Forum Geometricorum
Volume 14 (2014) 371–380.

FORUM GEOM
ISSN 1534-1178

Optimal Packings of Two Ellipses in a Square

Thierry Gensane and Pascal Honvault

Abstract. For each real number E in $]0, 1]$, we describe the densest packing $\mathcal{P}_E$ of two non-overlapping congruent ellipses of aspect ratio E in a square. We find three different patterns as E belongs to $]0, 1/2]$, $[1/2, E_0]$ where $E_0 = \sqrt{(6\sqrt{3}-3)/11}$, and $[E_0, 1]$. The technique of unavoidable sets – used by Friedman for proving the optimality of square packings – allows to prove the optimality of each packing $\mathcal{P}_E$.

1. Introduction

We consider the following generalization of the classical disk packing problem in a compact convex domain K: Let $n \in \mathbb{N}$ and $E \in]0, 1]$, what is the densest packing of n non-overlapping congruent ellipses of aspect ratio E in K?

In this paper, we describe for each aspect ratio E in $]0, 1]$, the densest packing $\mathcal{P}_E$ of two congruent unit ellipses of aspect ratio E in the square $K = [0, 1]^2$ and we prove the optimality of these packings. In Figure 1, we display six representative optimal packings of two congruent ellipses. For $E = 1$, the optimal packing $\mathcal{P}_1$ is composed of two disks lying in opposite corners, see [4] for a large list of dense packings of congruent disks in the square. An introductory bibliography on disk packing problems can be found in [1, 3]. When E decreases from 1 to $E_0 = \sqrt{(6\sqrt{3}-3)/11} \approx 0.8198$, the ellipses of optimal packings $\mathcal{P}_E$ flatten by keeping a constant tilted angle equals to $-\pi/4$. For $E \in [1/2, E_0]$, the angle of the two ellipses of $\mathcal{P}_E$ decreases and when $E = 1/2$ the ellipses reach a third side of the square. When E decreases from $1/2$ to 0, the ellipses slide along the sides and move towards the diagonal.

In all the following we consider only *unit ellipses* that is, ellipses whose equation is $x^2 - y^2/E^2 = 1$ when their major and minor axes coincide with the cartesian axes. We can reformulate our problem: What is the side length $s_2(E)$ of the smallest square which contains two non-overlapping unit ellipses of aspect ratio E?

In order to prove the optimality of square packings, Friedman [2] used *sets of unavoidable points*. We adapt his definition to the case of ellipse packings: Let $E \in]0, 1]$ and let P be a set of $n-1$ points in the square $K_s = [0, s]^2$ with $s > 0$. We say that P is a *set of unavoidable points* in K_s if any unit ellipse of aspect ratio E in K_s contains an element of P (possibly on its boundary). If P is a set of unavoidable points in K_s, then $s_n(E) \geq s$. For the convenience of the reader, we recall the proof given by Friedman: Shrinking the square K_s by a factor of $1 - \varepsilon/s$ gives a set P' of $n-1$ points in a square $K_{s-\varepsilon}$ so that any unit ellipse in $K_{s-\varepsilon}$ contains an element of P' in its interior. Therefore no more than

Publication Date: November 13, 2014. Communicating Editor: Paul Yiu.

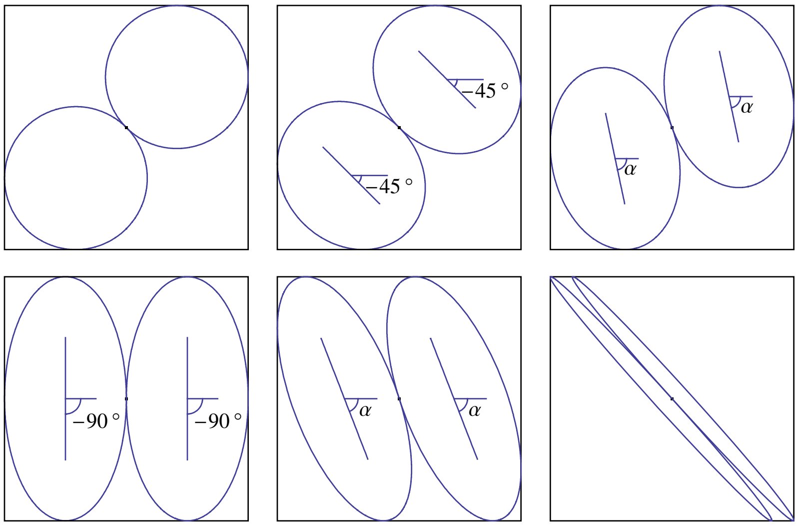


Figure 1. Six optimal packings $\mathcal{P}_E$ of two ellipses for $E = 1, 0.85, 0.69, 0.5, 0.4, 0.05$.

