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Generalized Archimedean Arbelos Twins

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Abstract. We generalize the well known Archimedean arbelos twins by extending the notion of arbelos, and we construct an infinite number of Archimedean circles.

1. Archimedean arbelos

On a segment AB we take an arbitrary point P and with diameters AP, PB, AB we construct the semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, where $R = R_1 + R_2$. If we cut from the large semicircle the small ones then the resulting figure is called from antiquity (Archimedes) arbelos (the shoemaker's knife). The perpendicular at P to AB meets the large semicircle at Q and divides the arbelos in two mixtilinear triangles with equal incircles.

Theorem 1 (Archimedean arbelos twins)**.** *The two circles $K_1(r_1)$ and $K_2(r_2)$ that are inside the arbelos and are tangent to the arbelos and the line PQ have equal radii $r_1 = r_2 = \frac{R_1R_2}{R_1+R_2}$* (see Figure 1)*. Equivalently,*

$$\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.$$

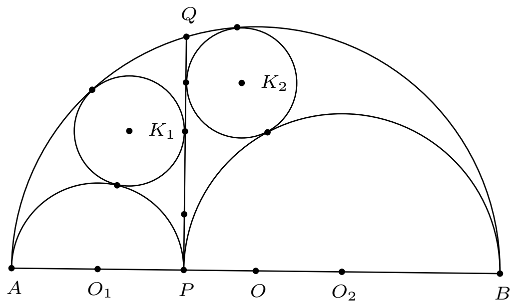

Figure 1.

A circle in the arbelos with radius r_1 is called an Archimedean circle ([6, p.61]).

We can find infinitely many such twin incircles where the above case is a limit special case. We generalize the notion of the arbelos as a triangle whose sides are in general circular arcs. We will investigate some special cases of these arbeloi.

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2. Generalized arbelos

2.1. *Soddy type arbelos with 3 vertices tangency points of the arcs.* Let U_X, U_Y, U_Z (counter clockwise direction) be the vertices of the arbelos whose sides are arcs of circles tangent to each other at the vertices, with centers X, Y, Z and radii r_X, r_Y, r_Z. If the rotation from U_Y to U_Z on the arc with radius r_X is clockwise relative to U_X, then r_X is positive, otherwise it is negative. Similarly we characterize the radii r_Y, r_Z relative to the movement on the appropriate arcs from U_Z to U_X, and from U_X to U_Y. Hence, we can have 8 different cases of arbeloi with the same vertices and the same radii in absolute value. If we change the sign of the radii of the arbelos, then we get the complementary arbelos. Denote by Δ the area of the triangle XYZ and r the radius of a circle tangent to the arcs-sides of the arbelos.

Theorem 2. *The radius r of the circle that is tangent to the sides of the above arbelos is given by*

$$\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} + \frac{2\Delta}{r_X r_Y r_Z},$$

or

$$\frac{1}{r} = \frac{1}{r_X} + \frac{1}{r_Y} + \frac{1}{r_Z} - \frac{2\Delta}{r_X r_Y r_Z}.$$

One of these corresponds to the complementary arbelos.

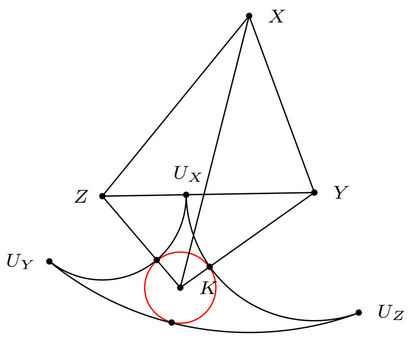

Figure 2a

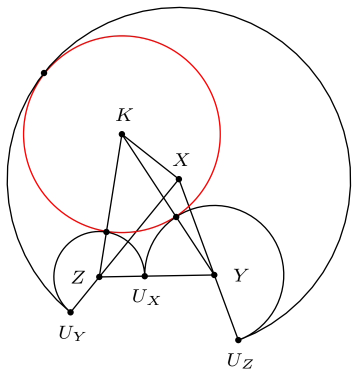

Figure 2b

Proof. If $K(r)$ is the circle tangent to the sides of the arbelos, then in Figure 2a, the radius r_X is negative, the sides of triangle XYZ are $a = |r_Y + r_Z|$, $b = |r_Z + r_X|$, $c = |r_X + r_Y|$, and the tripolar coordinates of K relative to XYZ and the area Δ of triangle XYZ are

