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On a Circle Containing the Incenters of Tangential Quadrilaterals

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Abstract. When we fix one side and draw different tangential quadrilaterals having the same side lengths but different angles we observe that their incenters lie on a circle. Based on a known formula expressing the incircle radius of a tangential quadrilateral by its tangent lengths, some older results will be presented in a new light and the equation of the before mentioned circle will appear. This circle encodes information about tangential and bicentric quadrilaterals that leads to an apparently new characterization of tangential quadrilaterals. Curiously enough, no trigonometric formulae are needed.

1. Introduction

Figure 1 shows a tangential quadrilateral $ABCD$, its incircle with incentre I and radius r. Let W, X, Y, Z, be the tangency points and denote the tangent lengths AW etc. by e, f, g and h. While the tangent lengths determine the sides of a tangential quadrilateral $AB = a = e + f$, $BC = b = f + g$, $CD = c = g + h$, $DA = d = h + e$, the tangent lengths cannot be derived unambiguously from the sides - in contrast to the triangle, where the tangent lengths are $e = s-a$, $f = s-b$, $g = s - c$ with semiperimeter s.

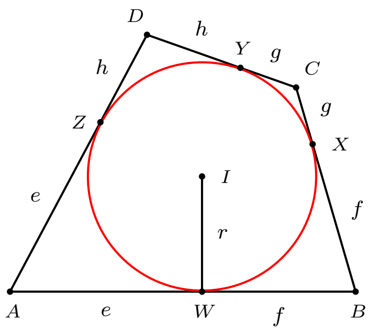

Figure 1

The reason is that the condition $a + c = b + d$ for the sides in a tangential quadrilateral produces one degree of freedom in the solutions of the equations for the tangent lengths. This dichotomy between ambiguous and unambiguous tangent lengths in quadrilaterals and triangles continues to hold in polygons of an even and odd number of vertices.

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2. Inradius and distances of incenter from vertices of the quadrilateral

Given the tangent lengths e, f, g, h, the radius r of the inscribed circle of the tangential quadrilateral is determined according to the formula

$$r^2 = \frac{fgh + egh + efh + efg}{e + f + g + h} \tag{1}$$

cf. [8], [9, Lemma 2], [13, (1)], [16]. In [8] this equation is derived from

$$\mathrm{Im}((r + ei)(r + fi)(r + gi)(r + hi)) = \sigma_1 r^3 - \sigma_3 r = 0, \tag{2}$$

wherein σ_k are the k-th degree elementary symmetric functions of e, f, g, h. Formula (2) says that the four right-angled triangles with the legs r and e, r and f, r and g, and r and h can be put together to form an angle of $180°$. From one pair of each of these triangles one can form a tangential quadrilateral as shown in Figure 2.

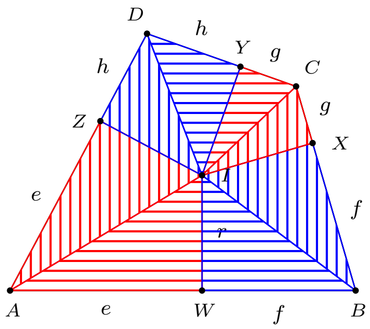

Figure 2

The generalization of formula (2) for arbitrary polygons can be resolved unambiguously for $r > 0$ only in the case of the triangle and the tangential quadrilateral and leads for the triangle to the formula

$$r^2 = \frac{efg}{e + f + g} = \frac{(s - a)(s - b)(s - c)}{s},$$

which expresses the radius r of its incircle by the tangent lengths or the side lengths. This also gives a short proof of Heron's triangle area formula [18] in the spirit of "proofs without words". From (1) we obtain

$$AI^2 = r^2 + e^2 = \frac{fgh + egh + efh + efg}{e + f + g + h} + e^2 = \frac{(e + f)(e + g)(e + h)}{e + f + g + h}, \tag{3}$$

compare with [13, (5)]. Either multiply out the right hand side, or observe that $\sigma_1(r^2 + e^2) = \sigma_3 + e^2\sigma_1$ is a monic third degree polynomial in e vanishing for $e = -f$, $e = -g$ and $e = -h$. Similar formulae hold for the other distances between the vertices and the incenter I.

For later use, we note

$$\frac{AI \cdot CI}{BI \cdot DI} = \frac{e + g}{f + h}, \tag{4}$$

which is an immediate consequence of (3). Similar formulae are in [7, Theorem 8].