$n-1$ non-overlapping unit ellipses can be packed into a square of side $s-\varepsilon$, and $s_n(E) > s-\varepsilon$. Since this is true for all $\varepsilon > 0$, we must have $s_n(E) \geq s$. The upper bound $s_n(E) \leq s$ is obtained by constructing a packing of n non-overlapping unit ellipses in K_s.

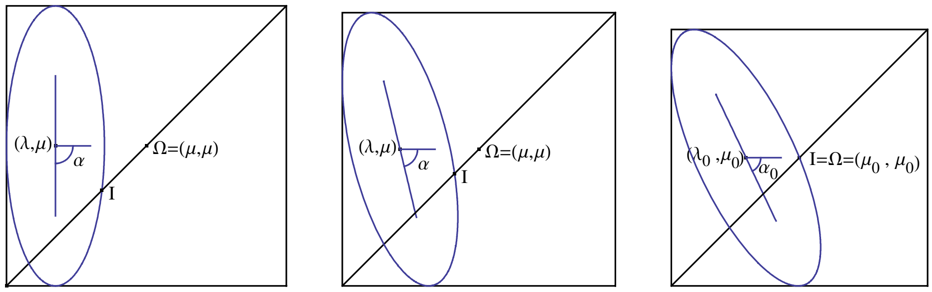
In the case of $n = 2$ ellipses and in order to get $s_2(E) \geq s$, it suffices to show that the center Ω of $K_s = [0, s]^2$ belongs to each unit ellipse $\mathbf{e} \subset K_s$ of aspect ratio E. In fact, we will consider only unit ellipses $\mathbf{e}_\alpha = \mathbf{e}_{(\lambda,\mu),\alpha,E}$ centered at (λ, μ) with $\lambda > 0, \mu > 0$, tilted at an angle $\alpha \in [-\pi/2, -\pi/4]$ and which are tangent to the axes $x = 0$ and $y = 0$:

Fact 1. Let $K_s = [0, s]^2$ be a square of side length s and $\Omega = (s/2, s/2)$ its center. If for all $\alpha \in [-\pi/2, -\pi/4]$, the ellipse $\mathbf{e}_\alpha$ contains the point Ω, then all unit ellipses $\mathbf{e}$ included in K_s contain Ω.

This fact is trivially obtained by contraposition (if a unit ellipse $\mathbf{e} \subset K_s$ does not contain the point Ω, we apply a translation followed by a reflection or a rotation and we get an ellipse $\mathbf{e}_\alpha$ with $\alpha \in [-\pi/2, -\pi/4]$ which does not contain the point Ω). As we want to find the minimal value of s such that each ellipse $\mathbf{e}_\alpha \subset K_s$ contains the center Ω, we consider the intersection points $I = (x_I(\alpha), x_I(\alpha))$ and J of the diagonal $y = x$ and the ellipse $\mathbf{e}_\alpha$, the abscissa of I being larger than the one of J. In Section 2 and 3 we will prove that:

• If $0 < E \leq 1/2$, there exists a unique $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $x_I(\alpha_0) = \mu(\alpha_0) = \mu_0$. The center $\Omega = (\mu_0, \mu_0)$ is an unavoidable point in $K_{2\mu_0}$ and then $s_2(E) \geq 2\mu_0$. The square $K_{2\mu_0}$ is displayed on the right hand side of Figure 2.

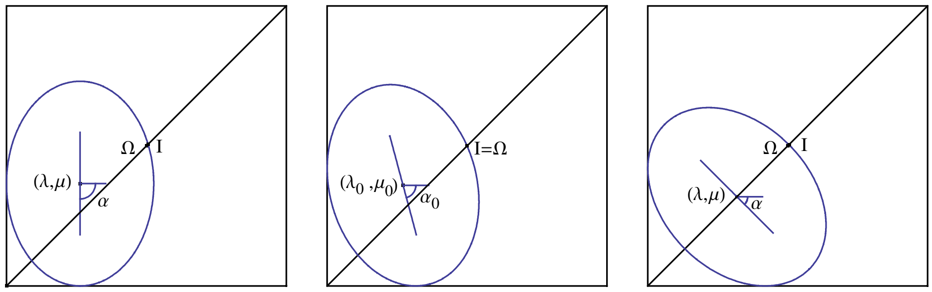
• If $1/2 < E < E_0$, the abscissa $x_I(\alpha)$ has a minimum value for a unique $\alpha_0 \in [-\pi/2, -\pi/4]$. The center $\Omega = (x_I(\alpha_0), x_I(\alpha_0))$ is an unavoidable point



$\alpha = -\frac{\pi}{2}$ $0 < \alpha < \alpha_0$ $\alpha = \alpha_0$

Figure 2. Three ellipses $\mathbf{e}_\alpha \subset K_{2\mu}$ in the case $E \leq \frac{1}{2}$

in $K_{2x_I(\alpha_0)}$ and then $s_2(E) \geq 2x_I(\alpha_0)$. The three squares in Figure 3 represent $K_{2x_I(\alpha_0)}$.