$$KX = \lambda = |r_X + r|, \quad KY = \mu = |r_Y + r|, \qquad KZ = \nu = |r_Z + r|,$$
$$\Delta = \sqrt{r_X r_Y r_Z (r_X + r_Y + r_Z)}.$$

If we substitute these values into the equality that the tripolar coordinates satisfy ([3]):

$$(\mu^2+\nu^2-a^2)^2\lambda^2+(\nu^2+\lambda^2-b^2)^2\mu^2+(\lambda^2+\mu^2-c^2)^2\nu^2$$
$$-(\mu^2+\nu^2-a^2)(\nu^2+\lambda^2-b^2)(\lambda^2+\mu^2-c^2)=\lambda^2\mu^2\nu^2,$$

we get

$$\left(\frac{1}{r}-\frac{1}{r_X}-\frac{1}{r_Y}-\frac{1}{r_Z}\right)^2=\frac{4\Delta^2}{r_X^2r_Y^2r_Z^2}.$$

Therefore,

$$\frac{1}{r}=\frac{1}{r_X}+\frac{1}{r_Y}+\frac{1}{r_Z}\pm\frac{2\Delta}{r_Xr_Yr_Z}.$$

□

Remarks. (1) The Arcimedean arbelos is of Soddy type with collinear vertices where $r_X=R_1$, $r_Y=R_2$, $r_Z=-R=-(R_1+R_2)$, $\Delta=0$. In this case, there is a double solution. Hence, the inradius of the Archimedean arbelos is given by

$$\frac{1}{r}=\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_1+R_2}.$$

(2) If the radii r_X, r_Y, r_Z are positive, then the radii r refer to the inner and outer Soddy circles ([2]).

In the sequel, we shall adopt the following notation: For a line ℓ, Π_ℓ denotes the *orthogonal projection map* onto ℓ.

Lemma 3. *Let $K(r)$ be a circle tangent externally or internally to the circles $O_1(R_1)$ and $O_2(R_2)$, where R_1, R_2 may assume any real values. For a point F on the radical axis of the circles,*

$$r=\frac{\overrightarrow{O_1O_2}\cdot\Pi_{O_1O_2}(\overrightarrow{FK})}{R_1-R_2}. \quad (1)$$

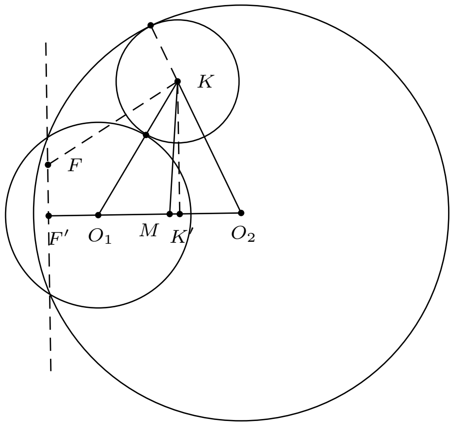

Figure 3.

Proof. Let M be the midpoint of O_1O_2 (Figure 3). We have

$$R_2^2 - R_1^2 = FO_2^2 - FO_1^2 = (\overrightarrow{FO_2} - \overrightarrow{FO_1}) \cdot (\overrightarrow{FO_2} + \overrightarrow{FO_1}) = 2\overrightarrow{O_1O_2} \cdot \overrightarrow{FM},$$

$$|R_1 + r|^2 - |R_2 + r|^2 = KO_1^2 - KO_2^2 = (\overrightarrow{KO_1} - \overrightarrow{KO_2}) \cdot (\overrightarrow{KO_1} + \overrightarrow{KO_2}) = 2\overrightarrow{O_1O_2} \cdot \overrightarrow{MK}.$$

