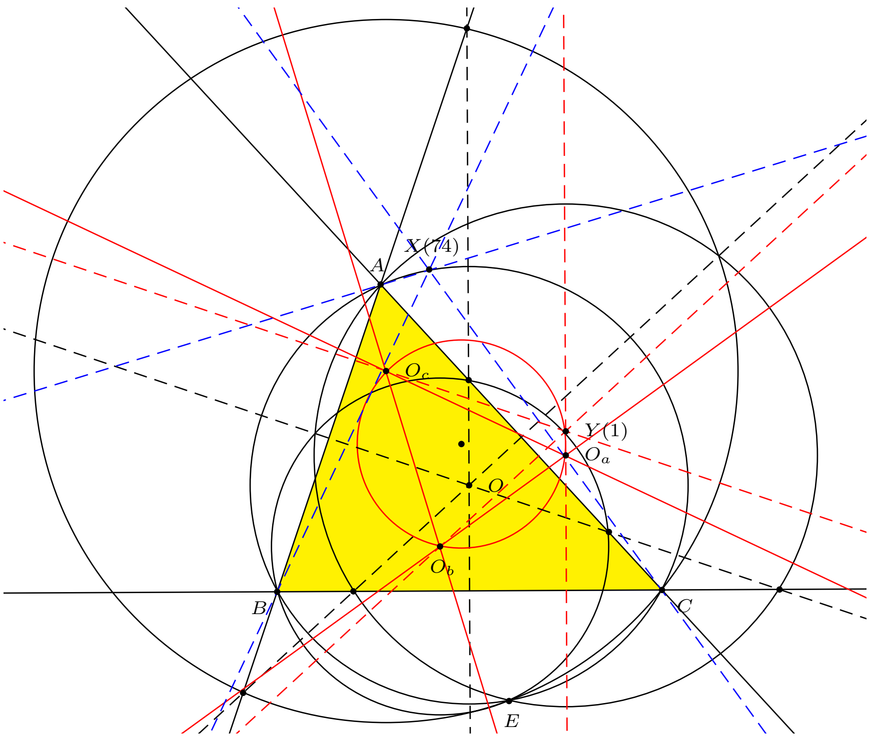

Figure 6

where

$$f_a = b^2c^2(S_B - S_C)(a^4S_A^3 + a^2S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - (8S_A + S_B + S_C)S_{BB}S_{CC}),$$

and f_b, f_c are defined cyclically,
(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $O_aO_bO_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_A(S_B - S_C)}{a^2S_{AA} - S_A(S_B - S_C)^2 - a^2S_{BC}} x = 0.$$

6.2. *The triangle $G_aG_bG_c$.*

Proposition 10. *The triangle $G_aG_bG_c$ is orthologic to ABC.*
(a) *The perpendiculars from G_a to BC, G_b to CA, and G_c to AB concur at the triangle center* [2]

$$Y_{(2)} := (a^2(S_{BB} + S_{BC} + S_{CC})S_{AA} + S_{BB}S_{CC}(2S_A - S_B - S_C) : \cdots : \cdots).$$

[2]The triangle center $Y_{(2)}$ does not appear in the current edition of [2]. It has (6-9-13)-search number $-5.65228493146\cdots$.

(b) *The perpendiculars from A to G_bG_c, B to G_cG_a, and C to G_aG_b concur at the triangle center*

$$X(1294) = \left(\frac{1}{S_{AA}(S_{BB} - S_{BC} + S_{CC}) - S_{BB}S_{CC}} : \cdots : \cdots\right)$$

on the circumcircle.

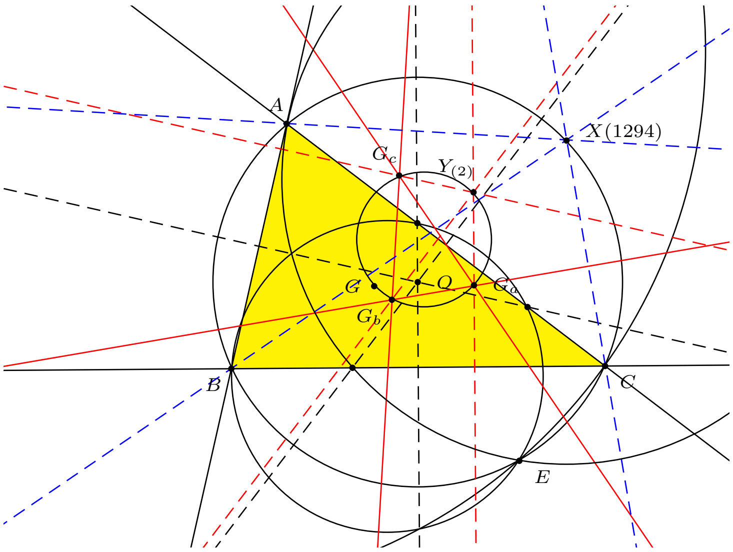

Figure 7

Remark. The circle $G_aG_bG_c$ contains the centroid G and the orthology center $Y_{(2)}$.