$\alpha = -\frac{\pi}{2} < \alpha_0$ $\alpha = \alpha_0$ $\alpha = -\frac{\pi}{4} > \alpha_0$

Figure 3. For $\frac{1}{2} < E < E_0$ and if $\alpha \neq \alpha_0$, the center of $K_{2x_I(\alpha_0)}$ belongs to the interior of $\mathbf{e}_\alpha$.

• If $E_0 \leq E \leq 1$, the abscissa $x_I(\alpha)$ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$. The center $\Omega = (x_I(-\pi/4), x_I(-\pi/4))$ is an unavoidable point in $K_{2x_I(-\pi/4)}$ and then $s_2(E) \geq 2x_I(-\pi/4)$, see Figure 4 in Section 3.

We finish the paper by remarking that among all the optimal packings $\mathcal{P}_E$, the densest optimal packings of two congruent ellipses in the square is $\mathcal{P}_{1/2}$.

2. Technical lemmas.

First we precise the coordinates of the center of the ellipse $\mathbf{e}_\alpha$ and the parametrization of $\mathbf{e}_\alpha$ used in Lemma 3.

Lemma 1. *(a) The coordinates of the center of the ellipse* $\mathbf{e}_\alpha$ *are equal to*

$$\lambda = \sqrt{\cos^2 \alpha + E^2 \sin^2 \alpha} \text{ and } \mu = \sqrt{\sin^2 \alpha + E^2 \cos^2 \alpha}. \tag{1}$$

The function μ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$ and we have $\mu(-\pi/2) = 1$ and $\mu(-\pi/4) = \sqrt{(1+E^2)/2}$.
(b) We have

$$(2\lambda\mu)^2 = 4E^2 + (1-E^2)^2 \sin^2 2\alpha. \tag{2}$$

(c) The ellipse $\mathbf{e}_\alpha$ can be parameterized by

$$\mathbf{e}_\alpha(t) = \left(\begin{array}{c} 2\lambda \cos^2\left(\frac{t+\varphi}{2}\right) \\ 2\mu \cos^2\left(\frac{t+\psi}{2}\right) \end{array} \right), \tag{3}$$

where the angles $\varphi \in [-\frac{\pi}{2}, 0]$ and $\psi \in [-\pi, -\frac{\pi}{2}]$ are the respective arguments of the complex numbers $\cos\alpha + iE\sin\alpha$ and $\sin\alpha - iE\cos\alpha$.

Proof. (a) Let us consider the parametrization of the ellipse $\mathbf{e}_\alpha$

$$\mathbf{e}_\alpha(t) = \left(\begin{array}{c} x(t) \\ y(t) \end{array} \right) = \left(\begin{array}{c} \lambda \\ \mu \end{array} \right) + \left(\begin{array}{cc} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{array} \right) \left(\begin{array}{c} \cos t \\ E\sin t \end{array} \right). \tag{4}$$

Let us recall that the *orthoptic curve* of $\mathbf{e}_\alpha$, i.e the locus of all points where the curve's tangents meets at right angles, is the circle centered at $\omega = (\lambda, \mu)$ with radius $\sqrt{1+E^2}$. Since the axes $x = 0$ and $y = 0$ are orthogonal, the origin $(0, 0)$ belongs to this circle and we have

$$\lambda^2 + \mu^2 = 1 + E^2. \tag{5}$$

The ellipse $\mathbf{e}_\alpha$ touches tangentially the axe $x = 0$. Therefore for some t we have $x(t) = x'(t) = 0$, which gives

$$\begin{aligned} \lambda &= -\cos\alpha\cos t + E\sin\alpha\sin t, & (6)\\ 0 &= -\cos\alpha\sin t - E\sin\alpha\cos t. & (7) \end{aligned}$$