By addition, we get

$$2r(R_1 - R_2) = 2\overrightarrow{O_1O_2} \cdot (\overrightarrow{FM} + \overrightarrow{MK}) = 2\overrightarrow{O_1O_2} \cdot \Pi_{O_1O_2}(\overrightarrow{FK}),$$

and the result follows. □

Remark. For the Archimedean twins (1) gives

$$r_1 = \frac{(2R_1 - r_1)R_2}{R_1 + R_2 + R_1} \implies \frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Theorem 4. *Let $U_XU_YU_Z$ be an arbelos with collinear centers X, Y, Z on a line $\mathcal{L}$, and $K(r)$ be the incircle of the arbelos.*

$$r = \left| \frac{\Pi_{\mathcal{L}}(\overrightarrow{U_YU_Z})}{\frac{r_X - r_Y}{XY} - \frac{r_X - r_Z}{XZ}} \right|.$$

Proof. Since U_Z, U_Y are points on the radical axes of the circle pairs $X(r_X)$, $Y(r_Y)$, and $X(r_X)$, $Z(r_Z)$ respectively, by Lemma 3 we have

$$r = \frac{\overrightarrow{XY} \cdot \Pi_{\mathcal{L}}(\overrightarrow{U_ZK})}{r_X - r_Y} = \frac{\overrightarrow{XZ} \cdot \Pi_{\mathcal{L}}(\overrightarrow{U_YK})}{r_X - r_Z}.$$

Since X, Y, Z are all on the line $\mathcal{L}$, $\Pi_{\mathcal{L}}(\overrightarrow{U_ZK}) = \frac{r(r_X - r_Y)}{XY}\overrightarrow{\jmath}$ and $\Pi_{\mathcal{L}}(\overrightarrow{U_YK}) = \frac{r(r_X - r_Z)}{XZ}\overrightarrow{\jmath}$, where $\overrightarrow{\jmath}$ is a unit vector on $\mathcal{L}$. Hence,

$$\begin{aligned}\Pi_{\mathcal{L}}(\overrightarrow{U_YU_Z}) &= \Pi_{\mathcal{L}}(\overrightarrow{U_YK}) - \Pi_{\mathcal{L}}(\overrightarrow{U_ZK}) \\ &= r\left(\frac{r_X - r_Z}{XZ} - \frac{r_X - r_Y}{XY}\right)\overrightarrow{\jmath}\end{aligned}$$

From this the result follows. □

2.2. *Arbelos of type A.* On a line $\mathcal{L}$ we take the consecutive points U'_Z, U_Y, U_Z, U'_Y, and construct on the same side of the line the semicircles (U'_ZU_Z), $(U_YU'_Y)$, and (U_YU_Z). Let U_X be the intersection of the first two semicircles (see Figure 4). The arbelos $U_XU_YU_Z$ is of type A. It has arc U_YU_Z positive, and U_ZU_X, U_XU_Y both negative. The diameter U_YU_Z is the base of the arbelos.

2.2.1. *The incircle.* Let $K(r)$ be the incircle of this arbelos and A, B, C the points of tangency. If S is the external center of similitude of the semicircles (U'_ZU_Z) and $(U_YU'_Y)$, then the line BC passes through S, and the inversion with pole S and power $d^2 = SU_Y \cdot SU_Z = SU_X^2 = SB \cdot SC$ swaps the semicircles (U'_ZU_Z) and $(U_YU'_Y)$, and leaves the circle $K(r)$ and the semicircle (U_YU_Z) invariant. Hence, $SA^2 = SB \cdot SC$, and the SA is tangent at A to both $K(r)$ and (U_YU_Z). If the line SA meets the perpendiculars to $\mathcal{L}$ at U_Y, U_Z at D and E respectively, then D is the radical center of $K(r)$, (U_YU_Z), $(U_YU'_Y)$, and E is the radical center of $K(r)$, (U_YU_Z), (U'_ZU_Z). Hence, $DC = DA$ and $EB = EA$.