3. Construction of the bicentric quadrilateral from its side lengths

With given side lengths $AB = a$, $BC = b$, $CD = c$, $DA = d$, that fulfill the condition $a+c = b+d$, we can construct in general different incongruent tangential quadrilaterals. Among them, the tangential quadrilateral with the greatest area, and therefore with the greatest radius of its incircle, is the bicentric quadrilateral with sides a, b, c, d, (cf. [3, p.135, (5)], [4, p. 238, (35)], [20]). The conditions for its constructability $a < b + c + d$ etc. as a cyclic quadrilateral from its side lengths are obviously satisfied and the construction can be carried out by use of the circle of Apollonius as in [11, pp. 82-83], or by joining two similar cyclic quadrilaterals such that they form a trapezoid with the same angles, but in a different order (see Figure 3). It seems that Bretschneider must have had this gluing method of similar geometric figures in mind when he produced this impressive series of formulas for triangles and quadrilaterals in [3] and [4]. Recently, Varverakis used this trick in his article [20] to have a short and visual proof of the maximum area property for cyclic quadrilaterals.

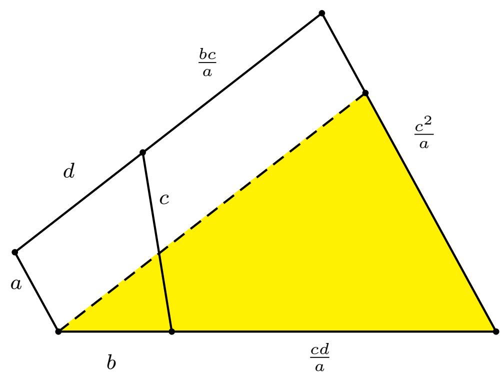

Figure 3

The triangle inequalities for the shaded triangle in Figure 3, i.e. the constructability conditions for the trapezoid and hence for the bicentric quadrilateral are satisfied. Consider, for example, this one

$$\left(d + \frac{bc}{a}\right) + \left(\frac{c^2}{a} - a\right) > \left(b + \frac{dc}{a}\right).$$

It is equivalent to $(c - a)(a + b + c - d) > 0$ and is satisfied if the dilated quadrilateral is glued to the larger of the sides a and c. In the same way the other triangle inequalities are proved. For $c = a$, there is no need to paste two similar quadrilaterals together since cyclic quadrilaterals with a pair of equal opposite sides are trapezoids.

The area K of a bicentric quadrilateral is $K = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{abcd}$ by Brahmagupta's formula (cf. [3, p.135, (5)], [4, p. 238, (35)], [5, Th.

3.22], [6, pp.62-63], [10], [11, Th. 109], [15]), and the radius of its incircle

$$r_{\max} = \frac{K}{s} = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{b+d} \tag{5}$$

is the largest among the incircle radii of all tangential quadrilaterals with sides a, b, c, d.

4. Construction of a tangential quadrilateral from given tangent lengths and conclusions

If we want to construct a tangential quadrilateral with given tangent lengths e, f, g, h, we could determine its inradius r with some intricate segment multiplications and divisions based on formula (1) and then assemble the right-angled triangles with the legs r and e, r and f, r and g, and r and h as in Figure 2. Another way that takes us further and leads to the main formula (7), starts from the bicentric quadrilateral with side lengths $a = e + f$, $b = f + g$, $c = g + h$, $d = h + e$. We get the same bicentric quadrilateral with sides a, b, c, d from different tangent lengths, the degree of freedom being the choice of one of them, e.g. e. Then the other tangent lengths are

$$\begin{aligned} BW = {} & BX = f = a - e, \\ CX = {} & CY = g = b - a + e, \\ DY = {} & DZ = b = d - e; \end{aligned} \tag{6}$$

see Figure 4.

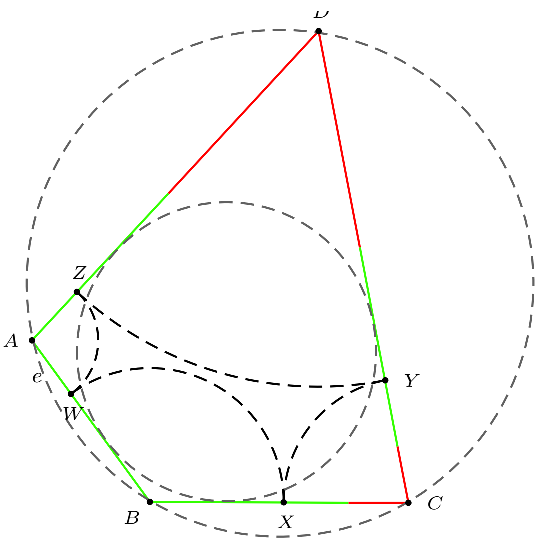

Figure 4

The right hand sides are positive, if a is the smallest side and $e < a$. The green zones indicate where the endpoints of the segments of lengths e, f, g, h are situated, the red zones are excluded. According to (1), the incircle radius for a tangential quadrilateral with the tangent lengths from (6) is