Proposition 11. *The triangle $G_aG_bG_c$ is orthologic with the pedal triangle of P if and only if P lies on the line*

$$\sum_{\text{cyclic}} b^2c^2(S_B - S_C)(a^2S_A - S_{BC})x = 0,$$

which passes through O and $X(64) = \left(\frac{a^2}{a^2S_A - S_{BC}} : \frac{b^2}{b^2S_B - S_{CA}} : \frac{c^2}{c^2S_C - S_{AB}}\right)$.

If P lies on this line,

(a) *the perpendiculars from G_a, G_b, G_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$6S_{ABC} \sum_{\text{cyclic}} (S_B - S_C)(a^2S_A - S_{BC})yz$$
$$- (x + y + z)\left(\sum_{\text{cyclic}} S_A(S_B - S_C)(S_{AA}(S_{BB} - 3S_{BC} + S_{CC}) - S_{BB}S_{CC})x\right) = 0.$$

(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $G_aG_bG_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_A(S_B - S_C)(a^2S_A - S_{BC})}{a^2S_A - 2S_{BC}}x = 0.$$

6.3. *The triangle $H_aH_bH_c$.*

Proposition 12. *The triangle $H_aH_bH_c$ is orthologic to ABC.*
(a) *The perpendiculars from H_a to BC, H_b to CA, and H_c to AB concur at the orthocenter H of $\mathbf{T}$.*
(b) *The perpendiculars from A to H_bH_c, B to H_cH_a, and C to H_aH_b concur at*

$$X(1300) = \left(\frac{1}{S_A(a^2(S_{AA} - S_{BC}) - S_A(S_B - S_C)^2)} : \cdots : \cdots\right)$$

on the circumcircle of $\mathbf{T}$.

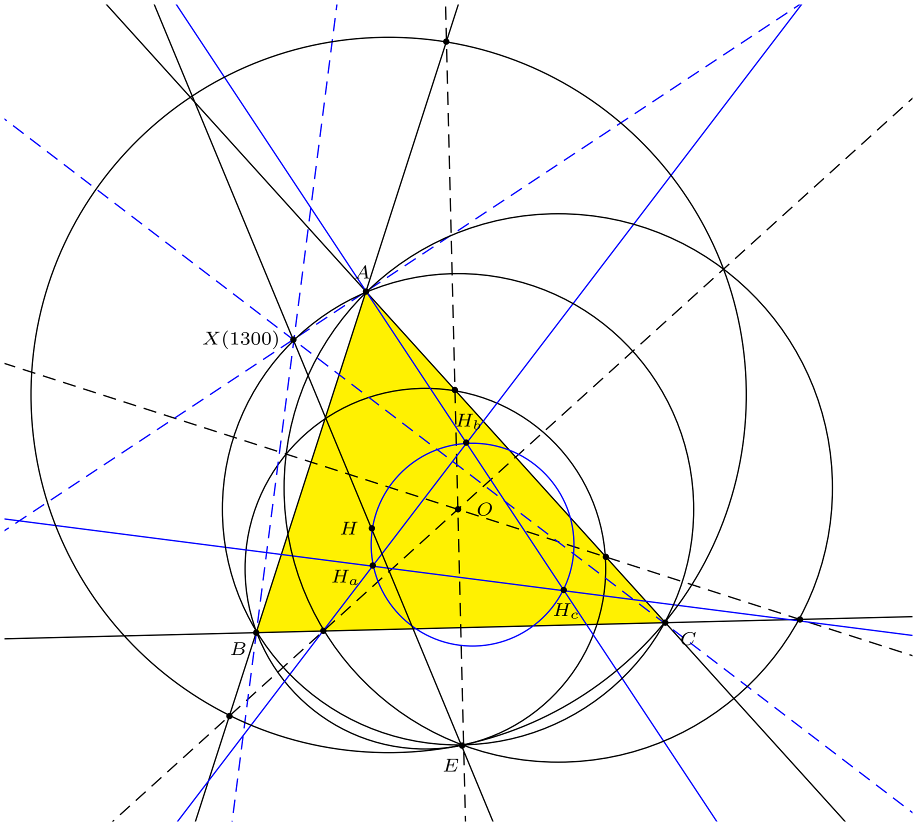

Figure 8

Remark. $X(1300)$ is the second intersection of the circumcircle with the line EH.