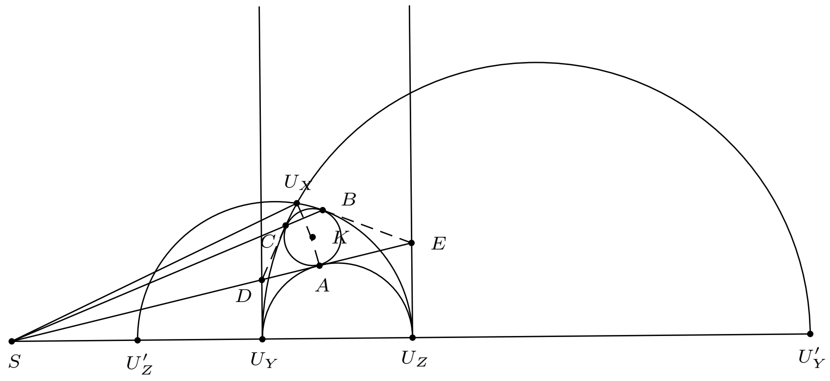

Figure 4.

Construction of the incircle. We construct the external center of similitude S of the semicircles (U'_ZU_Z) and $(U_YU'_Y)$, and take a point A on the semicircle (U_YU_Z) such that $SA = SU_X$. Let the line SA meet the perpendiculars to $\mathcal{L}$ through U_Y, U_Z at D, E respectively. We take on (U'_ZU_Z) the point B such that $EB = EA$, and on $(U_YU'_Y)$ the point C such that $DC = DA$. The circumcircle of ABC is the incircle of the arbelos.

The radius of the incircle. If we take $U'_ZU_Y = 2y$, $U_YU_Z = 2R$, $U_ZU'_Y = 2z$, then $r_X = R$, $r_Y = -R-y$, $r_Z = -R-z$, $XY = -y$, and $XZ = z$. From Theorem 4, we have

$$r = \left| \frac{2R}{\frac{2R+y}{-y} - \frac{2R+z}{z}} \right| = \frac{Ryz}{R(y+z)+yz},$$

or

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} + \frac{1}{z}. \tag{2}$$

2.3. *Arbelos of type B.* On a line $\mathcal{L}$ we take the consecutive points U'_Y, U'_Z, U_Y, U_Z, and construct on the same side of the line the semicircles (U'_ZU_Z), (U'_YU_Y), and (U_YU_Z). Let U_X be the intersection of the first two semicircles (see Figure 5). The arbelos $U_XU_YU_Z$ is of type B. It has arc U_YU_Z positive and the arcs U_ZU_X, U_XU_Y of different signs. The base of the arbelos is U_YU_Z.

Construction of the incircle. The construction is the same as in type A, but now S is the internal point of similitude of the semicircles (U'_ZU_Z) and (U'_YU_Y).

The radius of the incircle. We have the same formula as (2), but now since U'_Y is not on the right hand side of U_Z, the distance z must be negative, and so we have

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{y} - \frac{1}{z}. \tag{3}$$

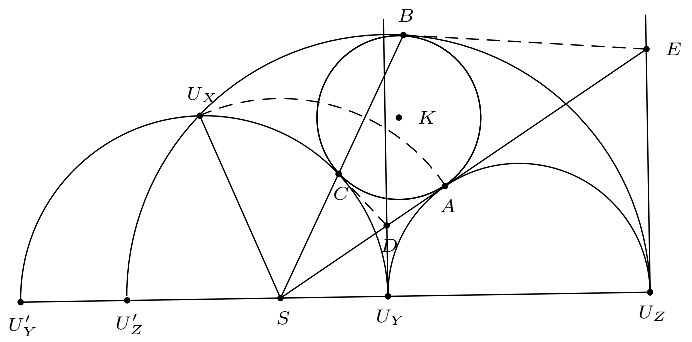

Figure 5.