By adding the squares of equations (6-7), we find $\lambda^2 = \cos^2\alpha + E^2\sin^2\alpha$. The value of μ comes from (5).
(b) We have

$$\begin{aligned} (\lambda\mu)^2 &= (\cos^2\alpha + E^2\sin^2\alpha)(\sin^2\alpha + E^2\cos^2\alpha) \\ &= E^2(\cos^4\alpha + \sin^4\alpha) + (1+E^4)\cos^2\alpha\sin^2\alpha \\ &= E^2(1 - 2\sin^2\alpha\cos^2\alpha) + (1+E^4)\cos^2\alpha\sin^2\alpha, \end{aligned}$$

which gives the result.
(c) By definition of φ, we have $\lambda(\cos\varphi + i\sin\varphi)$=$\cos\alpha + iE\sin\alpha$. We get

$$x(t) = \lambda + \cos\alpha\cos t - \sin\alpha E\sin t = \lambda + \lambda\cos(t+\varphi) = 2\lambda\cos^2\left(\frac{t+\varphi}{2}\right).$$

The expression of $y(t)$ is obtained similarly. □

In the proof of Lemma 3, we change the variable α to the variable T that we now define:

Lemma 2. *Let us consider the angles φ and ψ associated to the ellipse $\mathbf{e}_\alpha$ and defined in Lemma 1. We set $\delta = \frac{\psi-\varphi}{2} \in [-\frac{\pi}{4}, 0[$ and $T = -\cot\delta$. Then the real number T decreases monotonically from 1 at $\alpha = -\pi/2$ to E at $\alpha = -\pi/4$.*

Proof. The definitions of φ and ψ give

$$\begin{aligned}\frac{\mu}{\lambda}e^{i(\psi-\varphi)} &= \frac{\sin(\alpha)-iE\cos(\alpha)}{\cos\alpha+iE\sin\alpha}\\ &= \frac{1}{\lambda^2}\left(\sin\alpha\cos\alpha(1-E^2)-iE\right).\end{aligned}$$

Hence, for $\alpha \in [-\pi/2, -\pi/4]$ we have

$$\cos(\psi-\varphi) = \frac{\sin 2\alpha(1-E^2)}{2\lambda\mu} < 0 \text{ and } \sin(\psi-\varphi) = \frac{-E}{\lambda\mu} < 0. \tag{8}$$

By (8) and the formula $\tan u = (1-\cos 2u)/\sin 2u$, we find

$$\tan\left(\frac{\psi-\varphi}{2}\right) = \frac{1-\frac{\sin 2\alpha(1-E^2)}{2\lambda\mu}}{\frac{-E}{\lambda\mu}} = \frac{2\lambda\mu-\sin 2\alpha(1-E^2)}{-2E}. \tag{9}$$

With Lemma 1 (b) we find

$$\tan\left(\frac{\psi-\varphi}{2}\right) = \frac{1}{2E}\left(-\sqrt{4E^2+(1-E^2)^2\sin^2 2\alpha}+(1-E^2)\sin 2\alpha\right).$$

As $\sin 2\alpha$ decreases monotonically from 0 to -1 on the interval $[-\pi/2, -\pi/4]$, we find that $\tan((\psi-\varphi)/2)$ decreases from -1 to $-1/E$. Thus the real number $T = -1/\tan((\psi-\varphi)/2)$ decreases from 1 to E. □

Let us recall that $I = (x_I, x_I)$ is the intersection point of the diagonal $y = x$ and the ellipse $\mathbf{e}_\alpha$ with a maximal x_I.

Lemma 3. *Let us consider* $E_0 = \sqrt{(6\sqrt{3}-3)/11} \approx 0.8198$.
(a) If $E \in [E_0, 1]$*,* x_I *is decreasing for* $\alpha \in [-\pi/2, -\pi/4]$.
(b) If $E \in]1/2, E_0[$*, the function* x_I *reaches a unique minimal value for a unique real number* $\alpha_0 \in [-\pi/2, -\pi/4]$.
(c) If $E \in [0, 1/2]$*, the function* x_I *is increasing for* $\alpha \in [-\pi/2, -\pi/4]$.

Proof. First we prove that

$$x_I(\alpha) = \frac{E}{\sqrt{T}\left(\sqrt{ET^2+(1+E^2)T+E}-T\sqrt{2E}\right)}, \tag{10}$$

where $T = -\cot\delta$ has been defined in Lemma 2. We use the parametrization of the ellipse $\mathbf{e}_\alpha$ given by (3) and we look for some $t \in [0, 2\pi]$ such that

$$\sqrt{\lambda}\cos\left(\frac{t+\varphi}{2}\right) = \varepsilon\sqrt{\mu}\cos\left(\frac{t+\psi}{2}\right), \tag{11}$$