Proposition 13. *The triangle $H_aH_bH_c$ is orthologic with the pedal triangle of P if and only if P lies on the line*

$$\frac{S_B - S_C}{a^2 S_A}x + \frac{S_C - S_A}{b^2 S_B}y + \frac{S_A - S_B}{c^2 S_C}z = 0$$

joining the circumcenter O to $X(394) = (a^2 S_{AA} : b^2 S_{BB} : c^2 S_{CC})$.

If P lies on this line,
(a) *the perpendiculars from H_a, H_b, H_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$4S^2 \sum_{\text{cyclic}} S_{BC}(S_B - S_C)yz + (x + y + z)\left(\sum_{\text{cyclic}} a^2 S_A(S_B - S_C)(S^2 - S_{AA})x\right) = 0,$$

with center at the nine-point center N;
(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $H_aH_bH_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{S_A(a^4 S_{AA} - a^2 S_A(S_{BB} + S_{CC}) - S_{BC}(S_B - S_C)^2)}x = 0.$$

6.4. *The triangle $O'_aO'_bO'_c$.*

Proposition 14. *The triangle $O'_aO'_bO'_c$ is orthologic to ABC.*
(a) *The perpendiculars from O'_a to BC, O'_b to CA, and O'_c to AB concur at the orthocenter H of* $\mathbf{T}$.
(b) *The perpendiculars from A to $O'_bO'_c$, B to $O'_cO'_a$, and C to $O'_aO'_b$ concur at the triangle center*

$$X(265) = \left(\frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}}\right)$$

(see Proposition 6).

Proposition 15. *The triangle $O'_aO'_bO'_c$ is orthologic with the pedal triangle of P if and only if P lies on the Euler line. If P lies on this line,*
(a) *the perpendiculars from O'_a, O'_b, O'_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$\begin{aligned}&4S^2 \sum_{\text{cyclic}} a^2 S_A(S_B - S_C)yz\\ &+ a^2b^2c^2(x + y + z)(S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z) = 0,\end{aligned}$$

with center N,
(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $O'_aO'_bO'_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{a^2 S_A - 2S_{BC}}x = 0.$$

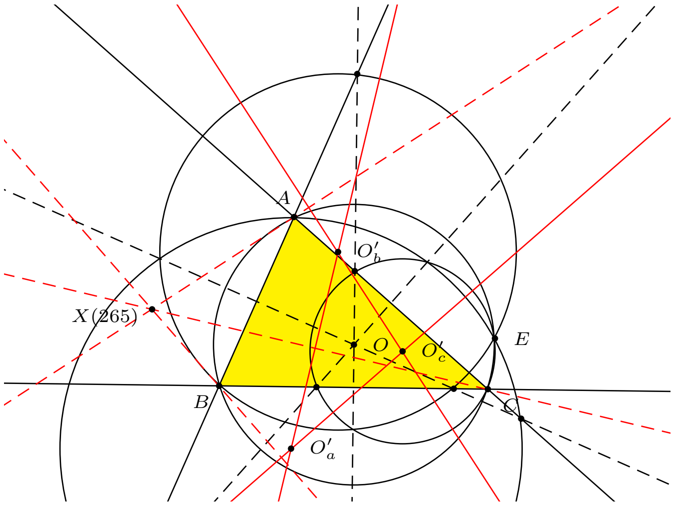

Figure 9

6.5. *The triangle $G'_aG'_bG'_c$.*

Proposition 16. *The triangle $G'_aG'_bG'_c$ is orthologic to ABC.*
(a) *The perpendiculars from G'_a to BC, G'_b to CA, and G'_c to AB concur at the triangle center*

$$X(381) = \left(a^2S_A + 4S_{BC} : b^2S_B + 4S_{CA} : c^2S_C + 4S_{AB}\right)$$

on the Euler line.
(b) *The perpendiculars from A to $G'_bG'_c$, B to $G'_cG'_a$, and C to $G'_aG'_b$ concur at the triangle center*

$$X(265) = \left(\frac{S_A}{S^2 - 3S_{AA}} : \frac{S_B}{S^2 - 3S_{BB}} : \frac{S_C}{S^2 - 3S_{CC}}\right)$$

(see Proposition 6).