Remark. For the incircle $K(r)$ of the Archimedean arbelos, since $U_X = A$, $U_Y = P$, $U_Z = B$ with base $2R_2$, $y = R_1$, $z = -R_1 - R_2$, (3) gives

$$\frac{1}{r} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2}.$$

3. Generalized Archimedean arbelos twins

We shall construct in the Archimedean arbelos a generalized pair of inscribed equal circles.

Theorem 5. *In the Archimedean arbelos* (Figure 6) *where $AP = 2R_1$, $PB = 2R_2$, $AB = 2R_1 + 2R_2$, we extend AB (to left and right) with equal segments $AA' = 2x = BB'$. The semicircle $(A'P)$ divides the arbelos in two arbeloi with incircles $K_1(r_1)$, $K_3(r_3)$, and the semicircle (PB') divides the arbelos in two arbeloi with incircles $K_2(r_2)$, $K_4(r_4)$. We have a couple of twin circles: $r_1 = r_2$ and $r_3 = r_4$, with*

$$\frac{1}{r_1} = \frac{1}{x} + \frac{1}{R_1} + \frac{1}{R_2} \quad \text{and} \quad \frac{1}{r_3} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1 + R_2 + x}.$$

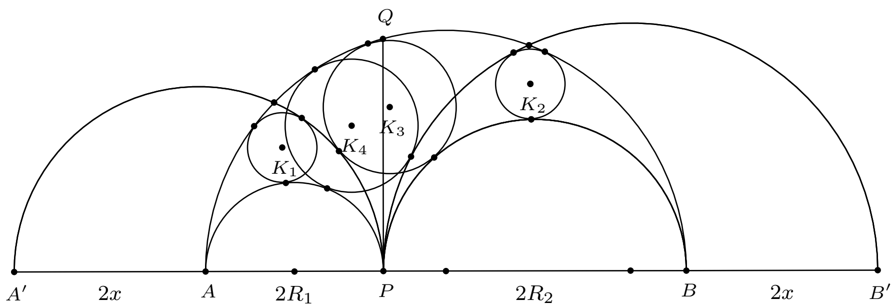

Figure 6.

Proof. The circle $K_1(r_1)$ is the incircle of an arbelos of type A with base AP. Hence, $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} + \frac{1}{R_2}$.

The circle $K_2(r_2)$ is the incircle of an arbelos of type A with base PB. Hence, $\frac{1}{r_2} = \frac{1}{R_2} + \frac{1}{x} + \frac{1}{R_1}$.

The circle $K_3(r_3)$ is the incircle of an arbelos of type B with base PB. Hence, $\frac{1}{r_3} = \frac{1}{R_2} + \frac{1}{R_1} - \frac{1}{R_1+R_2+x}$.

The circle $K_4(r_4)$ is the incircle of an arbelos of type B with base AP. Hence, $\frac{1}{r_3} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_1+R_2+x}$.

Therefore, $r_1 = r_2$ and $r_3 = r_4$. □

Remark. If $x \to \infty$, then the semicircles $(A'P)$ and (PB') tend to the perpendicular semiline PQ, and the four incircles tend to the Archimedean twin circles.

4. Bisectors of the Archimedean Arbelos

We have seen that in the Archimedean arbelos, the semiline PQ divides the arbelos in two arbeloi with equal incircles. This semiline as a degenerate semicircle is like a bisector from P of the arbelos that produces the twins (Archimedean circles) with radius r_P such that

$$\frac{1}{r_P} = \frac{1}{R_1} + \frac{1}{R_2}.$$

We find the other two bisectors of the arbelos from the vertices A and B (Figure 7).