$$\begin{aligned} r^2 = & \frac{(f+g)eh + (h+e)fg}{(f+g)+(h+e)} \\ = & \frac{be(d-c) + d(a-e)(b-a+e)}{b+d} \\ = & -\left(e - \frac{ad}{b+d}\right)^2 + \frac{abcd}{(b+d)^2} \\ = & -\left(e - \frac{ad}{s}\right)^2 + r_{\max}^2. \end{aligned} \tag{7}$$

This formula has the following consequences.

Theorem 1. *A tangential quadrilateral $ABCD$ with sides a, b, c, d, and semiperimeter $s = a + c = b + d$ is cyclic if and only if its tangent lengths are*

$$e = \frac{ad}{s}, \quad f = \frac{ab}{s}, \quad g = \frac{bc}{s}, \quad h = \frac{cd}{s}. \tag{8}$$

With these formulae, it is easy to deduce the characterization of bicentrics in Problem 10804 in the MONTHLY [19] or in Hajja's article [9, Lemma 1].

Corollary 2. *A tangential quadrilateral is bicentric if and only if $eg = fh$.*

One direction is obvious from (8). If, on the other hand, $eg = fh$, then $eb = e(f+g) = f(h+e) = fd$, and together with $e + f = a$, we find $e = \frac{ad}{b+d}$, $f = \frac{ab}{b+d}$, i.e. (8).

From $\frac{AC}{BD} = \frac{ad+bc}{ab+cd}$, a companion formula of Ptolemy's theorem, valid for all cyclic quadrilaterals (cf. [1, p.130], [2, Lemma 2], [11, p. 85]) we get immediately from (8) the second criterion of [9].

Corollary 3. *A tangential quadrilateral $ABCD$ is bicentric only if*

$$\frac{AC}{BD} = \frac{e+g}{f+h}. \tag{9}$$

To prove the converse of Corollary 3, we invert the points A, B, C, D, with respect to the incircle and insert (4) into the formulae

$$AC = A'C' \cdot \frac{AI \cdot CI}{r^2}, \qquad BD = B'D' \cdot \frac{BI \cdot DI}{r^2}$$

for the distances of the images A', B', C', D', and get

$$\frac{AC}{BD} = \frac{e+g}{f+h} \cdot \frac{A'C'}{B'D'}. \tag{10}$$

Formula (10) says that under condition (9) the diagonals of the parallelogram $A'B'C'D'$ are of equal lengths, so that $A'B'C'D'$ is a rectangle, hence cyclic, and the pre-images A, B, C, D, are cyclic too.

Writing (7) as $r^2 + \left(e - \frac{ad}{s}\right)^2 = r_{\max}^2$, we obtain the following Theorem 4, or more visually, Theorem 4′ as announced in the abstract.

Theorem 4. *For different choices of the tangent lengths e, f, g, h, having the same sums $a = e + f$, $b = f + g$, $c = g + h$, $d = h + e$, we make the side AB of the bicentric quadrilateral $ABCD$ with sides a, b, c, d to the x-axis of a coordinate system with origin A. The points with coordinates (e, r) move on a circle around the point W of tangency of the incircle with AB, which goes through the incenter I of $ABCD$.*

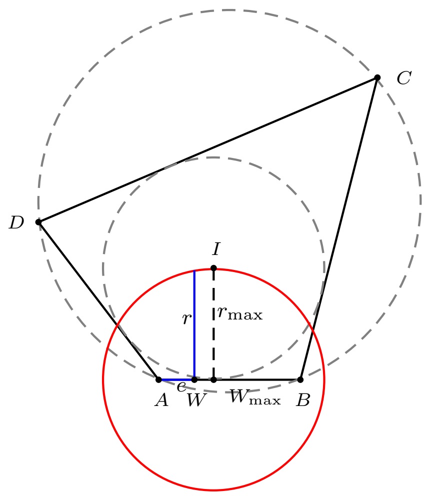

Figure 5

Theorem 4′. *When we fix one side and draw different tangential quadrilaterals having the same side lengths by changing the angles, the incenters will move on a circle.*