Proposition 17. *The triangle $G'_aG'_bG'_c$ is orthologic with the pedal triangle of P if and only if P lies on the Euler line. If P lies on this line,*
(a) *the perpendiculars from G'_a, G'_b, G'_c to the corresponding sides of the pedal triangle of P are concurrent at a point on the conic*

$$6\sum_{\text{cyclic}} a^2S_A(S_B - S_C)yz + (x+y+z)\left(\sum_{\text{cyclic}} b^2c^2(S_B - S_C)x\right) = 0$$

with center at the centroid G,
(b) *the perpendiculars from the vertices of the pedal triangle of P to the corresponding sides of $G'_aG'_bG'_c$ are concurrent at a point on the line*

$$\sum_{\text{cyclic}} \frac{S_B - S_C}{a^2S_A - 2S_{BC}}x = 0.$$

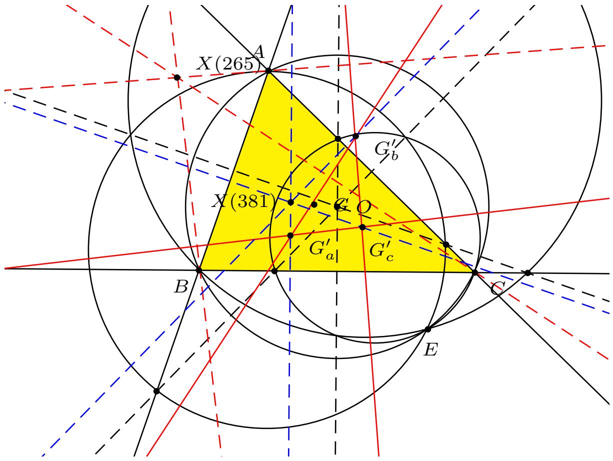

Figure 10

7. The triangles $\mathbf{T}(P)$ and $\mathbf{T}(Q)$

Proposition 18. *The triangles $\mathbf{T}(P)$ and $\mathbf{T}(Q)$ are perspective if and only if the line PQ contains the circumcenter O. In this case, the triangles are homothetic at O.*

Proof. Let $P = (x : y : z)$ and $Q = (u : v : w)$. The line containing A_P and A_Q has equation

$$a^2 S_A(wy - vz)\mathbb{X} + (c^2 S_C(vx - uy) - S_B(S_C - S_A)(uz - wx))\mathbb{Y} + (b^2 S_B(uz - wx) + S_C(S_A - S_B)(vx - uy))\mathbb{Z} = 0.$$

Similarly, we have the equations of the lines $B_P B_Q$ and $C_P C_Q$. The three lines are concurrent if and only if $f \cdot g = 0$, where

$$f := (c^2 S_C v - b^2 S_B w)x + (a^2 S_A w - c^2 S_C u)y + (b^2 S_B u - a^2 S_A v)z,$$

$$g := (u + v + w)^2 \left(\sum_{\text{cyclic}} a^4 S_{AA} yz \right) - (x + y + z) \left(\sum_{\text{cyclic}} ((c^2 S_C v + b^2 S_B w)^2 - 4S_{ABC} \cdot S_A vw)x \right).$$

Note that
(i) the equation $f = 0$ represents the line OQ;
(ii) the equation $g = 0$ represents a conic with center Q. Since the conic also contains Q and intersects the sidelines at imaginary points, its represents only the point Q.