where $\varepsilon = \pm 1$. Since the point I occurs for some $t \in [0, \pi]$ (and J for some $t \in [\pi, 2\pi]$), we have $(t+\varphi)/2 \in [-\pi/4, \pi/2]$, $(t+\psi)/2 \in [-\pi/2, \pi/4]$ and we get $\varepsilon = +1$ because the two cosines in (11) are positive. In this equality we expand $\cos((t+\psi)/2) = \cos((t+\varphi)/2+\delta)$ and we find

$$\sqrt{\lambda}\cos\left(\frac{t+\varphi}{2}\right) = \sqrt{\mu}\left(\cos\left(\frac{t+\varphi}{2}\right)\cos\delta - \sin\left(\frac{t+\varphi}{2}\right)\sin\delta\right),$$

which gives

$$\tan\left(\frac{t+\varphi}{2}\right)=\frac{\sqrt{\mu}\cos\delta-\sqrt{\lambda}}{\sqrt{\mu}\sin\delta}$$

and

$$x_I=2\lambda\cos^2\left(\frac{t+\varphi}{2}\right)=\frac{2\lambda}{1+\tan^2\left(\frac{t+\varphi}{2}\right)}=\frac{2\lambda\mu\sin^2\delta}{\lambda+\mu-2\sqrt{\lambda\mu}\cos\delta}. \tag{12}$$

By (8) and Lemma 1 (b) and since $\sin^2 2\delta=1-\cos^2(\psi-\varphi)$, we have

$$\sin^2 2\delta=\frac{(2\lambda\mu)^2-(1-E^2)^2\sin^2 2\alpha}{(2\lambda\mu)^2}=\frac{4E^2}{(2\lambda\mu)^2}.$$

The previous equality gives $2\lambda\mu=-2E/\sin 2\delta$ and by (5) we have $\lambda+\mu=(1+E^2+2\lambda\mu)^{1/2}$. Substituting these values in (12) we find

$$\begin{aligned} x_I &= \frac{-E\sin\delta}{\cos\delta\left(\sqrt{1+E^2-\frac{E}{\sin\delta\cos\delta}}-\sqrt{\frac{-2E\cos\delta}{\sin\delta}}\right)} \\ &= \frac{E}{\sqrt{T}\left(\sqrt{(1+E^2)T+\frac{E}{\sin^2\delta}}-\sqrt{T}\sqrt{T}\sqrt{2E}\right)}. \end{aligned}$$

It remains to use $1/\sin^2\delta=1+T^2$ and we get (10). Let us denote by $f(T)$ the denominator of the right hand side of (10). We find

$$f'(T)=\frac{g(T)-h(T)}{2\sqrt{T}\sqrt{ET^2+(1+E^2)T+E}},$$

where $h(T)=3\sqrt{2E}\,T\sqrt{ET^2+(1+E^2)T+E}$ and $g(T)=3ET^2+2(1+E^2)T+E$. Since the functions $g(T)$ and $h(T)$ are positive, the sign of $f'(T)$ is equal to the one of the polynomial $P(T)=g^2(T)-h^2(T)$, that is

$$P(T)=-9E^2T^4-6E(1+E^2)T^3+4(E^4-E^2+1)T^2+4E(1+E^2)T+E^2. \tag{13}$$

We get $P'(T)=-36E^2T^3-18E(1+E^2)T^2+8(E^4-E^2+1)T+4E(1+E^2)$. The discriminant of $P''(T)=-4(27E^2T^2+9E(1+E^2)T-2(E^4-E^2+1))$ is $\Delta=16\cdot 27E^2(11E^4-2E^2+11)>0$. We remark that $P''(0)>0$ and $\lim_{T\to\infty}P''(T)=-\infty$, then $P''(T)$ has a unique positive root T_2. Since $P'(0)>0$ and $\lim_{T\to\infty}P'(T)=-\infty$, the polynomial $P'(T)$ has a unique root $T_1>T_2$ and $P'(T)\geq 0$ for all $T\in[0,T_1]$. Finally, $P(0)=E^2>0$ implies that the polynomial $P(T)$ has a unique positive root T_0.

Moreover, $P(E)=-E^2(11E^4+6E^2-9)$ vanishes at a unique positive value $E_0=\sqrt{(-3+6\sqrt{3})/11}$. We remark that $T_0=E_0$ if and only if $E=E_0$. In the three following cases we conclude with Lemma 2:

(a) If $E\in[E_0,1]$, we have $P(E)\leq 0$. Then $T_0\leq E\leq 1$ which implies that $P(T)\leq 0$ for all $T\in[E,1]$. So $f(T)$ is decreasing on $[E,1]$ and then x_I is increasing with respect to T.