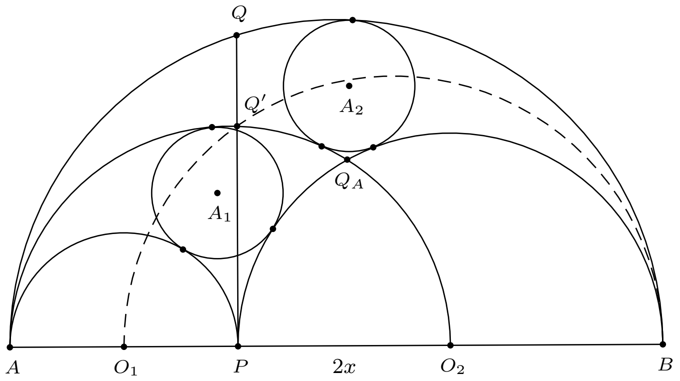

Figure 7.

Let the semicircle (AO_2) bisector of the arbelos meet the semicircle (PB) at Q_A, and $A_1(r_1)$, $A_2(r_2)$ be the incircles of the arbeloi Q_AAP, Q_AAB.

The arbelos Q_AAP with base AP is of type B, so we have $\frac{1}{r_1} = \frac{1}{R_1} + \frac{1}{x} - \frac{1}{R_1+R_2}$.

The arbelos Q_AAB with base AB is the complement of an arbelos of type A; so we have $\frac{1}{r_2} = -\left(\frac{1}{R_1+R_2} - \frac{1}{R_1} - \frac{1}{R_2-x}\right)$. In order to have $r_1 = r_2 = r_A$, we set $2x = R_2$. Hence the point O_2 is the midpoint of PB, and the bisector semicircle (AO_2)meets PQ at the point Q' such that $PQ' = \sqrt{AP \cdot PO_2} = \sqrt{2R_1R_2}$, and $\frac{1}{r_A} = \frac{1}{R_1} + \frac{2}{R_2} - \frac{1}{R_1+R_2}$.

Similarly, if O_1 is the midpoint of AP, then the semicircle (O_1B) is the bisector of the arbelos from B that passes also from Q', and $\frac{1}{r_B} = \frac{2}{R_1} + \frac{1}{R_2} - \frac{1}{R_1+R_2}$.

Hence, *the three bisectors of the Archimedean Arbelos are concurrent at Q'.*

5. Infinitely many Archimedean circles

There are many exciting constructions of Archimedean circles or families of these; see [4, 5].

Here we construct a more natural family of Archimedean circles that contains the original Archimedean twins. In the Arcimedean arbelos with diameters AP, PB, AB, and semicircles $O_1(R_1)$, $O_2(R_2)$, $O(R)$, we take at the left of A the point X, and at right of B the point Y such that $AX = BY = 2x$. We rotate clockwise P around A by an angle $\frac{\pi}{2}$ at A_1, and counterclockwise P around B by an angle $\frac{\pi}{2}$ at B_1. The line XA_1 meets the line PQ at D_1. The line YB_1 meets the line PQ at C_1. We take at the left of A the point C such that $CA = PC_1 = 2d_1$, and at right of B the point D such that $BD = PD_1 = 2d_2$. The semicircles (CP), (AY) meet at M, and the semicircles (PD), (XB) meet at N. We show that the incircles $K_1(r_1)$, $K_2(r_2)$ of the arbeloi MAP, NPB are both Archimedean circles.