From these we conclude that the triangles $\mathbf{T}(P)$ and $\mathbf{T}(Q)$ are perspective if and only if O, P, Q are collinear. Each of the lines $A_P A_Q$, $B_P B_Q$, $C_P C_Q$ contains O. The perspector is O. □

Theorem 19. *The triangle bounded by the lines A_PA_Q, B_PB_Q, C_PC_Q has circumcenter O.*

Proof. Since O is the orthocenter of the triangle $\mathbf{T}(P)$, the circles OB_PC_P, OC_PA_P, and OA_PB_P have equal radii. Note that O is also the incenter of each of triangles $\mathbf{T}_a$, $\mathbf{T}_b$, $\mathbf{T}_c$, and these triangles are all similar to $\mathbf{T}'$. Therefore, $\angle OA_PA_Q = \angle OB_PB_Q = \angle OC_PC_Q$. Let $A''B''C''$ be the triangle bounded by the lines A_PA_Q, B_PB_Q and C_PC_Q. Applying the law of sines to triangles $OA''C_P$, $OB''A_P$, $OC''B_P$, we conclude that $OA'' = OB'' = OC''$. $\square$

Proposition 20. *The triangles $\mathbf{T}'(P)$ and $\mathbf{T}'(Q)$ are perspective if and only if Q lies on the line HP. If the condition is satisfied, the triangles are homothetic at O.*

8. The triangles $\mathbf{T}(P)$ and $\mathbf{T}'(P)$

Theorem 21. *The triangles $\mathbf{T}(P)$ and $\mathbf{T}'(P)$ are perspective if and only if P lies on*
(a) *the Jerabek hyperbola or*
(b) *the line* $\sum_{\text{cyclic}} \frac{x}{S_A(S_B-S_C)} = 0$.

Remarks. (1) If P lies on the Jerabek hyperbola, the perspector is the circumcenter O.

(2) The line in (b) contains the Jerabek center $X(125)$ and $X(122)$ (also $X(684)$, $X(1650)$, $X(2972)$). In this case the lines $A_PA'_P$, $B_PB'_P$, $C_PC'_P$ are parallel.

Theorem 22. *Let P be a point with coordinates $(x:y:z)$ in $\mathbf{T}$.*
(a) *The circumcenter of $\mathbf{T}(P)$ has coordinates $(x:y:z)$ in triangle $O'_aO'_bO'_c = \mathbf{T}'(O) = A'_OB'_OC'_O$.*
(b) *The circumcenter of $\mathbf{T}'(P)$ has coordinates $(x:y:z)$ in $O_aO_bO_c$, which is $\mathbf{T}(Q)$ for*

$$Q = X(1147) = (a^4S_A(S^2-S_{AA}) : b^4S_B(S^2-S_{BB}) : c^4S_C(S^2-S_{CC})).$$

Proof. (a) The point with coordinates $(x:y:z)$ with respect to $O'_aO'_bO'_c$ is

$$\begin{aligned}\overline{O} = (&S_{BC}(4S^2\cdot S_A - a^2b^2c^2)x + c^2a^2S_{CCA}y + a^2b^2S_{ABB}z\\ :\ &b^2c^2S_{BCC}x + S_{CA}(4S^2\cdot S_B - a^2b^2c^2)y + a^2b^2S_{AAB}z\\ :\ &b^2c^2S_{BBC}x + c^2a^2S_{CAA}y + S_{AB}(4S^2\cdot S_C - a^2b^2c^2)z).\end{aligned}$$

Its square distance from A_P is

$$\frac{a^2b^2c^2Q(x,y,z)}{16S^2\cdot(S_{ABC})^2(x+y+z)^2},$$

where

$$\begin{aligned}Q(x,y,z) = {}&b^2c^2S_{BB}S_{CC}x^2 + c^2a^2S_{CC}S_{AA}y^2 + a^2b^2S_{AA}S_{BB}z^2\\ &- 2S_{ABC}(a^2S_{AA}yz + b^2S_{BB}zx + c^2S_{CC}xy).\end{aligned}$$

The symmetry of Q in S_A, S_B, S_C and x, y, z shows that this is the same if A_P is replaced by B_P or C_P. The point $\overline{O}$ therefore is the circumcenter of $A_PB_PC_P$. $\square$

Theorem 23. (a) *Triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are perspective at the circumcenter O of triangle ABC.*
(b) *The six circumcenters O_a, O_b, O_c, O'_a, O'_b, O'_c are concyclic. The center of the circle containing them is $X(156)$, the nine-point center of the tangential triangle.*