This theorem allows us to construct the inradius $r = \sqrt{\frac{fge+egh+efh+efg}{e+f+g+h}}$ of a tangential quadrilateral with the tangent lengths e, f, g, h from the bicentric quadrilateral $ABCD$ with the sides $e+f$, $f+g$, $g+h$, $h+e$. Having once determined r, we can easily construct the tangential quadrilateral with given tangent lengths. To perform this construction of r draw the perpendicular at the point W with $AW = e$ on the side AB of the bicentric quadrilateral $ABCD$. It intersects the above mentioned circle of the incenters around $W_{\max}$ at a point whose distance to AB is r. This construction can be carried out without any restriction if one starts with a tangency point within the green ranges of Figure 4. If, for the sake of simplicity, AB is the shortest side of the quadrilateral, the inequalities

$$AW_{\max} = \frac{ad}{s} \leq r_{\max} = \frac{\sqrt{abcd}}{s},$$
$$BW_{\max} = \frac{ab}{s} \leq r_{\max} = \frac{\sqrt{abcd}}{s}$$

guarantee that the segment AB lies within the circle with the center $W_{\max}$ and the radius $r_{\max}$, see Figure 5.

Another way to understand the formula (7) is the following characterization of tangential quadrilaterals, complementing the investigations of [14] and [17].

Theorem 5. *The points K, L, M, N are situated on the sides a, b, c, d, of a quadrilateral $ABCD$ such that they divide the respective sides in the ratio of the adjacent sides:*

$$\frac{AK}{KB}=\frac{d}{b},\quad \frac{BL}{LC}=\frac{a}{c},\quad \frac{CM}{MD}=\frac{b}{d},\quad \frac{DN}{NA}=\frac{c}{a}. \tag{11}$$

Then $ABCD$ is a tangential quadrilateral if and only if $KLMN$ is cyclic.

Proof. For a tangential quadrilateral we infer from (7) that the points K, L, M, N, lie on a circle around I with radius $r_{\max}=\frac{\sqrt{abcd}}{a+c}$.

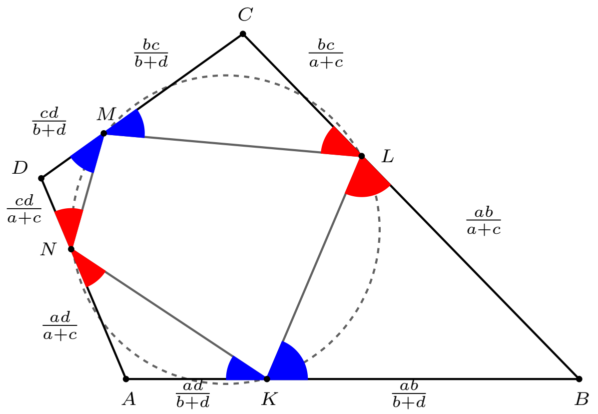

Figure 6.

In order to prove the converse, let $KLMN$ be a cyclic quadrilateral and suppose that $a+c>b+d$. Then $AK=\frac{ad}{b+c}>\frac{ad}{a+c}=AN$ implies that $\angle AKN<\angle KNA$. Similar inequalities hold for the other angles between the sides of $KLMN$ and the sides of $ABCD$. Therefore the sum of the red angles at points N and L, i.e. $\angle KNA+\angle DNM+\angle MLC+\angle BLK$ is greater than the sum of the corresponding blue angles at points K and M. From this we get a contradiction because for a cyclic quadrilateral $KLMN$ both sums must be $180°$. Hence we conclude that the quadrilateral $ABCD$ is tangential. $\square$

It is noteworthy that the radii $r_{\max}=\frac{K}{s}=\frac{\sqrt{abcd}}{a+c}$ of the circles through points K, L, M and N, for incongruent tangential quadrilaterals with equal side lengths but different angles do not depend on the latter. This permits another construction of the bicentric quadrilateral with side lengths a, b, c, d, $a+c=b+d$. This construction is as follows: Draw any tangential quadrilateral $ABCD$ with the side lengths a, b, c, d. Divide its sides in the ratio of the adjacent sides to get the points K, L, M and N (cf. (11)). Measure the distance of the incenter I from one of the points K, L, M and N and construct a circle with this distance as radius, tangent to AB in K. Then complete the construction by drawing tangents from A and B

to this circle whereon the segments $AD = d$ and $BC = b$ are given the prescribed lengths.

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