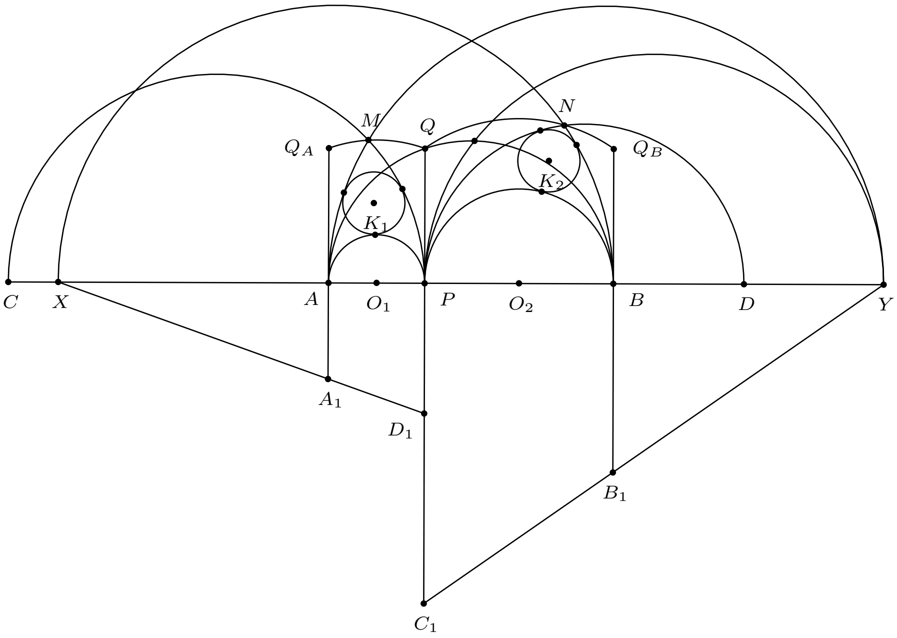

Figure 8.

Proof. Since in triangle XPD_1, PA_1 is a bisector and AA_1 is parallel to PD_1 (Figure 8), we know that $\frac{1}{PD_1} + \frac{1}{XP} = \frac{1}{AA_1}$. Hence,

$$\frac{1}{d_2} + \frac{1}{x+R_1} = \frac{1}{R_1}. \tag{4}$$

Similarly, from triangle PYC_1, we have

$$\frac{1}{d_1} + \frac{1}{x+R_2} = \frac{1}{R_2}. \tag{5}$$

The arbelos MAP is of type A; so we have $\frac{1}{r_1}=\frac{1}{R_1}+\frac{1}{d_1}+\frac{1}{x+R_2}=\frac{1}{R_1}+\frac{1}{R_2}$ from (5).

The arbelos NPB is also of type A; so we have $\frac{1}{r_2}=\frac{1}{R_2}+\frac{1}{d_2}+\frac{1}{x+R_1}=\frac{1}{R_2}+\frac{1}{R_1}$ from (4).

Hence $r_1=r_2$ is the radius of the Archimedean circle. □

For $x=0$ the circles $K_1(r_1)$, $K_2(r_2)$ coincide with the original Archimedean twin circles since the semicircles (AY), (XB) coincide with the semicircle (AB), and the semicircles (CP), (PD) coincide with the line PQ. If M_1, M_2 are the midpoints of CP, AY, then applying Stewart's theorem to triangle MM_1M_2, we have

$$O_1M^2\cdot M_1M_2=M_1M^2\cdot O_1M_2+M_2M^2\cdot O_1M_1-O_1M_1\cdot O_1M_2\cdot M_1M_2,$$

or

$$(d_1+R_2+x)O_1M^2=(d_1+R_1)^2(R_2+x)+(R_1+R_2+x)^2d_1-d_1(R_2+x)(d_1+R_2+x).$$

Substituting $d_1=\frac{R_2(x+R_2)}{x}$, we get

$$O_1M^2=R_1^2+4R_1R_2=O_1P^2+PQ^2=O_1Q^2.$$

Hence, the locus of M is the circular arc $O_1(Q)$ from the point Q to Q_A on the perpendicular Q_AA to AB. Similarly, we can prove that the locus of N is the circular arc $O_2(Q)$ from the point Q to Q_B on the perpendicular Q_BB to AB.

6. A special generalization of arbelos with arcs of angle 2ϕ

If we substitute the semicircles in Archimedean arbelos with arcs of angle 2ϕ, i.e., $AP=2R_1\sin\phi$, $PB=2R_2\sin\phi$, $AB=2(R_1+R_2)\sin\phi$ (Figure 9) [1], the points A, O_1, O are collinear; so are the points O, O_2, B. The tangent to the arc (AP) at P meets the arc (AB) at Q_A, and the tangent to the arc (PB) at P meets the arc (AB) at Q_B. Let $K_1(r_1)$, $K_2(r_2)$ be the incircles of the arbeloi Q_BAP and Q_APB. If $2\phi=\pi$, then we have the classical arbelos, and Q_A, Q_B coincide with the point Q. We prove that $r_1=r_2$.