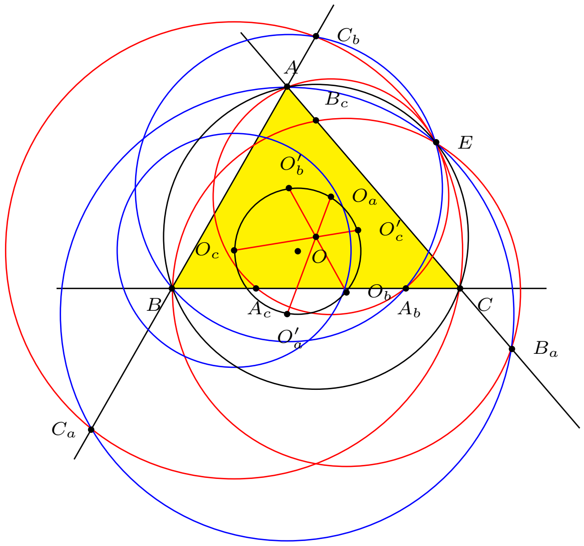

Figure 11

Proof. With respect to $O_aO_bO_c$,

$$O'_a = (-a^2S_{BC} : S_C(a^2S_A + 2S_{BC}) : S_B(a^2S_A + 2S_{BC})) = \left(\frac{-a^2}{a^2S_A + 2S_{BC}} : \frac{b^2}{b^2S_B} : \frac{c^2}{c^2S_C}\right),$$

$$O'_b = (S_C(b^2S_B + 2S_{CA}) : -b^2S_{BC} : S_A(b^2S_B + 2S_{CA})) = \left(\frac{a^2}{a^2S_A} : \frac{-b^2}{b^2S_B + 2S_{CA}} : \frac{c^2}{c^2S_C}\right),$$

$$O'_c = (S_B(c^2S_C + 2S_{AB}) : S_A(c^2S_C + 2S_{AB}) : -c^2S_{BC}) = \left(\frac{a^2}{a^2S_A} : \frac{b^2}{b^2S_B} : \frac{-c^2}{c^2S_C + 2S_{AB}}\right).$$

These expressions show that
(a) triangles $O_aO_bO_c$ and $O'_aO'_bO'_c$ are perspective at the orthocenter of $O_aO_bO_c$,
(b) O'_a, O'_b, O'_c are on the the circumcircle of $O_aO_bO_c$. □

The orthocenter of $O_aO_bO_c$ is the circumcenter O. The circumcenter is $X(156)$, which is the nine-point center of the tangential triangle.

Proposition 24. *The triangle $O_aO_bO_c$ is perspective with the inferior triangle of the tangential triangle at a point on its circumcircle.*

Proof. With respect to $O_aO_bO_c$, the midpoint of the A-side of the tangential triangle has coordinates

$$\begin{aligned}&(a^2S_{AA}(2S_{BB} - 7S_{BC} + 2S_{CC}) + S_{ABC}(3S_{BB} - 2S_{BC} + 3S_{CC}) + a^2S_{BB}S_{CC})\\ &: a^2b^2S_A(c^2S_C - 2S_{AB}) : c^2a^2S_A(b^2S_B - 2S_{CA}));\end{aligned}$$

similarly for the midpoints of the other two sides of the tangential triangle. From these coordinates, it is clear that the medial triangle of the tangential triangle is perspective to $O_aO_bO_c$ at the point with coordinates

$$\left(\frac{a^2}{a^2S_A - 2S_{BC}} : \frac{b^2}{b^2S_B - 2S_{CA}} : \frac{c^2}{c^2S_C - 2S_{AB}}\right).$$

Since triangle $O_aO_bO_c$ is similar to $\mathbf{T}$, this perspector is a point on the circumcircle of $O_aO_bO_c$. It is the isogonal conjugate (with respect to $O_aO_bO_c$) of the infinite point of the Euler line of the triangle. □

Remark. With respect to $\mathbf{T}$, this perspector has coordinates

$$(a^2(S_{AA}(S_{BB} + S_{BC} + S_{CC}) - S_{BB}S_{CC}) : \cdots : \cdots).$$

See Proposition 8.

Proposition 25. *The triangle $O'_aO'_bO'_c$ is perspective with the tangential triangle at*

$$X(195) = (a^2(-3a^2S_A^3 + (5S_{BB} + 6S_{BC} + 5S_{CC})S_{AA} + 9a^2S_{ABC} + 4S_{BB}S_{CC}) : \cdots : \cdots).$$

Remark. $X(195)$ is the circumcenter of the triangle of reflections (see [3]).