(b) If $E \in]1/2, E_0[$, we have $P(E) > 0$ and $P(1) = 2(E+1)^2(2E-1)(E-2) < 0$. Then $E < T_0 < 1$ which implies that $f(T)$ is increasing on $[E, T_0]$ and decreasing on $[T_0, 1]$. Thus x_I is decreasing for $T \in [E, T_0]$ and increasing for $T \in [T_0, 1]$.

(c) If $E \in]0, 1/2]$, we have $P(E) \geq 0$ and $P(1) \geq 0$ which implies that $P(T) \geq 0$ on $[E, 1]$. Thus x_I is decreasing for $T \in [E, 1]$. □

3. Calculation of $s_2(E)$

Now we can describe the various optimal packings of two ellipses in the square and the corresponding side lengths $s_2(E)$.

Theorem 4. *If $E \leq 1/2$, then*

$$s_2(E) = \sqrt{(1+E)^2 + \sqrt{(1+E)^4 - 8E^2}}. \tag{14}$$

The minimum value $s_2(E)$ is obtained for two parallel ellipses $\mathbf{e}_1 = \mathbf{e}_{\alpha_0}$ and $\mathbf{e}_2$ with

$$\alpha_0 = -\arccos \frac{1}{2}\sqrt{\frac{4 - s_2^2(E)}{1 - E^2}}, \tag{15}$$

and where $\mathbf{e}_2$ is the reflection of $\mathbf{e}_1$ through the center of the square.

Proof. If $\alpha = -\pi/2$, the center $\Omega = (\mu, \mu)$ of $K_{2\mu}$ does not belong to $\mathbf{e}_\alpha$ (except for $E = 1/2$). If $\alpha = -\pi/4$, the center Ω is also the center of the ellipse $\mathbf{e}_\alpha$. We know by Lemma 1 (a) and Lemma 3 (c) that $x_I - \mu$ is increasing for $\alpha \in [-\pi/2, -\pi/4]$. Then there exists a unique angle $\alpha_0 \in [-\pi/2, -\pi/4]$ such that $I = \Omega_0 = (\mu_0, \mu_0)$ with $\mu_0 = \mu(\alpha_0)$. We note that the center Ω_0 belongs to the boundary of the ellipse $\mathbf{e}_{\alpha_0}$, see Figure 2. Let us show that in the square $K_{2\mu_0}$, the center Ω_0 is an unavoidable point. By Fact 1, it suffices to show that any ellipses $\mathbf{e}_\alpha$ included in $K_{2\mu_0}$ contain Ω_0:

- If $\alpha < \alpha_0$, the ellipse $\mathbf{e}_\alpha$ is not contained in $K_{2\mu_0}$ because it intersects the upper side $y = 2\mu_0$ (μ is decreasing for $\alpha \in [-\pi/2, -\pi/4]$).
- If $\alpha > \alpha_0$, the point Ω_0 belongs to the interior of the ellipse $\mathbf{e}_\alpha$ because x_I is increasing for $\alpha \in [\alpha_0, -\pi/4]$.

It remains to calculate α_0 and μ_0. First, we show that

$$\mu_0 - \lambda_0 = \frac{E}{\mu_0}. \tag{16}$$

We have by (4) that $\mathbf{e}_\alpha(t) = (\mu, \mu)$ if and only if

$$\begin{aligned} \mu - \lambda &= \cos\alpha\cos t - E\sin\alpha\sin t, \\ 0 &= \sin\alpha\cos t + E\cos\alpha\sin t. \end{aligned}$$

Substituting $-E\cos\alpha\sin t/\sin\alpha$ for $\cos t$ in the first equality, we find

$$\begin{aligned} -E\sin(t) &= (\mu - \lambda)\sin\alpha, \\ \cos t &= (\mu - \lambda)\cos\alpha, \end{aligned}$$

which implies $E^2 = (\mu-\lambda)^2(\sin^2\alpha + E^2\cos^2\alpha) = (\mu-\lambda)^2\mu^2$, and then (16) since $\mu \geq \lambda$ for $\alpha \in [-\pi/2, -\pi/4]$. By (5) we get for $\alpha = \alpha_0$,

$$1 + E^2 + 2\lambda_0\mu_0 = (\lambda_0+\mu_0)^2 = (\lambda_0 - \mu_0 + 2\mu_0)^2 = \left(2\mu_0 - \frac{E}{\mu_0}\right)^2.$$

Since (16) implies $\mu_0^2 - E = \lambda_0\mu_0$, we have

$$2(\mu_0^2 - E) = 4\mu_0^2 + \frac{E^2}{\mu_0^2} - 4E - (1+E^2).$$

This equation in μ_0^2 leads to

$$4\mu_0^2 = (1+E)^2 + \varepsilon\sqrt{(1+E)^4 - 8E^2},$$

where $\varepsilon = \pm 1$. The case $\varepsilon = -1$ leads to $4\lambda_0\mu_0 = 4\mu_0^2 - 4E = (1-E)^2 - \sqrt{(1+E)^4 - 8E^2} \leq 0$ what is impossible.