Proof. The point A is on the radical axis of $O_1(R_1)$, $O_-(R_1+R_2))$. From (1), we have

$$r_1=\frac{\overrightarrow{O_1O}\cdot\Pi_{O_1O}(\overrightarrow{AK_1})}{r_{O_1}-r_O}=\frac{R_2\cdot\Pi_{O_1O}(AK_1)}{R_1+R_1+R_2}.$$

Also, $r_1=\Pi_{O_1O}(K_1P)$. Hence,

$$2R_1\sin^2\phi=\ \Pi_{O_1O}(AK_1)+\Pi_{O_1O}(K_1P)=\ r_1\cdot\frac{2R_1+R_2}{R_2}+r_1=\ r_1\cdot\frac{2(R_1+R_2)}{R_2},$$

or $r_1=\frac{R_1R_2\sin^2\phi}{R_1+R_2}$. Similarly,

$$r_2=\frac{\overrightarrow{OO_2}\cdot\Pi_{OO_2}(\overrightarrow{BK_2})}{r_O-r_{O_2}}=\frac{R_1\cdot\Pi_{OO_2}(BK_2)}{-R_1-R_2-R_2},$$

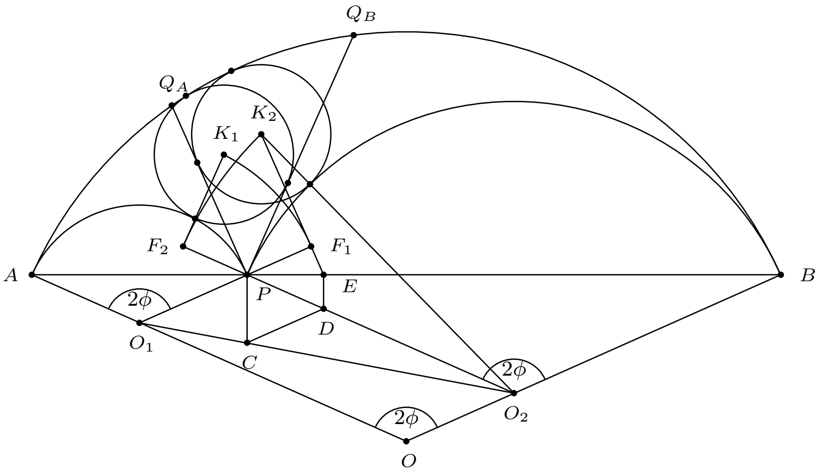

Figure 9.

and $r_2 = \Pi_{OO_2}(PK_2)$. Hence,

$$2R_2 \sin^2 \phi = \Pi_{OO_2}(PK_2) + \Pi_{OO_2}(K_2B) = r_2 + r_2 \cdot \frac{R_1 + 2R_2}{R_1} = r_2 \cdot \frac{2(R_1 + R_2)}{R_1},$$

or $r_2 = \frac{R_1R_2 \sin^2 \phi}{R_1+R_2}$.

If $2\phi = \pi$, then we have the radius of the Arcimedean circle. □

Construction of the twins. The perpendicular from P to AB meets O_1O_2 at the point C, and the parallel from C to PO_1 meets PO_2 at D. Since PC is a bisector in triangle PO_1O_2, we have $\frac{1}{CD} = \frac{1}{R_1} + \frac{1}{R_2}$. Hence we need the construction of $r_1 = CD \cdot \sin^2 \phi$. The perpendicular from D to AB meets AB at E and the perpendicular from E to PO_1 meets this line at the point F_1. The symmetric of F_1 in PC is the point F_2 on the line PO_2. The perpendicular at F_2 to PF_2 meets the circle $O_1(F_1)$ at the point K_1 and the line EF_1 meets the circle $O_2(F_2)$ at the point K_2. These points are the centers of the twin incircles and the construction of these circles is obvious.

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