9. A family of circumcircles of $\mathbf{T}(P)$ containing the Euler reflection point

Proposition 26. *Let $P = (x : y : z)$. The circumcircle of $\mathbf{T}(P) = A_PB_PC_P$ contains the Euler reflection point E if and only if P lies on the conic*

$$a^4S_{AA}yz + b^4S_{BB}zx + c^4S_{CC}xy = 0.$$

Proof. The coordinates of A_P, B_P, C_P are given at the beginning of §3. Computing the coordinates of E with respect to triangle $A_PB_PC_P$, we have

$$\begin{aligned}
& x' : y' : z' \\
= & S_{BC}\left(\mu\left(\sum_{\text{cyclic}} a^4S_{AA}yz\right) + \tau(x+y+z)\left(\frac{S_{BC}x}{S_B - S_C} + \frac{2S_{AAC}y}{b^2(S_A - S_B)} + \frac{2S_{AAB}z}{c^2(S_C - S_A)}\right)\right) \\
: & S_{CA}\left(\mu\left(\sum_{\text{cyclic}} a^4S_{AA}yz\right) + \tau(x+y+z)\left(\frac{2S_{BBC}x}{a^2(S_A - S_B)} + \frac{S_{CA}y}{S_C - S_A} + \frac{2S_{ABB}z}{c^2(S_B - S_C)}\right)\right) \\
: & S_{AB}\left(\mu\left(\sum_{\text{cyclic}} a^4S_{AA}yz\right) + \tau(x+y+z)\left(\frac{2S_{BCC}x}{a^2(S_C - S_A)} + \frac{2S_{CCA}y}{b^2(S_B - S_C)} + \frac{S_{AB}z}{S_A - S_B}\right)\right)
\end{aligned}$$

where

$$\begin{aligned}
\mu = &\ a^2S_{BC} + b^2S_{CA} + c^2S_{AB} - 6S_{ABC}, \\
\tau = &\ (S_{BB} - S_{CC})(S_{CC} - S_{AA})(S_{AA} - S_{BB}).
\end{aligned}$$

This is a point on the circumcircle of triangle $A_PB_PC_P$ (which is similar to ABC) if and only if

$$(S_B + S_C)y'z' + (S_C + S_A)z'x' + (S_A + S_B)x'y' = 0.$$

This reduces to

$$\mu^2 \cdot \left(\sum_{\text{cyclic}} a^4 S_{AA} yz\right) \cdot G = 0,$$

where

$$\begin{aligned} G = & \sum_{\text{cyclic}} (b^2c^2 S_{BB}S_{CC}x^2 - 2S_{ABC} \cdot a^2 S_{AA} yz) \\ = & \; S^2 \left(\sum_{\text{cyclic}} S_{BC}^2 x^2 - 2S_{CA}S_{AB} yz\right) + S_{ABC}^2 (x+y+z)^2 \end{aligned}$$

$G = 0$ is the equation of a conic with center O. Since $G(O) = 0$, we conclude that $G = 0$ defines only the point O. Therefore, the circle $A_PB_PC_P$ contains E if and only if P lies on the circumconic $\sum_{\text{cyclic}} a^4 S_{AA} yz$. □

Theorem 27. *Let Q be a triangle center on the circumcircle. The circle $Q_aQ_bQ_c$ contains the Euler reflection point E.*

Proof. Let $Q = \left(\frac{a^2}{(b^2-c^2)(a^2+t)} : \frac{b^2}{(c^2-a^2)(b^2+t)} : \frac{c^2}{(a^2-b^2)(c^2+t)}\right)$ be a triangle center on the circumcircle. For

$$P = \left(\frac{a'^2}{(b'^2 - c'^2)(a'^2+t)} : \frac{b'^2}{(c'^2 - a'^2)(b'^2+t)} : \frac{c'^2}{(a'^2 - b'^2)(c'^2+t)}\right)$$

we have $Q_a = A_P$, $Q_b = B_P$, $Q_c = C_P$. The circle $Q_aQ_bQ_c$ is the same as $A_PB_PC_P$. With respect to triangle $O'_aO'_bO'_c$, the center of $A_PB_PC_P$ has coordinates given above. Therefore the locus of the center of $Q_aQ_bQ_c$ is the circle $O'_aO'_bO'_c$, which is the nine-point circle of the tangential triangle of triangle ABC.

Note that P lies on the conic $\sum_{\text{cyclic}} a^4 S_{AA} yz = 0$. By Proposition 26, the circle $A_PB_PC_P$ contains the Euler reflection point E. □

Theorem 28. *The circumcircle of $\mathbf{T}'(P)$ contains the Euler reflection point if and only if P lies on the circumcircle.*

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