Then $s_2(E) \geq 2\mu_0 = \sqrt{(1+E)^2 + \sqrt{(1+E)^4 - 8E^2}}$. We can pack in $K_{2\mu_0}$ the reflection of $\mathbf{e}_{\alpha_0}$ through Ω_0 and we get the equality (14). We finally obtain the angle (15) by considering $\mu_0^2 = \sin^2\alpha_0 + E^2\cos^2\alpha_0$ and $s_2(E) = 2\mu_0$. □

Theorem 5. *If* $1/2 < E < E_0$*, then*

$$s_2(E) = \frac{2E}{\sqrt{T_0}\left(\sqrt{ET_0^2 + (1+E^2)T_0 + E} - T_0\sqrt{2E}\right)}, \tag{17}$$

where T_0 *is the unique positive root of (13). The minimum value* $s_2(E)$ *is obtained for two parallel ellipses* $\mathbf{e}_1 = \mathbf{e}_{\alpha_0}$ *and* $\mathbf{e}_2$ *with*

$$\alpha_0 = -\frac{1}{2}\left(\pi + \arcsin\frac{E(T_0^2-1)}{T_0(1-E^2)}\right), \tag{18}$$

and where the ellipse $\mathbf{e}_2$ *is the reflection of* $\mathbf{e}_1$ *through the center of the square.*

Proof. We denote by α_0 the unique angle α such that $T_0 = T(\alpha)$ and by I_0 the intersection of $\mathbf{e}_{\alpha_0}$ with $y = x$. Since the continuous function x_I is decreasing for $\alpha \in [-\pi/2, \alpha_0]$ and increasing for $\alpha \in [\alpha_0, -\pi/4]$, the point I_0 belongs to the interior of $\mathbf{e}_\alpha$ if $\alpha \neq \alpha_0$. Then any ellipse $\mathbf{e}_\alpha$ in $K_{2x_{I_0}}$ contains I_0. As Fact 1 gives that I_0 is an unavoidable point, we have $s_2(E) \geq 2x_{I_0}$. We can pack the reflection of the ellipse $\mathbf{e}_{\alpha_0}$ through I_0 and we get $s_2(E) \leq 2x_{I_0}$. The value of x_{I_0} is given by (10).

By (2), (9) and since $-1/T = \tan(\psi - \varphi)/2$, we have

$$(2\lambda\mu)^2 = \left(\frac{2E}{T} + (1-E^2)\sin 2\alpha\right)^2 = 4E^2 + (1-E^2)^2\sin^2 2\alpha,$$

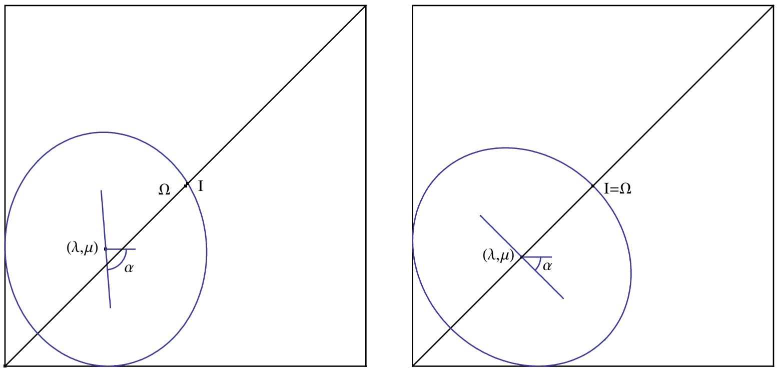
which gives $\sin 2\alpha = E(T^2-1)/((1-E^2)T)$ and (18) for $2\alpha \in [-\pi, -\pi/2]$. □

Theorem 6. *If* $E_0 \leq E \leq 1$*, then*

$$s_2(E) = \sqrt{2}\left(\sqrt{1+E^2} + E\right). \tag{19}$$

The minimum value $s_2(E)$ is obtained for two parallel ellipses $\mathbf{e}_1 = \mathbf{e}_\alpha$ and $\mathbf{e}_2$ with $\alpha = -\pi/4$ and where $\mathbf{e}_2$ is the reflection of $\mathbf{e}_1$ through the center of the square.

Proof. We consider again the intersection point I_0 of the ellipse $\mathbf{e}_{-\pi/4}$ and the diagonal $y = x$. Since x_I is decreasing for $\alpha \in [-\pi/2, -\pi/4]$, any ellipse $\mathbf{e}_\alpha$ in $K_{2x_{I_0}}$ contains I_0. As Fact 1 gives that I_0 is an unavoidable point, we get $s_2(E) \geq 2x_{I_0} = 2x_I(-\pi/4) = \sqrt{2}(\sqrt{1+E^2}+E)$. As in the two previous cases, we can pack the reflection of the ellipse $\mathbf{e}_{-\pi/4}$ through I_0 and we get the equality (19). □



$-\frac{\pi}{2} \leq \alpha < -\frac{\pi}{4}$ $\qquad\qquad$ $\alpha = -\frac{\pi}{4}$

Figure 4. For $E_0 \leq E \leq 1$ and $\frac{\pi}{2} \leq \alpha < -\frac{\pi}{4}$, the center of $K_{2x_I(-\frac{\pi}{4})}$ belongs to the interior of $\mathbf{e}_\alpha$.

It is not surprising that among all the optimal packings $\mathcal{P}_E$, the densest one is $\mathcal{P}_{1/2}$, see Figure 1. We denote by $d(E)$ the density of $\mathcal{P}_E$ and we have for all E in $]0, 1]$,

$$d(E) = \frac{2\pi E}{s_2^2(E)}.$$

The formulas (14), (17), (19) for $s_2(E)$ give that $d(E)$ equals to

$$\begin{aligned}
d_1(E) = &\quad \frac{2\pi E}{(1+E)^2 + \sqrt{(1+E)^4 - 8E^2}} \text{ if } E \in]0, \frac{1}{2}],\\
d_2(E) = &\quad \frac{\pi}{2E} T_0 \left(\sqrt{E T_0^2 + (1+E^2) T_0 + E} - T_0 \sqrt{2E} \right)^2 \text{ if } E \in]\frac{1}{2}, E_0[,\\
d_3(E) = &\quad \frac{\pi E}{\left(\sqrt{1+E^2} + E\right)^2} \text{ if } E \in [E_0, 1].
\end{aligned}$$

The optimality of $\mathcal{P}_E$ for all E on each interval $]0, 1/2],]1/2, E_0[, [E, 1]$ implies the continuity of $d(E)$ on $]0, 1]$. It is easy to verify that $d_1(1/2) = d_2(1/2) =$

$\pi/4$ and $d_2(E_0) = d_3(E_0) = \pi E_0 / \left(\sqrt{1+E_0^2}+E_0\right)^2$. We show that $d(E)$ is increasing on $]0,1/2]$ and decreasing on $[1/2,1]$. For $E \in]0,\frac{1}{2}]$, we get

$$d_1'(E) = \frac{2\pi(1-E^2)}{\sqrt{(E+1)^4-8E^2}\left(1+2E+E^2+\sqrt{(E+1)^4-8E^2}\right)} > 0$$

and for $E \in [E_0, 1]$,

$$d_3'(E) = \frac{\pi\left(\sqrt{1+E^2}-2E\right)}{\sqrt{1+E^2}\left(\sqrt{1+E^2}+E\right)^2} < 0.$$

In the case of Theorem 5, we have $s_2(E) = 2E/f(T_0)$ and $d_2(E) = (\pi/(2E))f^2(T_0)$. Since $T_0 = T_0(E)$ is a single root of (13), $T_0(E)$ is differentiable at $E \in]1/2, E_0[$ and we get

$$d_2'(E) = \frac{\pi}{2E^2}\left(2Ef(T_0)f'(T_0)\frac{dT_0}{dE} - f^2(T_0)\right).$$

As $f'(T_0) = 0$, we obtain $d_2'(E) < 0$.

References

[1] H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New-York, 1991.

[2] E. Friedman, Packing unit squares in squares: a survey and new results, *Electron. J. Combin.*, 7 (2000), ♯ DS7.

[3] H. Melissen, Packing and covering with circles, Ph.D. Thesis, Utrecht University, 1997.

[4] E. Specht, website, `http://hydra.nat.uni-magdeburg.de/packing/`.

Thierry Gensane: LMPA J. Liouville, B.P. 699, F-62228 Calais, Univ Lille Nord de France, F-59000 Lille, France

E-mail address: gensane@lmpa.univ-littoral.fr

Pascal Honvault: LMPA J. Liouville, B.P. 699, F-62228 Calais, Univ Lille Nord de France, F-59000 Lille, France

E-mail address: honvault@lmpa.univ-littoral.fr